

# Confluence of Layered Rewrite Systems

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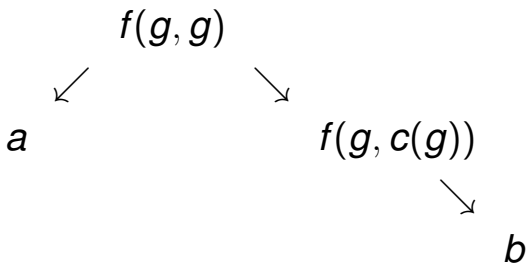


# I. Problem and Result

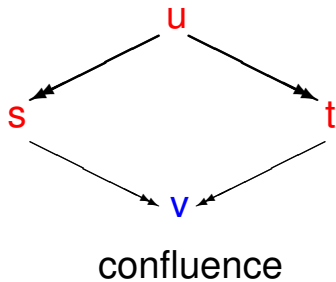
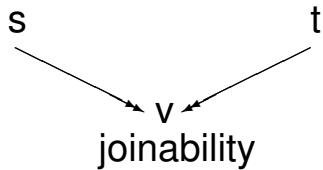
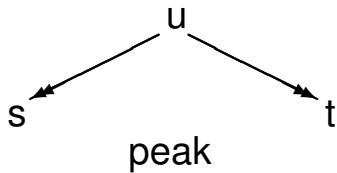
# Example

$$\left. \begin{aligned} f(x, x) &\rightarrow a \\ f(y, c(y)) &\rightarrow b \\ g &\rightarrow c(g) \end{aligned} \right\}$$

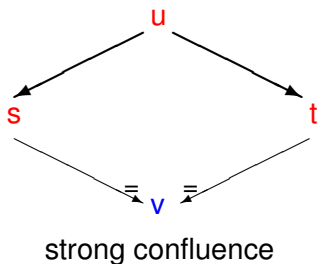
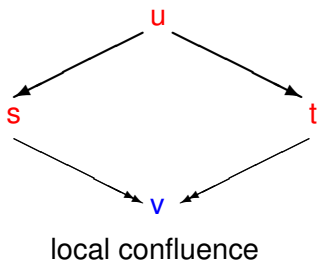
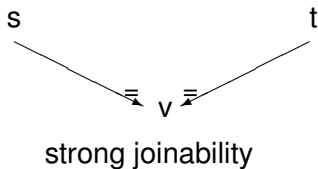
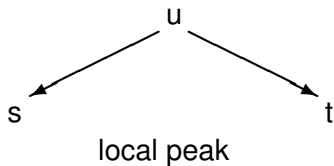
NKH is non-terminating, non-left-linear,  
right-ground, critical pair free, but **non-confluent**:



# Confluence of an abstract relation $\longrightarrow$



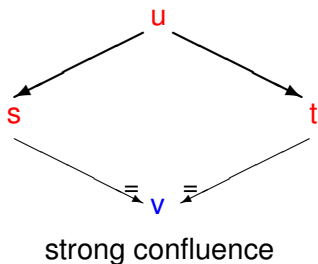
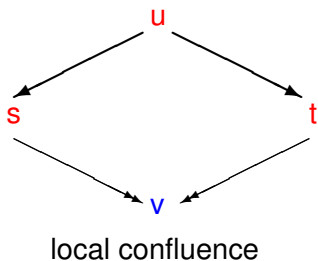
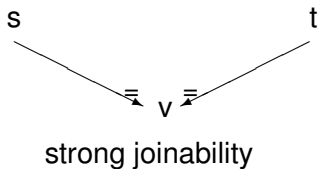
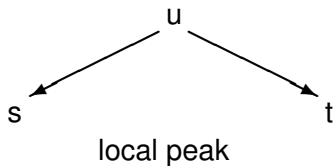
# Abstract Local Analysis



[Newman, 1942] Assuming  $\rightarrow$  terminates, local confluence implies confluence.

[Hindley, 1964] Strong confluence always implies confluence.

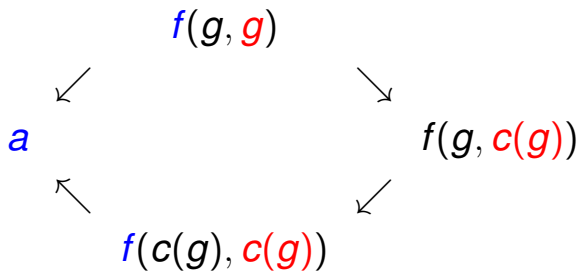
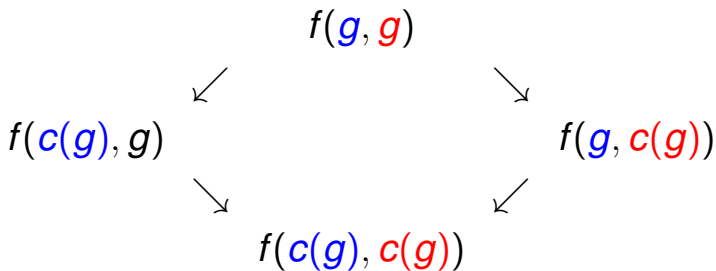
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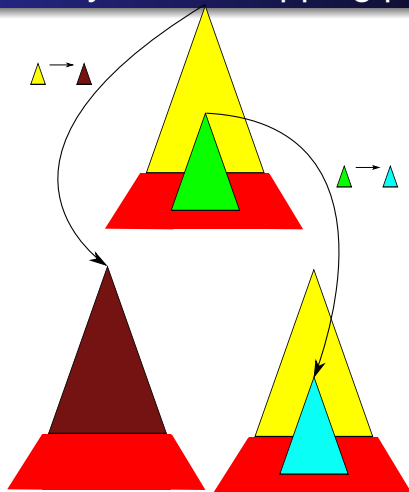
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# Concrete local analysis: disjoint and ancestor peaks



# Concrete local analysis: overlapping peaks

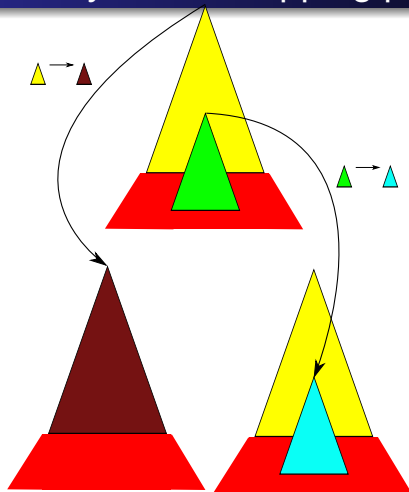


Joinability of *critical pairs* implies:

- joinability of all local peaks [Huet, 80]
- confluence of terminating systems [Knuth-Bendix, 69]

SJ of critical pairs implies SC of linear systems [Huet, 80]

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Find a class of rewriting systems, which

- is **syntactically** defined
- contains NKH and is **robust** to variations
- allows for **non-linearities**
- confluence checkable on **suitable critical pairs**

Existing classes containing NKH do not answer the question because they either

- are non-syntactic (relative termination)
- use linearity (right ground, simple right linear)
- lack robustness (depth-preserving)
- ban critical pairs (non-E-overlapping)

**Theorem:** a rank non-increasing layered rewrite system is confluent if its cyclic critical pairs have cyclic-joinable decreasing diagrams.

**Fact:** cyclic critical pair free rank increasing layered rewrite systems need not be confluent.

**Conjecture:** a rank non-increasing layered rewrite system is confluent iff its realizable cyclic critical pairs have cyclic-joinable decreasing diagrams.

## II. Layered rewrite systems

# Layered rewrite system

Linearized overlaps in lefthand sides are allowed at disjoint positions, but not at prefix positions of one another.

$$\{h(f(x, x)) \rightarrow a \\ f(y, c(y)) \rightarrow b \\ g \rightarrow c(g)\}$$

is layered, but the following system is not:

$$\{f(h(x)) \rightarrow x \\ h(a) \rightarrow a \\ a \rightarrow b\}$$

The rank of a term  $t$  is the maximum number of linearized redexes traversed from the root to some leaf of  $t$ , counting 1 for an overlapping redex.

Let again

$$\{h(f(x, x)) \rightarrow a$$
$$f(y, c(y)) \rightarrow b$$
$$g \rightarrow c(g)\}$$

Then  $h(f(c(g), c(g)))$  has rank **2**, not **3** !

A layered system is **rank non-increasing** if the rank does not increase along rewrites.

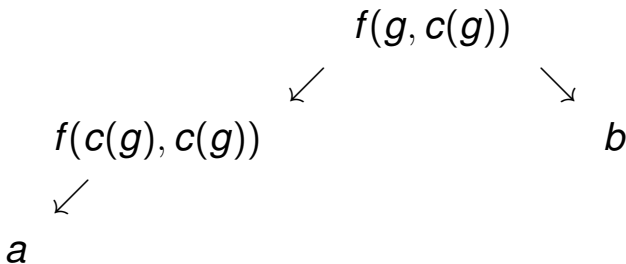
All layered rewrite systems listed so far are rank non-increasing. The following is rank increasing:

$$\{d(x, x) \rightarrow 0 \quad f(y) \rightarrow d(y, f(y)) \quad c \rightarrow f(c)\}$$

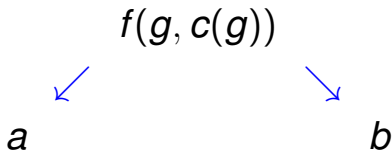
### III. Sub-rewriting

# Sub-rewriting

*Sub-rewriting* uses preparatory rewriting below the lhs of a rule until it becomes an actual redex, hence transforming the non-local rewrite peak



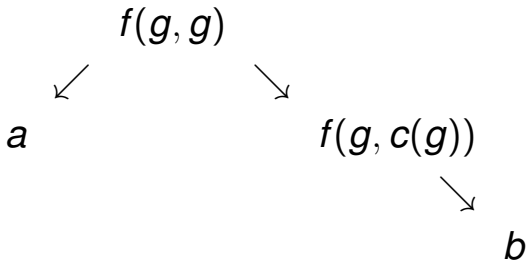
into the local sub-rewriting peak:



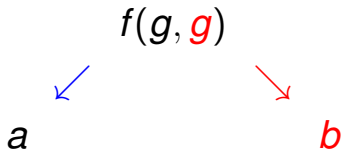


# Sub-rewriting

Note that our starting non-local rewrite peak



cannot be transformed as before into :



because  $g$  is not in the substitution of  $f(y, c(y))$ .

It must actually be considered as a composition of an ancestor rewrite peak and the previous sub-rewriting overlapping peak:

$$\begin{array}{c} f(g, g) \\ \swarrow \quad \searrow \\ a \leftarrow f(c(g), c(g)) \leftarrow f(g, c(g)) \\ \quad \quad \quad \searrow \\ \quad \quad \quad b \end{array}$$

## IV. Cyclic unification

serves computing sub-rewriting critical pairs.

*Cyclic solved forms* of a unification problem  $P$  different from  $\perp$  are unification problems

$\bar{x} = \bar{u} \wedge \bar{y} = \bar{v}$  such that

(i) variables in  $\bar{x}, \bar{y}$  are all different

(ii)  $\forall u \in \bar{u}, \mathcal{V}ar(u) \cap (\bar{x} \cup \bar{y}) = \emptyset$

(iii)  $\forall v \in \bar{v}, v \notin \mathcal{X}, \mathcal{V}ar(v) \cap \bar{x} = \emptyset, \mathcal{V}ar(v) \cap \bar{y} \neq \emptyset$

Equations in  $\bar{y} = \bar{v}$  are called *cyclic*

$$f(x, x) \rightarrow a \quad f(y, c(y)) \rightarrow b \quad g \rightarrow c(g)$$

Cyclic unification:

$$f(x, x) = f(y, c(y)) \Rightarrow$$

$$x = y \wedge x = c(y) \Rightarrow$$

$$x = y \wedge y = c(y) \Rightarrow$$

$$x = c(y) \wedge y = c(y)$$

Most general cyclic unifier:

$$\langle \{ \}, \{ x \rightarrow c(y), y \rightarrow c(y) \} \rangle$$

A *cyclic rewrite system* is a set of rules  $R = \{\bar{y} \rightarrow \bar{v}\}$  such that  $\mathcal{V}ar(v) \cap \bar{y} \neq \emptyset$  and  $v \notin \mathcal{X}$ . Variables in  $R$  are treated as constants.

A *cyclic unifier* of a unification problem  $P$  is a pair  $\langle \eta, R \rangle$  made of an idempotent substitution  $\eta$  and a cyclic rewrite system  $R = \{\bar{y} \rightarrow \bar{v}\}$  st:

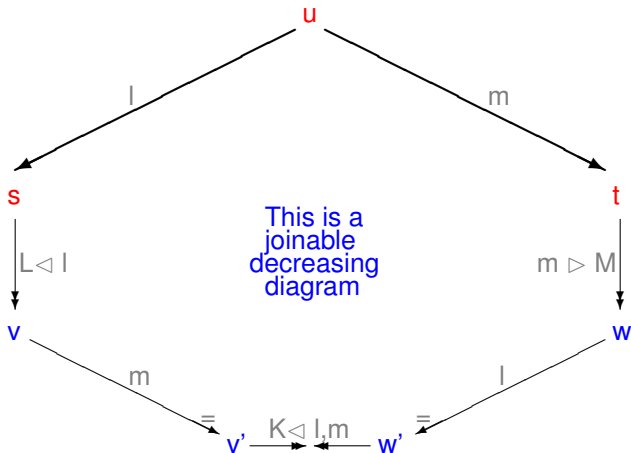
- (i)  $P$  and  $P \wedge R$  have identical sets of solutions ;
- (ii)  $(\forall u = v \in P) u\eta =_{R\eta}^{cc} v\eta$ .

**Theorem:** Given a solved form  $S := \bar{x} = \bar{u} \wedge \bar{y} = \bar{v}$  for  $P$ , the *canonical  $S$ -based cyclic unifier*  $\langle \eta_S := \bar{x} \mapsto \bar{u}, R_S := \bar{y} \rightarrow \bar{v} \rangle$  is most general among the set of cyclic unifiers of  $P$ .

## V. Decreasing diagrams

# Joinable decreasing diagrams

Given: a labelled relation  $\longrightarrow$  on a set  $S$ , which set of labels is ordered by the partial well-founded order  $\triangleright$ .



**Theorem: [van Oostrom]** If every local peak for  $\longrightarrow$  has a joinable decreasing diagram, then  $\longrightarrow$  **confluent**.



Let  $l \xrightarrow{i} r, g \xrightarrow{j} d \in R$  and  $p \in \mathcal{FP}os(l)$  such that  $l|_p = g$  has a cyclic solved form  $S$ . Then,

- $\langle r\eta_S, l[d]_p\eta_S \rangle$  is called a *cyclic critical pair*
- the CCP  $\langle r\eta_S, l[d]_p\eta_S \rangle$  has a *cyclic-joinable decreasing diagram* if

$$r\eta_S \xrightarrow{I} s \stackrel{CC}{=}_{R_S} t \xleftarrow{J} l[d]_p\eta_S$$

whose sequences of rule indices  $I$  and  $J$  satisfy the decreasing diagram condition.

**Theorem:** A rank non-increasing layered rewrite system is confluent if its cyclic critical pairs have cyclic-joinable decreasing diagrams.

**Proof:** by induction on the rank of terms, using van Oostrom's theorem for the 3 kinds of peaks. The critical peak case uses a lifting argument to show that critical decreasing diagrams remain decreasing under substitution instance.

## VI. Tightness

# Rank non-increasingness is necessary

$$\{d(x, x) \rightarrow 0 \quad f(y) \rightarrow d(y, f(y)) \quad c \rightarrow f(c)\}$$

This rank increasing layered system is non-confluent since:

$$f(fc) \longrightarrow d(fc, ffc) \longrightarrow d(ffc, ffc) \longrightarrow 0$$

$$f(fc) \longrightarrow f(d(c, fc)) \longrightarrow f(d(fc, fc)) \longrightarrow f0$$

But  $f0$  generates the regular tree language  $\{S \rightarrow d(0, S), S \rightarrow f0\}$  not containing 0.

The **only if** direction would require that all critical peaks have ground instances:

- 1 cyclic critical pairs are *realizable* by a ground substitution  $\theta$ :

$$\forall y \rightarrow v \in R_S \quad (y\theta \twoheadrightarrow_R \overset{\leftarrow}{R} v\theta)$$

- 2 realizable critical peaks have ground instances.

# Conclusion

# A substantial new result answering Huet's problem

But our main interest is not in plain first-order rewriting par se, but in

- dependent type theories with inductive types and universes, where confluence on typed terms, strong normalization and type preservation become inter-related. Our goal now is to reduce their confluence on untyped terms to the existence of decreasing diagrams for their critical pairs. In this respect, our progress looks very modest, since
  - Confluence in dependent type theories is **modulo conversion** and  $\alpha$ -renaming
  - Reductions (both  $\beta$  and recursors) are **rank-increasing** (but hopefully **left-linear**).
- Rewriting cyclic terms with sharing (applications to algebraic topology)