



**HAL**  
open science

# A characterization of switched linear control systems with finite L2-gain

Yacine Chitour, Paolo Mason, Mario Sigalotti

► **To cite this version:**

Yacine Chitour, Paolo Mason, Mario Sigalotti. A characterization of switched linear control systems with finite L2-gain. 2015. hal-01198394v1

**HAL Id: hal-01198394**

**<https://inria.hal.science/hal-01198394v1>**

Preprint submitted on 12 Sep 2015 (v1), last revised 7 Apr 2016 (v3)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# A characterization of switched linear control systems with finite $L_2$ -gain

Yacine Chitour, Paolo Mason \*, Mario Sigalotti<sup>†‡</sup>

September 12, 2015

## Abstract

Motivated by an open problem posed by J.P. Hespanha, we extend the notion of Barabanov norm and extremal trajectory to classes of switching signals that are not closed under concatenation. We use these tools to prove that finiteness of the  $L_2$ -gain is equivalent, for a large set of switched linear control systems, to the condition that the generalized spectral radius associated with minimal realizations of the original switched system is smaller than one.

## 1 Introduction

Let  $n, m, p$  be integers and  $\tau$  a positive real number. Consider the switched linear control system

$$\dot{x} = A_\sigma x + B_\sigma u, \quad y = C_\sigma x + D_\sigma u, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m, \quad y \in \mathbb{R}^p,$$

where  $\sigma$  is in the class  $\Sigma_\tau$  of piecewise constant signals with dwell time  $\tau$  taking values in a fixed finite set  $P$  of indices. Define the  $L_2$ -gain as

$$\gamma_2(\tau) = \sup \left\{ \frac{\|y_{u,\sigma}\|_2}{\|u\|_2} \mid u \in L_2([0, \infty), \mathbb{R}^m) \setminus \{0\}, \sigma \in \Sigma_\tau \right\},$$

where  $y_{u,\sigma}$  is the output corresponding to the trajectory of the system associated with  $u$  and  $\sigma$  starting at the origin at time  $t = 0$ . In [7], Hespanha posed Problem 4.1, where the following questions are asked: (i) under which conditions is the function  $\tau \mapsto \gamma_2(\tau)$  bounded

---

\*Y. Chitour and P. Mason are with Laboratoire des Signaux et Systèmes (L2S, UMR CNRS 8506), CNRS - CentraleSupélec - Université Paris-Sud, 3, rue Joliot Curie, 91192, Gif-sur-Yvette, France, [yacine.chitour@lss.supelec.fr](mailto:yacine.chitour@lss.supelec.fr), [paolo.mason@lss.supelec.fr](mailto:paolo.mason@lss.supelec.fr)

<sup>†</sup>M. Sigalotti is with INRIA Saclay, Team GECO & CMAP, École Polytechnique, Palaiseau, France, [mario.sigalotti@inria.fr](mailto:mario.sigalotti@inria.fr)

<sup>‡</sup>This research was partially supported by the iCODE Institute, research project of the IDEX Paris-Saclay, and by the Hadamard Mathematics LabEx (LMH) through the grant number ANR-11-LABX-0056-LMH in the “Programme des Investissements d’Avenir”.

over  $(0, \infty)$ ? (ii) when  $\gamma_2$  is not a bounded function over  $(0, \infty)$ , how to compute  $\tau_{\min}$ , the infimum of the dwell-times  $\tau > 0$  for which  $\gamma_2(\tau)$  is finite? (iii) how regular is  $\gamma_2(\cdot)$ ?

In [8], Hespanha proved the surprising fact that, in general,  $\lim_{\tau \rightarrow \infty} \gamma_2(\tau) > \max_{\sigma \in P} \gamma_2^\sigma$ , where  $\gamma_2^\sigma$  denotes the  $L_2$ -gain of the time-invariant control system where  $\sigma(\cdot) \equiv \sigma$ . These results have been improved in [10] assuming a minimal realization assumption. In that reference,  $\gamma_2(\tau)$  is characterized in terms of a suitable switched Riccati equation, in the spirit of  $H_\infty$ -theory and worst-case switching laws. Similar questions have been considered in [9] for other classes of switching signals (average dwell-time, persistent-dwell time, ...). In the latter work a restrictive hypothesis for a class of switching signals has been put forward, namely, that of closure under concatenation.

Indeed, the main obstacle to address Hespanha's problem lies in the fact that  $\Sigma_\tau$  is not closed under concatenation for  $\tau > 0$ . This issue has already been addressed by Wirth in [15], where the author proves the continuity of the map  $\tau \mapsto \rho(\tau)$ , where  $\rho(\tau)$  denotes the generalized spectral radius associated with the switched linear system  $\dot{x} = A_\sigma x$ ,  $\sigma \in \Sigma_\tau$ . The approach followed by Wirth consists of building parameterized families of norms behaving as extremal norms on suitable subsets of  $\Sigma_\tau$ . The construction adopted here to tackle the lack of closure under concatenation, although reminiscent of that of [15], follows a different path. We require that the class of switching signals  $\Sigma$  admits a subset  $\hat{\Sigma}$  large enough to encompass the asymptotic properties of the switched linear system  $\dot{x} = A_\sigma x$ ,  $\sigma \in \Sigma_\tau$ , and well-behaved with respect to concatenation. The flows associated with  $\hat{\Sigma}$  define a semigroup of matrices, which we call *quasi-Barabanov semigroup*. Its analysis allows us to point out the role played by special trajectories, which we call *quasi-extremal* and which describe the asymptotic worst-case behavior of the original switched system, which naturally extend the notions of Barabanov norm and extremal trajectory (see [2] and also [14]).

Several of our results hold true under a suitable observability condition, that we call *uniform observability*, see Definition 21, given for classes  $\Sigma$  of switching signals which are not closed under concatenation. In the context of Hespanha's problem, i.e., for the class  $\Sigma_\tau$  of switching signals with dwell-time larger than or equal to  $\tau > 0$ , the uniform observability assumption amounts to require every pair  $(A_\sigma, C_\sigma)$ ,  $\sigma \in P$ , to be observable and therefore this assumption is weaker than the minimal realization assumption of [10].

Combining the notions of quasi-Barabanov semigroup and quasi-extremal trajectory with the characterization of controllability and observability for switched linear control systems given in [13] we prove, under the uniform observability condition, that  $\gamma_2$  is finite if and only if  $\rho_{\min} < 1$ , where  $\rho_{\min}$  is the generalized spectral radius associated with a minimal realization of the original switched control system (analogously to the situation in the unswitched case). This result allows one to reduce the specific questions posed by Hespanha in [7, Problem 4.1] about  $L_2$ -gain issues to the computation of the above mentioned generalized spectral radius. When the uniform observability assumption does not hold true, the situation is more complicated as we illustrate by means of an example: we construct a 3-dimensional switched linear control system with piecewise constant arbitrary switching laws, which is controllable, observable, has finite  $L_2$ -gain, and whose uncontrolled dynamics have generalized spectral radius equal to one.

The last result of the paper establishes the right-continuity of the map  $\tau \mapsto \gamma_2(\tau)$ , providing a partial answer to Question (i) and to Question (iii) in [7, Problem 4.1].

## 2 Preliminaries

### 2.1 Notations

If  $n, m$  are nonnegative integers, the set of  $n \times m$  matrices with real entries is denoted  $M_{n,m}(\mathbb{R})$  and simply  $M_n(\mathbb{R})$  if  $n = m$ . We use  $\text{Id}_n$  to denote the  $n \times n$  identity matrix. A norm on  $\mathbb{R}^n$  is denoted  $\|\cdot\|$  and similarly for the induced operator norm on  $M_n(\mathbb{R})$ . A subset  $\mathcal{M}$  of  $M_n(\mathbb{R})$  is said to be irreducible if the only invariant subspaces by the elements of  $\mathcal{M}$  are  $\{0\}$  or  $\mathbb{R}^n$ . For every  $s, t \geq 0$  and  $A(\cdot) \in L_\infty([s, s+t], M_n(\mathbb{R}))$ , denote by  $\overrightarrow{\exp} \int_s^{s+t} A(\tau) d\tau \in M_n(\mathbb{R})$  the flow (or fundamental matrix) of  $\dot{x}(\tau) = A(\tau)x(\tau)$  from time  $s$  to time  $s+t$ . Given two signals  $A_j : [0, t_j] \rightarrow \mathcal{M}$ ,  $j = 1, 2$ , we denote by  $A_1 * A_2 : [0, t_1 + t_2] \rightarrow \mathcal{M}$  the *concatenation of  $A_1(\cdot)$  and  $A_2(\cdot)$* , i.e., the signal coinciding with  $A_1(\cdot)$  on  $[0, t_1]$  and with  $A_2(\cdot - t_1)$  on  $(t_1, t_1 + t_2]$ . Similarly, if  $\mathcal{A}$  and  $\mathcal{B}$  are two subsets of signals, we use  $\mathcal{A} * \mathcal{B}$  to denote the set of signals obtained by concatenation of a signal of  $\mathcal{A}$  and a signal of  $\mathcal{B}$ .

Let  $n, p$  and  $m$  be integers. Consider now switched linear control systems of the type

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad y(t) = C(t)x(t) + D(t)u(t), \quad (1)$$

where  $x \in \mathbb{R}^n$ ,  $u \in \mathbb{R}^m$ ,  $y \in \mathbb{R}^p$ ,  $(A(\cdot), B(\cdot), C(\cdot), D(\cdot))$  belongs to a class  $\mathcal{T}$  of measurable switching laws taking values in a bounded set of quadruples of matrices  $\mathcal{M} \subset M_n(\mathbb{R}) \times M_{n,m}(\mathbb{R}) \times M_{p,n}(\mathbb{R}) \times M_{p,m}(\mathbb{R})$ .

For  $t \geq 0$  and a switching law  $(A(\cdot), B(\cdot), C(\cdot), D(\cdot)) \in \mathcal{T}$ , the controllability and observability Gramians in time  $t$  are defined respectively as

$$\begin{aligned} & \int_0^t (\overrightarrow{\exp} \int_s^0 A(\tau) d\tau) B(s) B(s)^T (\overrightarrow{\exp} \int_s^0 A(\tau) d\tau)^T ds, \\ & \int_0^t (\overrightarrow{\exp} \int_0^s A(\tau) d\tau)^T C(s)^T C(s) (\overrightarrow{\exp} \int_0^s A(\tau) d\tau) ds. \end{aligned} \quad (2)$$

For  $T > 0$ , let  $L_2(0, T)$  be the Hilbert space of measurable functions  $u : [0, T] \rightarrow \mathbb{R}^m$  with finite  $L_2$ -norm, i.e., such that  $\|u\|_{2,T} := \left( \int_0^T \|u(t)\|^2 dt \right)^{1/2}$  is finite. If  $T = \infty$ , we simply use  $L_2$  and  $\|u\|_2$  to denote respectively the corresponding Hilbert space and  $L_2$ -norm.

For  $u(\cdot) \in L_2$  and  $\sigma := (A(\cdot), B(\cdot), C(\cdot), D(\cdot)) \in \mathcal{T}$ , let  $y_{u,\sigma}$  be the corresponding output of Eq. (1) and define the  $L_2$ -gain associated with  $\mathcal{T}$  by

$$\gamma_2(\mathcal{T}) := \sup \left\{ \frac{\|y_{u,\sigma}\|_2}{\|u\|_2} \mid u(\cdot) \in L_2 \setminus \{0\}, (A(\cdot), B(\cdot), C(\cdot), D(\cdot)) \in \mathcal{T} \right\}. \quad (3)$$

In this paper, we investigate qualitative properties of  $\gamma_2(\mathcal{T})$  and in particular we are interested in finding conditions ensuring its finiteness. We will therefore assume from now on with no loss of generality that  $D(\cdot) \equiv 0$ .

### 2.2 Classes of switching functions

We introduce in this section several classes of switching signals contained in  $L_\infty([0, \infty), \mathcal{M})$ , for some set  $\mathcal{M}$  in a finite dimensional vector space.

- $\mathcal{S}^{\text{arb}}(\mathcal{M})$  is the class of *arbitrarily switching* signals, i.e.,  $\mathcal{S}^{\text{arb}}(\mathcal{M}) = L_\infty([0, \infty), \mathcal{M})$ ;
- $\mathcal{S}^{\text{pc}}(\mathcal{M})$  is the class of piecewise constant signals;
- $\mathcal{S}^{\text{d},\tau}(\mathcal{M})$  is the class of piecewise constant signals with *dwell-time*  $\tau > 0$ , i.e., such that the distance between two switching times is at least  $\tau$  (notice that  $\mathcal{S}^{\text{pc}}(\mathcal{M})$  can be identified as  $\mathcal{S}^{\text{d},0}(\mathcal{M})$ );
- $\mathcal{S}^{\text{av-d},\tau,N_0}(\mathcal{M})$  is the class of piecewise constant signals which satisfy the  $(\tau, N_0)$  *average dwell-time* condition with  $\tau > 0$  and  $N_0$  a positive integer: for every  $s, t \geq 0$ , the number of switching times in  $[s, s+t]$  is bounded from above by  $N_0 + t/\tau$ ;
- for  $\mathcal{M}$  of the type  $\{M_\delta = (1 - \delta)M_0 + \delta M_1 \mid \delta \in [0, 1]\}$ ,

$$\mathcal{S}^{\text{pe},T,\mu}(\mathcal{M}) = \left\{ M_{\alpha(\cdot)} \mid \alpha \in L_\infty([0, \infty), [0, 1]), \int_t^{t+T} \alpha(s) ds \geq \mu \forall t \geq 0 \right\}$$

is the class of  $(T, \mu)$ -*persistently exciting* signals with  $0 < \mu \leq T$ ;

- $\mathcal{S}^{\text{lip},L}(\mathcal{M})$  is the class of Lipschitz signals with Lipschitz constant  $L > 0$ ;
- $\mathcal{S}^{\text{BV},T,\nu}(\mathcal{M})$  is the class of  $(T, \nu)$ -*BV signals*, i.e., the signals whose restriction to every interval of length  $T$  has total variation at most  $\nu$ , that is,  $A(\cdot) \in \mathcal{S}^{\text{BV},T,\nu}(\mathcal{M})$  if and only if

$$\sup_{\substack{t \geq 0, k \in \mathbb{N} \\ t=t_0 \leq t_1 \leq \dots \leq t_k = t+T}} \sum_{i=1}^k \|A(t_i) - A(t_{i-1})\| \leq \nu.$$

Most of these classes have been already considered in [9]. For the class of persistently exciting signals, see for instance [3, 5, 12] and references therein. Notice that all the classes in the above list are shift-invariant.

Rather than addressing the issues at stake for each class of switching signal given above, we develop a unifying framework which can also be applied to other classes. For that purpose, we adopt an axiomatic approach which singles out and exploits some useful common properties satisfied by the classes above.

### 3 Adapted norms for switched linear systems with concatenable subfamilies

We consider in this section a switched linear system

$$\dot{x}(t) = A(t)x(t)$$

where  $A(\cdot)$  belongs to a class  $\mathcal{S}$  of measurable switching laws taking values in a bounded set of matrices  $\mathcal{M} \subset M_n(\mathbb{R})$ .

From now on the family  $\mathcal{S}$  is assumed to be shift-invariant, and we define the corresponding *generalized spectral radius* (see e.g. [14]) as

$$\rho(\mathcal{S}) = \limsup_{t \rightarrow +\infty} \sup_{A(\cdot) \in \mathcal{S}} \left\| \overrightarrow{\exp} \int_0^t A(\tau) d\tau \right\|^{1/t}. \quad (4)$$

Notice that, since  $\mathcal{M}$  is bounded then  $\rho(\mathcal{S})$  is finite.

The main difficulty in extending the original Barabanov norm construction beyond the class of arbitrary switching is that the set of all flows  $\overrightarrow{\exp} \int_s^{s+t} A(\tau) d\tau$ , for  $A(\cdot) \in \mathcal{S}$  and  $s, t \geq 0$ , does not form a semigroup, since in general signals in  $\mathcal{S}$  cannot be combined arbitrarily by concatenation within  $\mathcal{S}$ . A key object in what follows is then the identification of a subclass  $\mathcal{F}_\infty$  of  $\mathcal{S}$ , constructed by concatenating in an arbitrary way some signals defined on finite intervals. What we are doing in the following is to attach to  $\mathcal{F}_\infty$  a semigroup of fundamental matrices that captures the asymptotic behaviour of  $\mathcal{S}$  if  $\mathcal{F}_\infty$  is large enough.

### 3.1 Concatenable subfamilies

Consider a set  $\mathcal{F} = \cup_{t \geq 0} \mathcal{F}_t$  with  $\mathcal{F}_t \subset L_\infty([0, t], \mathcal{M})$ ,  $t \in [0, \infty)$ . Define

$$\mathcal{F}_\infty = \left\{ A_1 * A_2 * \dots * A_k * \dots \mid A_k(\cdot) \in \mathcal{F}_{t_k} \text{ for } k \in \mathbb{N}, \sum_{k \in \mathbb{N}} t_k = \infty \right\} \quad (5)$$

and  $\overrightarrow{\exp}(\mathcal{F}) = \cup_{t \geq 0} \overrightarrow{\exp}(\mathcal{F}_t)$ , where, for every  $t \geq 0$ ,

$$\overrightarrow{\exp}(\mathcal{F}_t) = \left\{ \overrightarrow{\exp} \int_0^t A(\tau) d\tau \mid A(\cdot) \in \mathcal{F}_t \right\}.$$

Let, moreover,

$$\mu(\mathcal{F}) = \limsup_{t \rightarrow +\infty} \left( \sup \left\{ \|R_t\|^{1/t} \mid R_t \in \overrightarrow{\exp}(\mathcal{F}_t) \right\} \right),$$

with the convention that the quantity inside the parenthesis is equal to 0 if  $\mathcal{F}_t$  is empty. Notice that  $\mu(\mathcal{F}) \leq \rho(\mathcal{F}_\infty)$ , but the converse is in general not guaranteed since the computation of  $\rho(\mathcal{F}_\infty)$  takes into account all intermediate instants between two concatenation times, unlike the one of  $\mu(\mathcal{F})$ .

We list below some useful assumptions on the pair  $(\mathcal{S}, \mathcal{F})$  that will be exploited in the sequel.

**A1 (concatenability)**  $\mathcal{F}_s * \mathcal{F}_t \subset \mathcal{F}_{s+t}$  for every  $s, t \geq 0$ ;

**A2 (irreducibility)**  $\overrightarrow{\exp}(\mathcal{F})$  is irreducible;

**A3 (fatness)**  $\mathcal{F}_\infty \subset \mathcal{S}$  and there exist two constants  $C, \Delta \geq 0$  and a compact subset  $\mathcal{K}$  of  $\text{GL}(n)$  such that for every  $t \geq 0$  and  $A(\cdot) \in \mathcal{S}$ , there exist  $K \in \mathcal{K}$ ,  $\hat{t} \in [t, t + \Delta]$ , and  $R \in \overrightarrow{\exp}(\mathcal{F}_{\hat{t}})$  such that

$$\left\| \overrightarrow{\exp} \int_0^{\hat{t}} A(\tau) d\tau K R^{-1} \right\| \leq C. \quad (6)$$

Moreover, if  $A(\cdot) \in \mathcal{F}_\infty$ , one can take  $K = \text{Id}_n$  in (6).

**Remark 1.** As a consequence of the definition of  $\mathcal{F}_\infty$ , if  $A(\cdot) \in \mathcal{F}$  and  $B(\cdot) \in \mathcal{F}_\infty$ , then  $A * B(\cdot) \in \mathcal{F}_\infty$ . Moreover, by Assumption **A1**, one has that  $\overrightarrow{\text{exp}}(\mathcal{F}_s)\overrightarrow{\text{exp}}(\mathcal{F}_t) \subset \overrightarrow{\text{exp}}(\mathcal{F}_{s+t})$  for every  $s, t \geq 0$ . Hence  $\overrightarrow{\text{exp}}(\mathcal{F})$  is a semigroup and Assumption **A2** above is equivalent to

$$\forall x \in \mathbb{R}^n \setminus \{0\}, \text{ the linear span of } \overrightarrow{\text{exp}}(\mathcal{F})x \text{ is equal to } \mathbb{R}^n. \quad (7)$$

As in [14], one then says that  $\overrightarrow{\text{exp}}(\mathcal{F})$  is an irreducible semigroup. Note that [14] considers special classes of irreducible semigroups which verify the additional assumption

$$\text{(decomposability)} \quad \mathcal{F}_{s+t} = \mathcal{F}_s * \mathcal{F}_t \text{ for every } s, t \geq 0.$$

The decomposability assumption trivially implies that  $\overrightarrow{\text{exp}}(\mathcal{F}_s)\overrightarrow{\text{exp}}(\mathcal{F}_t) = \overrightarrow{\text{exp}}(\mathcal{F}_{s+t})$  for every  $s, t \geq 0$ .

**Remark 2.** Recall that the map  $L_\infty([0, t], \mathcal{M}) \ni A(\cdot) \mapsto \overrightarrow{\text{exp}} \int_0^t A(\tau) d\tau \in M_n(\mathbb{R})$  is continuous with respect to the weak- $\star$  topology in  $L_\infty([0, t], \mathcal{M})$  (see, for instance, [3, Proposition 21]). In particular **A3** holds true for any  $A(\cdot)$  in the weak- $\star$  closure of  $\mathcal{S}$ .

**Lemma 3.** If **A3** holds true then  $\rho(\mathcal{S}) = \mu(\mathcal{F})$ . If moreover **A1** holds true then the family  $\mathcal{F}$  verifies the following version of Fenichel's uniformity lemma: assume that there exists  $m > 0$  such that, for every sequence  $t_j$  tending to infinity with  $R_{t_j} \in \overrightarrow{\text{exp}}(\mathcal{F}_{t_j})$ , one has  $\lim_{j \rightarrow \infty} m^{-t_j} R_{t_j} = 0$ . Then  $\mu(\mathcal{F}) < m$ .

*Proof.* The inequality  $\rho(\mathcal{S}) \geq \mu(\mathcal{F})$  is immediate. The opposite one readily comes from Assumption **A3** and the definitions of  $\rho(\mathcal{S})$  and  $\mu(\mathcal{F})$ . As regard the second part of the lemma, we can assume that  $m = 1$  by eventually replacing  $\mathcal{M}$  by the set  $\mathcal{M} - \log(m)\text{Id}_n$ . Let  $\mathcal{S}^*$  be the the closure of  $\mathcal{S}$  for the weak- $\star$  topology induced by  $L_\infty([0, \infty), \mathcal{M})$ . We first show that, under the hypotheses on  $\mathcal{F}$ , one has that every trajectory associated with a switching signal in  $\mathcal{S}^*$  tends to zero. We argue by contradiction. Hence there exists  $A(\cdot) \in \mathcal{S}^*$ ,  $x \in \mathbb{R}^n$ ,  $\varepsilon > 0$  and a sequence  $t_j$  tending to infinity such that

$$\|(\overrightarrow{\text{exp}} \int_0^{t_j} A(\tau) d\tau)x\| \geq \varepsilon.$$

By Remark 2, Assumption **A3** actually extends to any switching signal in  $\mathcal{S}^*$ . Hence, for every  $j \geq 0$  applying Assumption **A3** to the switching signal  $A(\cdot)$  and the time  $t_j$  yields the inequality

$$\|R_{\hat{t}_j} K_j^{-1} x\| \geq \varepsilon / C$$

for some  $R_{\hat{t}_j} \in \overrightarrow{\text{exp}}(\mathcal{F}_{\hat{t}_j})$ , with  $\hat{t}_j \in [t_j, t_j + \Delta]$ , and  $K_j$  belonging to a given compact of  $\text{GL}(n)$ . According to the hypothesis of the lemma the left-hand side of the above inequality tends to 0 as  $j$  goes to infinity, which is a contradiction.

Since the switching laws of  $\mathcal{S}$  take values in the bounded set  $\mathcal{M}$ , one has that the class  $\mathcal{S}^*$  is weak- $\star$  compact. Recalling that  $\mathcal{S}$  is shift-invariant, we notice that all the assumptions of Fenichel's uniformity lemma are verified by the standard linear flow defined on  $\mathbb{R}^n \times \mathcal{S}^*$  (cf. [6]). Therefore the convergence of trajectories of  $\mathcal{S}$  to 0 is uniformly exponential, i.e.,  $\rho(\mathcal{S}) < 1$ . By the first part of the lemma we deduce that  $\mu(\mathcal{F}) < 1$ .  $\square$

We associate with each class of switching signals considered in the previous section a corresponding concatenable subfamily as listed below:

- $\mathcal{F}^{\text{arb}}(\mathcal{M})$ : arbitrarily switching signals on finite intervals;
- $\mathcal{F}^{\text{pc}}(\mathcal{M})$ : piecewise constant signals on finite intervals;
- $\mathcal{F}^{\text{d},\tau}(\mathcal{M})$ : piecewise constant signals on finite intervals with dwell-time  $\tau$  and such that the first and last subintervals on which the signal is constant have length at least  $\tau$  (notice that  $\mathcal{F}_t^{\text{d},\tau}(\mathcal{M}) = \emptyset$  for  $t < \tau$ );
- $\mathcal{F}^{\text{av-d},\tau,N_0}(\mathcal{M})$ : piecewise constant signals on finite intervals satisfying the  $(\tau, N_0)$  average dwell-time condition and such that the first and last subintervals on which the signal is constant have length at least  $N_0\tau$ ;
- $\mathcal{F}^{\text{pe},T,\mu}(\mathcal{M})$ :  $(T, \mu)$ -persistently exciting signals on finite intervals  $[0, t]$  for which  $t \geq T - \mu$  and the signal is constantly equal to  $M_1$  on  $[t - T + \mu, t]$ , where we recall that  $\mathcal{M} = \{M_\delta = (1 - \delta)M_0 + \delta M_1 \mid \delta \in [0, 1]\}$ ;
- $\mathcal{F}^{\text{lip},L}(\mathcal{M})$ : Lipschitz signals on finite interval starting and ending at some  $\bar{M} \in \mathcal{M}$  fixed independently on the signal;
- $\mathcal{F}^{\text{BV},T,\mu}(\mathcal{M})$ :  $(T, \nu)$ -BV signals on finite intervals  $[0, t]$ ,  $t \geq T$ , starting and ending at some  $\bar{M} \in \mathcal{M}$  fixed independently on the signal and constant on  $[t - T, t]$ .

With these choices of  $\mathcal{F}$  Assumption **A1** is automatically satisfied (with **B1** holding only for  $\mathcal{F}^{\text{arb}}(\mathcal{M})$  and  $\mathcal{F}^{\text{pc}}(\mathcal{M})$ ). Assumption **A3** is also easy to check. Indeed, every restriction  $A|_{[s,s+t]}(\cdot)$  of a signal in one of the classes  $\mathcal{S}$  introduced in Section 2.2 can be extended to a signal  $A_1 * A|_{[s,s+t]}(s + \cdot) * A_2$  in the corresponding class  $\mathcal{F}$ , with  $A_j : [0, t_j] \rightarrow \mathcal{M}$ ,  $j = 1, 2$ , and  $t_1, t_2 \leq t_*$  for some  $t_*$  uniform with respect to  $A(\cdot) \in \mathcal{S}$ , and  $s, t \geq 0$ .

To address the validity of Assumption **A2** for the previous classes of signals, we further introduce the following assumption on the set  $\mathcal{F}$ , which essentially says that the flow corresponding to any element in  $\mathcal{F}^{\text{pc}}(\mathcal{M})$  can be approached in a suitable sense by an analytic deformation of flows corresponding to elements in  $\mathcal{F}$ .

**A4 (Analytic propagation)** For every  $t, \varepsilon > 0$  and  $A(\cdot) \in \mathcal{F}_t^{\text{pc}}(\mathcal{M})$ , there exists a path  $(0, 1] \ni \lambda \mapsto (t_\lambda, A_\lambda(\cdot)) \in (0, \infty) \times \mathcal{F}_{t_\lambda}^{\text{arb}}(\mathcal{M})$  such that

- $\lambda \mapsto \overrightarrow{\text{exp}} \int_0^{t_\lambda} A_\lambda(\tau) d\tau$  is analytic;
- $\|\overrightarrow{\text{exp}} \int_0^{t_1} A_1(\tau) d\tau - \overrightarrow{\text{exp}} \int_0^t A(\tau) d\tau\| \leq \varepsilon$ ;
- the set  $\{\lambda \in (0, 1] \mid A_\lambda(\cdot) \in \mathcal{F}\}$  has nonempty interior.

We then have the following technical lemma.

**Lemma 4.** *Let  $\mathcal{S}$  be one of the classes introduced in Section 2.2 with corresponding  $\mathcal{F}$  as above. Assume moreover that*



- if  $\mathcal{S} = \mathcal{S}^{\text{lip},L}(\mathcal{M})$  then  $\mathcal{M}$  is rectifiably pathwise connected, that is, every two distinct points of  $\mathcal{M}$  can be connected by a Lipschitz-continuous curve lying in  $\mathcal{M}$ ;
- if  $\mathcal{S} = \mathcal{S}^{\text{BV},T,\nu}(\mathcal{M})$  then for every  $M_0, M_1 \in \mathcal{M}$ , there exists a finite sequence of points in  $\mathcal{M}$  whose first and last elements are  $M_0$  and  $M_1$ , respectively, and such that the distance between two subsequent elements is smaller than  $\nu$ .

Then Assumptions **A1**, **A3** and **A4** hold true.

*Proof.* Let  $t, \varepsilon > 0$  and  $A(\cdot) \in \mathcal{F}_t^{\text{pc}}(\mathcal{M})$ . For the cases  $\mathcal{S} = \mathcal{S}^{\text{d},\tau}(\mathcal{M})$  and  $\mathcal{S} = \mathcal{S}^{\text{av-d},\tau,N_0}(\mathcal{M})$  one can find the path  $\lambda \mapsto A_\lambda(\cdot)$  as follows. For every  $\delta > 0$ , consider the time-reparameterized signal  $A(\delta \cdot) \in \mathcal{F}_{t/\delta}^{\text{pc}}(\mathcal{M})$ . Then  $A(\delta \cdot) \in \mathcal{F}$  for  $\delta$  small enough and the function  $\delta \mapsto \overrightarrow{\text{exp}} \int_0^{t/\delta} A(\delta\tau) d\tau$  is analytic. Indeed, the function  $(0, \infty) \ni \delta \mapsto \overrightarrow{\text{exp}} \int_0^{t_0/\delta} A(\delta\tau) d\tau = \overrightarrow{\text{exp}} \int_0^{t_0} \frac{1}{\delta} A(s) ds$  is analytic, since the Volterra series associated with this flow define an analytic function of  $\delta$ .

We conclude by taking  $A_\lambda(\cdot) = A(\lambda \cdot)$ . For the cases  $\mathcal{S} = \mathcal{S}^{\text{pe},T,\mu}(\mathcal{M})$  and  $\mathcal{S} = \mathcal{S}^{\text{BV},T,\mu}(\mathcal{M})$ , the previous construction can be modified as follows. First notice that, up to adding some short intervals on which  $A(\cdot)$  is constant we can assume that

**Case  $\mathcal{S} = \mathcal{S}^{\text{pe},T,\mu}(\mathcal{M})$ :** the value of  $A(\cdot)$  on the last interval on which it is constant is  $M_1$ ;

**Case  $\mathcal{S} = \mathcal{S}^{\text{BV},T,\nu}(\mathcal{M})$ :** the distance between two subsequent values of  $A(\cdot)$  is smaller than  $\nu$ , and the first and last values of  $A(\cdot)$  are both equal to  $\bar{M}$ .

The modification of  $A(\cdot)$  can be taken so that the corresponding variation of  $\overrightarrow{\text{exp}} \int_0^t A(\tau) d\tau$  is smaller than  $\varepsilon$ . The argument now works as before in the case  $\mathcal{S} = \mathcal{S}^{\text{BV},T,\nu}(\mathcal{M})$ . In the case  $\mathcal{S} = \mathcal{S}^{\text{pe},T,\mu}(\mathcal{M})$  one can take  $A_\lambda(s) = A(s/\lambda)$  on  $[0, \lambda t_0]$  and  $A_\lambda(s) = M_1$  on  $[\lambda t_0, t]$ , where  $t_0$  is such that  $A|_{[t_0, t]} \equiv M_1$ .

We are left to discuss the case  $\mathcal{S} = \mathcal{S}^{\text{lip},L}(\mathcal{M})$ . Similarly to what is done above, we can first assume that  $A(\cdot)$  is equal to  $\bar{M}$  on the first and last interval on which it is constant. Then we modify  $A(\cdot)$  by adding, at each switching time, a Lipschitz continuous arc defined on a small time interval and bridging the discontinuity. These modifications can be done while keeping the corresponding variation of  $\overrightarrow{\text{exp}} \int_0^t A(\tau) d\tau$  smaller than  $\varepsilon$ . By a time-reparameterization  $\lambda \mapsto A(\lambda \cdot)$  we can lower the Lipschitz constant and complete the proof as above.  $\square$

As regards Assumption **A2**, we obtain the following proposition.

**Proposition 5.** *Consider a set  $\mathcal{F} = \cup_{t \geq 0} \mathcal{F}_t$  with  $\mathcal{F}_t \subset L_\infty([0, t], \mathcal{M})$ ,  $t \geq 0$ , where  $\mathcal{M}$  is an irreducible bounded set of matrices in  $M_n(\mathbb{R})$ . Then Assumption **A4** implies Assumption **A2**.*

*Proof.* Let  $x, z \in \mathbb{R}^n \setminus \{0\}$ . We have to prove that there exists  $R \in \overrightarrow{\text{exp}}(\mathcal{F})$  such that  $z^T R x \neq 0$ . Since  $\mathcal{M}$  is irreducible, it follows from [14, Lemma 3.1] that there exist  $t > 0$  and  $A(\cdot)$  in  $\mathcal{F}_t^{\text{pc}}(\mathcal{M})$  such that

$$z^T \left( \overrightarrow{\text{exp}} \int_0^t A(\tau) d\tau \right) x \neq 0. \quad (8)$$

Consider the path  $\lambda \mapsto A_\lambda(\cdot)$  provided by Assumption **A4**. The function

$$\lambda \mapsto z^T \left( \overrightarrow{\text{exp}} \int_0^{t_\lambda} A_\lambda(\tau) d\tau \right) x$$

is analytic and not identically equal to zero. It therefore vanishes at isolated values of  $\lambda$ , whence the conclusion with  $R$  of the type  $\overrightarrow{\text{exp}} \int_0^{t_\lambda} A_\lambda(\tau) d\tau$ .  $\square$

### 3.2 Quasi-Barabanov semigroups

The main goal of the section is to prove the result below.

**Theorem 6.** *Let  $(\mathcal{S}, \mathcal{F})$  satisfy Assumptions **A1–A3**. Then there exists a constant  $C \geq 1$  such that for any  $x_0 \in \mathbb{R}^n \setminus \{0\}$  there exists a trajectory  $x : t \mapsto \overrightarrow{\text{exp}} \int_0^t A(\tau) d\tau x_0$  with  $A(\cdot)$  belonging to the weak- $\star$  closure of  $\mathcal{F}_\infty$  such that, for every  $t \geq 0$ ,*

$$\frac{1}{C} \rho(\mathcal{S})^t \|x_0\| \leq \|x(t)\| \leq C \rho(\mathcal{S})^t \|x_0\|.$$

For that purpose, we first need the following definitions.

**Definition 7.** *Let  $\mathcal{M}$ ,  $\mathcal{F}$ ,  $\mathcal{F}_\infty$ , and  $\overrightarrow{\text{exp}}(\mathcal{F})$  as in the previous section.*

1. *We say that  $\overrightarrow{\text{exp}}(\mathcal{F})$  is a quasi-extremal semigroup if there exists  $C_{\text{qe}} > 0$  such that, for every  $t \geq 0$  and  $R \in \overrightarrow{\text{exp}}(\mathcal{F}_t)$ , one has*

$$\|R\| \leq C_{\text{qe}} \mu(\mathcal{F})^t. \quad (9)$$

*Moreover, a quasi-extremal semigroup  $\overrightarrow{\text{exp}}(\mathcal{F})$  is said to be extremal if there exists a norm  $w$  on  $\mathbb{R}^n$  such that the induced matrix norm  $\|\cdot\|_w$  satisfies, for every  $t \geq 0$  and  $R \in \overrightarrow{\text{exp}}(\mathcal{F}_t)$ ,*

$$\|R\|_w \leq \mu(\mathcal{F})^t. \quad (10)$$

*A norm  $w$  satisfying Eq. (10) is said to be extremal for  $\overrightarrow{\text{exp}}(\mathcal{F})$ . item A quasi-extremal semigroup  $\overrightarrow{\text{exp}}(\mathcal{F})$  is said to be quasi-Barabanov if there exists  $C_{\text{qb}} > 0$  such that for every  $x \in \mathbb{R}^n$  and  $t \geq 0$  there exist  $t' \geq t$  and  $R$  in  $\overrightarrow{\text{exp}}(\mathcal{F}_{t'})$  such that*

$$\|Rx\| \geq C_{\text{qb}} \mu(\mathcal{F})^{t'} \|x\|. \quad (11)$$

2. *Let  $\overrightarrow{\text{exp}}(\mathcal{F})$  be a quasi-extremal semigroup. A trajectory  $x : t \mapsto \overrightarrow{\text{exp}} \int_0^t A(\tau) d\tau x_0$  with  $x_0 \neq 0$  and  $A(\cdot)$  belonging to the weak- $\star$  closure of  $\mathcal{F}_\infty$  is said to be quasi-extremal with constant  $C_{\text{qx}} \geq 1$  if for every  $t \geq 0$ ,*

$$\frac{1}{C_{\text{qx}}} \mu(\mathcal{F})^t \|x_0\| \leq \|x(t)\| \leq C_{\text{qx}} \mu(\mathcal{F})^t \|x_0\|. \quad (12)$$

The notion of generalized spectral radius for a quasi-Barabanov semigroup  $\overrightarrow{\text{exp}}(\mathcal{F})$  is actually equivalent to the following adaptation of the definition of maximal Lyapunov exponent

$$\lambda(\mathcal{F}) = \sup \left\{ \limsup_{k \rightarrow \infty} \frac{\log \|R_{t_k} \cdots R_{t_1}\|}{t_1 + \cdots + t_k} \mid R_{t_k} \in \overrightarrow{\text{exp}}(\mathcal{F}_{t_k}) \text{ for } k \in \mathbb{N}, \sum_{k \in \mathbb{N}} t_k = \infty \right\},$$

as stated below.

**Proposition 8.** *Let  $\mathcal{F}$  be a family of switching laws satisfying the concatenability condition **A1** and assume that  $\overrightarrow{\text{exp}}(\mathcal{F})$  is a quasi-Barabanov semigroup. Then  $\mu(\mathcal{F}) = e^{\lambda(\mathcal{F})}$ .*

*Proof.* The inequality  $\mu(\mathcal{F}) \geq e^{\lambda(\mathcal{F})}$  easily comes from (9). In order to show the opposite inequality one observes that, by (11), for any  $x_0 \in \mathbb{R}^n$  and  $i \in \mathbb{N}$ , there exist  $t_i \geq i$  and  $R_{t_i} \in \overrightarrow{\text{exp}}(\mathcal{F}_{t_i})$  such that

$$\|R_{t_k} \cdots R_{t_1} x_0\| \geq (C_{\text{qb}})^k \mu(\mathcal{F})^{t_1 + \cdots + t_k} \|x_0\|.$$

In particular

$$\limsup_{k \rightarrow \infty} \frac{\log \|R_{t_k} \cdots R_{t_1}\|}{t_1 + \cdots + t_k} \geq - \lim_{k \rightarrow \infty} \frac{2|\log(C_{\text{qb}})|}{k+1} + \log(\mu(\mathcal{F})) = \log(\mu(\mathcal{F})),$$

which concludes the proof.  $\square$

**Lemma 9.** *Let Assumptions **A1** and **A3** be satisfied. If  $\overrightarrow{\text{exp}}(\mathcal{F})$  is a quasi-Barabanov semigroup then there exists  $C_{\text{qx}} \geq 1$  such that any nonzero point of  $\mathbb{R}^n$  is the initial condition of a quasi-extremal trajectory with constant  $C_{\text{qx}}$ .*

*Proof.* Let  $C_{\text{qe}}, C_{\text{qb}}$  be as in (9), (11) and  $C, \mathcal{K}$  as in **A3**. Let  $\kappa \geq 1$  verify  $\|K\|, \|K^{-1}\| \leq \kappa$  for every  $K \in \mathcal{K}$ .

Fix  $x_0 \neq 0$ . There exists an increasing sequence of times  $t_k$  going to infinity and signals  $A_k(\cdot)$  in  $\mathcal{F}_{t_k}$  such that  $\|R_k x_0\| \geq C_{\text{qb}} \mu(\mathcal{F})^{t_k} \|x_0\|$ , where  $R_k = \overrightarrow{\text{exp}} \int_0^{t_k} A_k(\tau) d\tau$ .

We now claim that, for every  $k \in \mathbb{N}$  and every  $s \in [0, t_k]$ , one has

$$\left\| \overrightarrow{\text{exp}} \int_0^s A_k(\tau) d\tau x_0 \right\| \geq \frac{1}{C_0} \mu(\mathcal{F})^s \|x_0\| \quad (13)$$

for some constant  $C_0 > 0$  independent of  $x_0$  and  $s$ . Indeed, applying (6) one gets that  $\overrightarrow{\text{exp}} \int_s^{t_k} A_k(\tau) d\tau = MRK^{-1}$  with  $\|M\| \leq C$ ,  $K \in \mathcal{K}$ , and  $R$  in  $\overrightarrow{\text{exp}}(\mathcal{F}_{t_k - s + \delta})$ , for some  $\delta \in [0, \Delta]$ . One can therefore write  $R_k = MRK^{-1} \overrightarrow{\text{exp}} \int_0^s A_k(\tau) d\tau$ . It follows that

$$\begin{aligned} \|R_k x_0\| &\leq \|M\| \left\| RK^{-1} \overrightarrow{\text{exp}} \int_0^s A_k(\tau) d\tau x_0 \right\| \leq CC_{\text{qe}} \mu(\mathcal{F})^{t_k - s + \delta} \left\| K^{-1} \overrightarrow{\text{exp}} \int_0^s A_k(\tau) d\tau x_0 \right\| \\ &\leq \kappa CC_{\text{qe}} \mu(\mathcal{F})^{t_k - s + \delta} \left\| \overrightarrow{\text{exp}} \int_0^s A_k(\tau) d\tau x_0 \right\|. \end{aligned}$$

---

<sup>1</sup>Remarque commentee

<sup>2</sup>Autre remarque commentee

On the other hand  $\|R_k x_0\| \geq C_{\text{qb}} \mu(\mathcal{F})^{t_k} \|x_0\|$ , which proves (13) with

$$C_0 = \frac{\kappa C C_{\text{qe}} \max(1, \mu(\mathcal{F})^\Delta)}{C_{\text{qb}}}.$$

Notice that each  $A_k(\cdot)$  is the restriction on  $[0, t_k]$  of a signal  $B_k(\cdot)$  in  $\mathcal{F}_\infty$ . Up to a subsequence we can assume that  $B_k(\cdot)$  weak- $\star$  converges to some  $B_\star(\cdot)$  in the weak- $\star$  closure of  $\mathcal{F}_\infty$ . Passing to the limit in (13) we deduce that for every  $s \geq 0$ ,

$$\left\| \overrightarrow{\text{exp}} \int_0^s B_\star(\tau) d\tau x_0 \right\| \geq \frac{1}{C_0} \mu(\mathcal{F})^s \|x_0\|. \quad (14)$$

We next prove that there exists  $C_1 > 0$  such that for every  $s \geq 0$ ,  $x_0 \in \mathbb{R}^n$  and  $B(\cdot)$  in the weak- $\star$  closure of  $\mathcal{F}_\infty$ , it holds

$$\left\| \overrightarrow{\text{exp}} \int_0^s B(\tau) d\tau x_0 \right\| \leq C_1 \mu(\mathcal{F})^s \|x_0\|. \quad (15)$$

For that purpose, consider a sequence  $B_k(\cdot)$  in  $\mathcal{F}_\infty$  weak- $\star$  converging to  $B(\cdot)$ . Applying (6), Remark 2 and a compactness argument, we get that  $\overrightarrow{\text{exp}} \int_0^s B(\tau) d\tau = MRK^{-1}$  with  $\|M\| \leq C$ ,  $K \in \mathcal{K}$  and  $R$  in the closure of  $\cup_{\delta \in [0, \Delta]} \overrightarrow{\text{exp}}(\mathcal{F}_{s+\delta})$ . It follows that

$$\left\| \overrightarrow{\text{exp}} \int_0^s B(\tau) d\tau x_0 \right\| \leq \|M\| \|RK^{-1} x_0\| \leq C_1 \mu(\mathcal{F})^s \|x_0\|,$$

where  $C_1 = \kappa C C_{\text{qe}} \max(1, \mu(\mathcal{F})^\Delta)$ , proving (15). Together with (14), this concludes the proof of the lemma with  $C_{\text{qx}} = \max(1, C_0, C_1)$ .  $\square$

Set

$$\mathcal{R}_\infty = \{R \mid \exists t_k \rightarrow \infty, R_{t_k} \in \overrightarrow{\text{exp}}(\mathcal{F}_{t_k}), \text{ such that } \mu(\mathcal{F})^{-t_k} R_{t_k} \rightarrow R\}.$$

The following result is the counterpart of [14, Proposition 3.2] in our setting.

**Proposition 10.** *Let  $(S, \mathcal{F})$  satisfy Assumptions **A1–A3** and define  $\mathcal{R}_\infty$  as above. Then*

- (i)  $\mathcal{R}_\infty$  is compact and nonempty,  $\mathcal{R}_\infty \neq \{0\}$ ;
- (ii)  $\mathcal{R}_\infty$  is a semigroup;
- (iii) for every  $t \geq 0$ ,  $T \in \overrightarrow{\text{exp}}(\mathcal{F}_t)$  and  $S \in \mathcal{R}_\infty$ , both  $e^{-t\mu(\mathcal{F})} TS$  and  $e^{-t\mu(\mathcal{F})} ST$  belong to  $\mathcal{R}_\infty$ ;
- (iv)  $\mathcal{R}_\infty$  is irreducible.

*Proof.* To prove the proposition one follows exactly the arguments provided in the proof of [14, Proposition 3.2] except for the fact that  $\mathcal{R}_\infty \neq \{0\}$ . In our setting this result is easily proved by using Lemma 3 and the definition of  $\mathcal{R}_\infty$ .  $\square$

The following result can be proven as in [14, Lemma 3.4].

**Proposition 11.** *Let  $(\mathcal{S}, \mathcal{F})$  satisfy Assumptions **A1–A3** and define  $\mathcal{R}_\infty$  as above. Let  $\hat{v} : \mathbb{R}^n \rightarrow (0, \infty)$  be defined by*

$$\hat{v}(x) = \max_{R \in \mathcal{R}_\infty} \|Rx\|. \quad (16)$$

Then  $\hat{v}$  is an extremal norm for  $\overrightarrow{\text{exp}}(\mathcal{F})$ .

**Remark 12.** *If Assumption **A1** is replaced by the stronger decomposability assumption (see Remark 1), then  $\hat{v}$  is a Barabanov norm (see e.g. [14]).*

We have the following result.

**Proposition 13.** *Let  $(\mathcal{S}, \mathcal{F})$  satisfy Assumptions **A1–A3**. Then  $\overrightarrow{\text{exp}}(\mathcal{F})$  is an extremal and quasi-Barabanov semigroup.*

*Proof.* The fact that  $\overrightarrow{\text{exp}}(\mathcal{F})$  is an extremal semigroup readily comes from Proposition 11. In order to show the second part of the statement, let us consider  $\kappa \geq 1$  as in the proof of Lemma 9. Without loss of generality, let us also assume that  $\kappa$  satisfies

$$\frac{1}{\kappa} \|x\| \leq \hat{v}(x) \leq \kappa \|x\| \text{ for all } x \in \mathbb{R}^n$$

and that  $\|M\| \leq C$  implies that  $\|M\|_{\hat{v}} \leq \kappa$ , where  $C$  is as in **A3**.

For every  $x \in \mathbb{R}^n$ , there exists a sequence  $(R_k)_{k \in \mathbb{N}}$  so that  $R_k \in \overrightarrow{\text{exp}}(\mathcal{F}_{t_k})$  with  $\lim_{k \rightarrow \infty} t_k = \infty$  and  $\hat{v}(x) = \lim_{k \rightarrow \infty} \mu(\mathcal{F})^{-t_k} \|R_k x\|$ . For every  $k \in \mathbb{N}$ , let  $A_k(\cdot) \in \mathcal{F}_{t_k}$  be such that  $R_k = \overrightarrow{\text{exp}} \int_0^{t_k} A_k(\tau) d\tau$ .

Fix now  $s \geq 0$ . For every  $k \in \mathbb{N}$  such that  $t_k \geq s$ , we deduce from Assumption **A3** that  $\overrightarrow{\text{exp}} \int_0^s A_k(\tau) d\tau = M_1^{(k)} Q_1^{(k)}$  and  $\overrightarrow{\text{exp}} \int_s^{t_k} A_k(s) ds = M_2^{(k)} Q_2^{(k)} (K_2^{(k)})^{-1}$  with  $\|M_i^{(k)}\| \leq C$ ,  $K_2^{(k)} \in \mathcal{K}$ ,  $Q_1^{(k)} \in \overrightarrow{\text{exp}}(\mathcal{F}_{s+\delta_1^{(k)}})$  and  $Q_2^{(k)} \in \overrightarrow{\text{exp}}(\mathcal{F}_{t_k-s+\delta_2^{(k)}})$ , where  $\delta_i^{(k)} \in [0, \Delta]$ , for  $i = 1, 2$ .

One can therefore write  $R_k = M_2^{(k)} Q_2^{(k)} (K_2^{(k)})^{-1} M_1^{(k)} Q_1^{(k)}$ . Using the extremality of  $\hat{v}$  and the requirements imposed on  $\kappa$ , it follows that

$$\mu(\mathcal{F})^{-t_k} \|R_k x_0\| \leq \kappa \mu(\mathcal{F})^{-t_k} \hat{v}(R_k x_0) \leq \kappa^6 \mu(\mathcal{F})^{-s+\delta_2^{(k)}} \hat{v}(Q_1^{(k)} x_0).$$

Taking limits as  $k$  tends to infinity and up to subsequences, one gets

$$\hat{v}(x_0) \leq \kappa^6 \mu(\mathcal{F})^{-s+\delta_2} \hat{v}(Q_1 x_0),$$

where  $Q_1$  is in the closure of  $\cup_{\delta \in [0, \Delta]} \overrightarrow{\text{exp}}(\mathcal{F}_{s+\delta})$ , and  $\delta_2$  belongs to  $[0, \Delta]$ .

In particular, there exist  $\hat{Q} \in \overrightarrow{\text{exp}}(\mathcal{F}_{s+\hat{\delta}})$  for some  $\hat{\delta} \in [0, \Delta]$  such that

$$\hat{v}(x_0) \leq 2\kappa^6 \mu(\mathcal{F})^{-s+\hat{\delta}_2} \hat{v}(\hat{Q} x_0).$$

One deduces that  $\hat{v}(\hat{Q} x_0) \geq C_0 \mu(\mathcal{F})^{s+\hat{\delta}_2} \hat{v}(x_0)$  with  $C_0 = \frac{\max(1, \mu(\mathcal{F})^{-2\Delta})}{2\kappa^6}$ .  $\square$

As a consequence of Lemma 9, we have the following corollary.

**Corollary 14.** *Let  $(\mathcal{S}, \mathcal{F})$  satisfy Assumptions **A1**–**A3**. Then there exists  $C_{\text{qx}} \geq 1$  such that any nonzero point of  $\mathbb{R}^n$  is the initial condition of a quasi-extremal trajectory with constant  $C_{\text{qx}}$ .*

Theorem 6 follows directly from Corollary 14 and Lemma 3.

Note that, as a consequence of **A3** and the above results, the right-hand side inequality in the statement of Theorem 6 holds true for any trajectory associated with a signal in  $\mathcal{S}$ , up to adapting the constant  $C$ .

**Remark 15.** *As a consequence of Proposition 8, Proposition 13 and Lemma 3 one easily deduces that  $\rho(\mathcal{S}) = e^{\lambda(\mathcal{S})}$ , where*

$$\lambda(\mathcal{S}) = \sup_{A(\cdot) \in \mathcal{S}} \limsup_{t \rightarrow +\infty} \frac{\log \|\overrightarrow{\exp} \int_0^t A(\tau) d\tau\|}{t}$$

is the maximal Lyapunov exponent associated with the family  $\mathcal{S}$ . This result was already obtained in [15] if the class  $\mathcal{S}$  is assumed to be weak- $\star$  closed.

## 4 $L_2$ -gain for switched linear control systems

Consider a switched linear control system of the type

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad y(t) = C(t)x(t), \quad (17)$$

where  $(x, u, y) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p$  and  $(A(\cdot), B(\cdot), C(\cdot))$  belongs to a class  $\mathcal{T}$  of measurable switching laws taking values in a bounded set of triples of matrices  $\mathcal{M} \subset M_n(\mathbb{R}) \times M_{n,m}(\mathbb{R}) \times M_{p,n}(\mathbb{R})$ . We denote by  $\pi_A$  and  $\pi_{A \times B}$  the projections from  $M_n(\mathbb{R}) \times M_{n,m}(\mathbb{R}) \times M_{p,n}(\mathbb{R})$  to its first and first two factors, respectively. We set

$$\mathcal{M}_A = \pi_A(\mathcal{M}) = \{A \mid \exists(A, B, C) \in \mathcal{M}\}, \quad \mathcal{T}_A = \pi_A(\mathcal{T}) = \{A(\cdot) \mid \exists(A(\cdot), B(\cdot), C(\cdot)) \in \mathcal{T}\}$$

and we define similarly  $\mathcal{M}_B$ ,  $\mathcal{M}_C$ ,  $\mathcal{T}_B$  and  $\mathcal{T}_C$ .

In the sequel, we also assume that  $\mathcal{T}$  contains a subset  $\mathcal{G}_\infty$  made of concatenations of signals in a family  $\mathcal{G} = \cup_{t \geq 0} \mathcal{G}_t$  as in Eq. (5). Useful properties on  $(\mathcal{T}, \mathcal{G})$  are the following:

**T1**  $(\mathcal{T}_A, \mathcal{G}_A)$  satisfies Assumption **A1**, where  $\mathcal{G}_A = \pi_A(\mathcal{G})$ .

**T2**  $(\mathcal{T}_A, \mathcal{G}_A)$  satisfies Assumption **A4**. Moreover, for every  $t^*, \varepsilon > 0$  and  $(A(\cdot), B(\cdot), C(\cdot)) \in \mathcal{F}_t^{\text{pc}}(\mathcal{M})$ , there exists a path  $[0, 1] \ni \lambda \mapsto (t_\lambda, A_\lambda(\cdot), B_\lambda(\cdot), C_\lambda(\cdot)) \in (0, \infty) \times \mathcal{F}_{t_\lambda}^{\text{arb}}(\mathcal{M})$  such that:

- $\lambda \mapsto (W_\lambda^c(t_\lambda), W_\lambda^o(t_\lambda))$  is analytic, where  $W_\lambda^c(t_\lambda)$  and  $W_\lambda^o(t_\lambda)$  denote, respectively, the controllability and observability Gramians in time  $t_\lambda$  associated with  $\dot{x}(t) = A_\lambda(t)x(t) + B_\lambda(t)u(t)$ ,  $y(t) = C_\lambda(t)x(t)$ ;
- $\|W_1^c(t_1) - W^c(t^*)\| \leq \varepsilon$  and  $\|W_1^o(t_1) - W^o(t^*)\| \leq \varepsilon$ , where  $W^c(t^*)$  and  $W^o(t^*)$  denote, respectively, the controllability and observability Gramians in time  $t^*$  associated with  $\dot{x}(t) = A(t)x(t) + B(t)u(t)$ ,  $y(t) = C(t)x(t)$ ;

- the set  $\{\lambda \in [0, 1] \mid (A_\lambda(\cdot), B_\lambda(\cdot), C_\lambda(\cdot)) \in \mathcal{G}\}$  has nonempty interior.

A trivial adaptation of the proof of Lemma 4 yields the following result.

**Lemma 16.** *Let  $\mathcal{S}$  be one of the classes introduced in Section 2.2 with corresponding  $\mathcal{F}$ . Assume moreover that if  $\mathcal{S} = \mathcal{S}^{\text{lip},L}(\mathcal{M})$  then  $\mathcal{M}$  is rectifiably pathwise connected and that if  $\mathcal{S} = \mathcal{S}^{\text{BV},T,\nu}(\mathcal{M})$  then for every  $M_0, M_1 \in \mathcal{M}$ , there exists a finite sequence of points in  $\mathcal{M}$  whose first and last elements are  $M_0$  and  $M_1$ , respectively, and such that the distance between two subsequent elements is smaller than  $\nu$ . Then  $(\mathcal{S}, \mathcal{F})$  satisfies Assumptions **T1** and **T2**.*

## 4.1 Minimal realization for switched linear control systems

We start with the following definitions.

**Definition 17.** 1. A point  $x \in \mathbb{R}^n$  is  $\mathcal{G}$ -reachable for the switched linear control system (17) if there exist  $t \geq 0$ , a switching law  $(A(\cdot), B(\cdot), C(\cdot)) \in \mathcal{G}_t$  and an input  $u \in L_2$  such that the corresponding trajectory  $x_u$  starting at 0 reaches  $x$  in time  $t$ . The reachable set  $\mathfrak{R}(\mathcal{G})$  is the set of all  $\mathcal{G}$ -reachable points. System (17) is said to be  $\mathcal{G}$ -controllable if  $\mathfrak{R}(\mathcal{G}) = \mathbb{R}^n$ .

2. A point  $x \in \mathbb{R}^n$  is  $\mathcal{G}$ -observable for the switched linear control system (17) if there exist  $t \geq 0$  and a switching law  $(A(\cdot), B(\cdot), C(\cdot)) \in \mathcal{G}_t$  such that the trajectory  $x_0$  associated with the zero input and starting at  $x$  gives rise to an output  $y(\cdot)$  verifying  $y(t) \neq 0$ . The observable set  $\mathfrak{D}(\mathcal{G})$  is the set of all  $\mathcal{G}$ -observable points. System (17) is said to be  $\mathcal{G}$ -observable if  $\mathfrak{D}(\mathcal{G}) = \mathbb{R}^n$ .

It is not immediate from their definitions that the reachable and observable sets  $\mathfrak{R}(\mathcal{G})$  and  $\mathfrak{D}(\mathcal{G})$  are linear subspaces. It has been shown in [13] that this is the case if  $\mathcal{T} = \mathcal{G}_\infty = \mathcal{S}^{\text{arb}}(\mathcal{M})$ , where  $\mathcal{M} = \{(A_1, B_1, C_1), \dots, (A_k, B_k, C_k)\}$  with  $k$  a positive integer. In addition, it is proved in the same reference that the state space admits direct sum decompositions in controllable (respectively, observable) part for the switched linear control system exactly as in the unswitched situation. More precisely, there exists a direct sum decomposition of the state space  $\mathbb{R}^n = \mathfrak{R}(\mathcal{G}) \oplus E$  and an invertible  $n \times n$  matrix  $P$  such that, if  $r = \dim \mathfrak{R}(\mathcal{G})$  and  $P^{-1}x = (x^c \ x^u)$  with  $x^c \in \mathfrak{R}(\mathcal{G})$ , one has for  $1 \leq i \leq k$ ,

$$P^{-1}A_iP = \begin{pmatrix} A_i^c & * \\ 0 & A_i^u \end{pmatrix}, \quad P^{-1}B_i = \begin{pmatrix} B_i^c \\ 0 \end{pmatrix}, \quad C_iP = (C_i^c \ *),$$

where each  $A_i^c$  and  $B_i^c$  belongs to  $M_r(\mathbb{R})$  and  $M_{r,m}(\mathbb{R})$  respectively. Moreover, the switched linear control system defined on  $\mathbb{R}^r$  associated with  $\mathcal{S}^{\text{arb}}(\mathcal{M}^c)$ , where

$$\mathcal{M}^c = \{(A_1^c, B_1^c, C_1^c), \dots, (A_k^c, B_k^c, C_k^c)\},$$

is  $\mathcal{F}^{\text{arb}}(\mathcal{M}^c)$ -controllable. We refer to  $\mathcal{M}^c$  as the *controllable part* of  $\mathcal{M}$ . Notice that  $\mathcal{M}^c = \Pi^c(\mathcal{M})$  where

$$\Pi^c(A, B, C) = ((\text{Id}_r \ 0_{r,n-r})P^{-1}AP(\text{Id}_r \ 0_{r,n-r})^T, (\text{Id}_r \ 0_{r,n-r})P^{-1}B, CP(\text{Id}_r \ 0_{r,n-r})^T). \quad (18)$$

Also notice that the output  $y(\cdot)$  corresponding to the original system is equal to  $y(\cdot) = C^c(\cdot)x^c(\cdot)$  and thus the original switched linear control system has the same  $L_2$ -gain as the one reduced to the reachable space.

Similarly, there exists a direct sum decomposition of the state space  $\mathbb{R}^n = \mathfrak{D}(\mathcal{G}) \oplus F$  and an invertible  $n \times n$  matrix  $Q$  such that, if  $s = \dim \mathfrak{D}(\mathcal{G})$  and  $Q^{-1}x = (x^o, x^u)$  with  $x^o \in \mathfrak{D}(\mathcal{G})$ , one has for  $1 \leq i \leq k$ ,

$$Q^{-1}A_iQ = \begin{pmatrix} A_i^o & 0 \\ * & A_i^u \end{pmatrix}, \quad Q^{-1}B_i = \begin{pmatrix} B_i^o \\ * \end{pmatrix}, \quad C_iQ = (C_i^o \quad 0),$$

where  $A_i^o$  and  $C_i^o$  belong to  $M_s(\mathbb{R})$  and  $M_{p,s}(\mathbb{R})$  respectively. Moreover, the switched linear control system defined on  $\mathbb{R}^s$  associated with  $\mathcal{S}^{\text{arb}}(\mathcal{M}^o)$ , where

$$\mathcal{M}^o = \{(A_1^o, B_1^o, C_1^o), \dots, (A_k^o, B_k^o, C_k^o)\},$$

is  $\mathcal{F}^{\text{arb}}(\mathcal{M}^o)$ -observable. We refer to  $\mathcal{M}^o$  as the *observable part* of  $\mathcal{M}$ . Notice that  $\mathcal{M}^o = \Pi^o(\mathcal{M})$  where

$$\Pi^o(A, B, C) = ((\text{Id}_s \ 0_{s,n-s})Q^{-1}AQ(\text{Id}_s \ 0_{s,n-s})^T, (\text{Id}_s \ 0_{s,n-s})Q^{-1}B, CQ(\text{Id}_s \ 0_{s,n-s})^T). \quad (19)$$

The corresponding output  $y(\cdot)$  being equal to  $y(\cdot) = C^o(\cdot)x^o(\cdot)$ , one deduces the equality of the  $L_2$ -gains of the original switched linear control system and of the one reduced to the observable space.

From the above, one can proceed as follows. Consider a switched linear control system (17) associated with  $\mathcal{S}^{\text{arb}}(\mathcal{M})$ , where  $\mathcal{M} = \{(A_1, B_1, C_1), \dots, (A_k, B_k, C_k)\}$ . One first reduces it to its reachable space  $\mathfrak{R}(\mathcal{F}^{\text{arb}}(\mathcal{M}))$ . We thus get a  $\mathcal{F}^{\text{arb}}(\Pi^c(\mathcal{M}))$ -controllable switched linear control system with same  $L_2$ -gain as that of the original system. Then, one reduces the latter system to its observable space  $\mathfrak{D}(\mathcal{F}^{\text{arb}}(\Pi^c(\mathcal{M})))$  to finally obtain a switched linear control system associated with  $\mathcal{S}^{\text{arb}}(\mathcal{M}^{\text{min}})$  where  $\mathcal{M}^{\text{min}} = \Pi^o(\Pi^c(\mathcal{M}))$ . The latter system is finally  $\mathcal{F}^{\text{arb}}(\mathcal{M}^{\text{min}})$ -controllable and  $\mathcal{F}^{\text{arb}}(\mathcal{M}^{\text{min}})$ -observable and

$$\gamma_2(\mathcal{S}^{\text{arb}}(\mathcal{M}^{\text{min}})) = \gamma_2(\mathcal{S}^{\text{arb}}(\mathcal{M})). \quad (20)$$

We refer to  $\mathcal{S}^{\text{arb}}(\mathcal{M}^{\text{min}})$  and the corresponding switched linear control system as a *minimal realization* for the linear switched linear control system associated with  $\mathcal{S}^{\text{arb}}(\mathcal{M})$  and (17).

**Remark 18.** Note that eventhough the dimension  $n'$  of  $\mathfrak{D}(\mathcal{F}^{\text{arb}}(\Pi^c(\mathcal{M})))$  is uniquely defined by the original switched linear control system, the minimal realization is not unique since it depends on the choice of supplementary spaces to  $\mathfrak{R}(\mathcal{F}^{\text{arb}}(\mathcal{M}))$  in  $\mathbb{R}^n$  and to  $\mathfrak{D}(\mathcal{F}^{\text{arb}}(\Pi^c(\mathcal{M})))$  in  $\mathfrak{R}(\mathcal{F}^{\text{arb}}(\mathcal{M}))$ . However, one deduces from Eq. (20) that  $\gamma_2(\mathcal{S}^{\text{arb}}(\mathcal{M}^{\text{min}}))$  does not depend on a particular choice of a minimal realization. Moreover, it can be shown that any two minimal realizations with switching signal value sets  $\mathcal{M}_1^{\text{min}}$  and  $\mathcal{M}_2^{\text{min}}$  are algebraically similar, i.e., there exists an invertible matrix  $G \in GL_{n'}(\mathbb{R})$  so that

$$\mathcal{M}_2^{\text{min}} = \{(G^{-1}AG, G^{-1}B, CG) \mid (A, B, C) \in \mathcal{M}_1^{\text{min}}\}. \quad (21)$$

All the results presented in this paragraph belong to the theme of realization theory of switched linear control systems and we refer to [11] for a thorough presentation of such a theory.



Finally, it must be recalled that [13] also provides a nice and explicit geometric description of  $\mathfrak{R}(\mathcal{F}^{\text{arb}}(\mathcal{M}))$  and  $\mathfrak{D}(\mathcal{F}^{\text{arb}}(\mathcal{M}))$  in terms of the data of the problem. For subsequent arguments, one needs precise statements relative to that geometric description. We will only provide details regarding  $\mathfrak{R}(\mathcal{F}^{\text{arb}}(\mathcal{M}))$  since all results for  $\mathfrak{D}(\mathcal{F}^{\text{arb}}(\mathcal{M}))$  are standardly derived by duality.

We first need the following notation. If  $A$  is an  $n \times n$  matrix and  $\mathcal{B}$  is a subspace of  $\mathbb{R}^n$ , let  $\Gamma_A \mathcal{B}$  be the subspace of  $\mathbb{R}^n$  given by

$$\Gamma_A \mathcal{B} = \mathcal{B} + A\mathcal{B} + \dots + A^{n-1}\mathcal{B}.$$

For  $1 \leq i \leq k$ , let  $\mathcal{D}_j = \text{Im} [B_i \ A_i B_i \ \dots \ A_i^{n-1} B_i]$ . Moreover (see [13, Section 3.1]), define recursively the sequence of subspaces of  $\mathbb{R}^n$  denoted  $\mathcal{V}_j$ ,  $j \geq 1$ , by

$$\begin{aligned} \mathcal{V}_1 &= \mathcal{D}_1 + \dots + \mathcal{D}_k, \\ \mathcal{V}_{j+1} &= \Gamma_{A_1} \mathcal{V}_j + \dots + \Gamma_{A_k} \mathcal{V}_j, \end{aligned}$$

and finally set  $\mathcal{V}(\mathcal{M}) = \sum_{j \geq 1} \mathcal{V}_j$ . From the variation of constants formula, it is not difficult to see that  $\mathfrak{R}(\mathcal{F}^{\text{arb}}(\mathcal{M}))$  is included in  $\mathcal{V}(\mathcal{M})$ . The converse inclusion is also true but more delicate to establish, cf. [13, Theorem 1] and the proof of Proposition 19 below. In the subsequent paragraphs, we will use these results to derive the existence of a minimal realization associated with (17) and any class  $\mathcal{T}$  considered in Section 2.2 together with its corresponding concatenation of subfamilies of switching signals introduced in Section 3.1.

We now generalize the above construction to a bounded set  $\mathcal{M} \subset M_n(\mathbb{R}) \times M_{n,m}(\mathbb{R}) \times M_{p,n}(\mathbb{R})$ . We associate a subspace  $\mathcal{V}(\mathcal{M})$  of  $\mathbb{R}^n$  as follows. First, consider

$$\mathcal{V}_1(\mathcal{M}) = \text{Span}\{A^j b_l \mid 0 \leq j \leq n-1, 1 \leq l \leq m, (A, [b_1 \dots b_m]) \in \pi_{A \times B}(\mathcal{M})\}.$$

Then, define recursively

$$\mathcal{V}_{j+1}(\mathcal{M}) = \text{Span}\{A^j v \mid 0 \leq j \leq n-1, A \in \pi_A(\mathcal{M}), v \in \mathcal{V}_j(\mathcal{M})\}, \quad j \geq 1,$$

and finally set  $\mathcal{V}(\mathcal{M}) = \sum_{j \geq 1} \mathcal{V}_j(\mathcal{M})$ . Taking a (finite) generating family of  $\mathcal{V}(\mathcal{M})$ , one can extract a finite subset  $\mathcal{M}^{\text{finite}}$  of  $\mathcal{M}$  such that  $\mathcal{V}(\mathcal{M}^{\text{finite}}) = \mathcal{V}(\mathcal{M})$ . Hence

$$\mathcal{V}(\mathcal{M}) \supset \mathfrak{R}(\mathcal{F}^{\text{arb}}(\mathcal{M})) \supset \mathfrak{R}(\mathcal{F}^{\text{arb}}(\mathcal{M}^{\text{finite}})) = \mathcal{V}(\mathcal{M}^{\text{finite}}) = \mathcal{V}(\mathcal{M}), \quad (22)$$

where the first inclusion is deduced from the variation of constants formula. Therefore, all the sets appearing in (22) coincide.

We thus prove the following proposition.

**Proposition 19.** *Consider a switched linear control system associated with (17) and a class  $\mathcal{T}$  of measurable switching laws taking values in a bounded set  $\mathcal{M} \subset M_n(\mathbb{R}) \times M_{n,m}(\mathbb{R}) \times M_{p,n}(\mathbb{R})$ . Let  $(\mathcal{T}, \mathcal{G})$  satisfy Assumption **T2**. Then  $\mathfrak{R}(\mathcal{G}) = \mathfrak{R}(\mathcal{F}^{\text{arb}}(\mathcal{M})) = \mathcal{V}(\mathcal{M})$  and there exist  $t^* > 0$  and a switching law  $(A(\cdot), B(\cdot), C(\cdot)) \in \mathcal{G}_{t^*}$  such that the range of  $W^c(t^*)$ , the controllability Gramian in time  $t^*$  associated with  $\dot{x}(t) = A(t)x(t) + B(t)u(t)$ , is equal to  $\mathcal{V}(\mathcal{M})$ .*

*Proof.* First notice that the equality  $\mathfrak{R}(\mathcal{F}^{\text{arb}}(\mathcal{M})) = \mathcal{V}(\mathcal{M})$  is contained in Eq.(22) and one has the trivial inclusion  $\mathfrak{R}(\mathcal{G}) \subset \mathcal{V}(\mathcal{M})$ . Let us prove the opposite inclusion.

As done above there exists a finite subset  $\mathcal{M}^{\text{finite}}$  of  $\mathcal{M}$  such that  $\mathcal{V}(\mathcal{M}^{\text{finite}}) = \mathcal{V}(\mathcal{M})$ . It is proved in [13, Theorem 1] that there exists a piecewise-constant periodic switching law  $(A(\cdot), B(\cdot), C(\cdot))$  taking values in  $\mathcal{M}^{\text{finite}}$  and a time  $t^* > 0$  such that the range of  $W^c(t^*)$ , the controllability Gramian in time  $t^* > 0$  associated with  $\dot{x}(t) = A(t)x(t) + B(t)u(t)$ , is equal to  $\mathcal{V}(\mathcal{M})$ . Fix  $\varepsilon > 0$  such that if an  $n \times n$  matrix  $W$  satisfies  $\|W - W^c(t^*)\| < \varepsilon$  then the rank of  $W$  is larger than or equal to the rank of  $W^c(t^*)$ . Let  $\lambda \mapsto (A_\lambda(\cdot), B_\lambda(\cdot), C_\lambda(\cdot))$  be the path provided by Assumption **T2**. Then there exists  $\lambda^* \in (0, 1]$  such that  $(A_{\lambda^*}(\cdot), B_{\lambda^*}(\cdot), C_{\lambda^*}(\cdot)) \in \mathcal{G}$  and the rank of the corresponding controllability Gramian  $W_{\lambda^*}^c(t_{\lambda^*})$  is larger than or equal to the rank of  $W^c(t^*)$ , itself equal to  $\dim \mathcal{V}(\mathcal{M})$ . Since the range of  $W_{\lambda^*}^c(t_{\lambda^*})$  is included in  $\mathfrak{R}(\mathcal{G})$  this concludes the proof of the proposition.  $\square$

An analogous statement is obtained regarding observability spaces and, following the first part of the section, one finally derives the subsequent result.

**Proposition 20.** *Consider a switched linear control system associated with (17) and a class  $\mathcal{T}$  of measurable switching laws taking values in a bounded set  $\mathcal{M} \subset M_n(\mathbb{R}) \times M_{n,m}(\mathbb{R}) \times M_{p,n}(\mathbb{R})$ . Let  $(\mathcal{T}, \mathcal{G})$  satisfy Assumption **T2**. Then, there exists a positive integer  $n' \leq n$  and a surjective linear mapping*

$$\Pi : M_n(\mathbb{R}) \times M_{n,m}(\mathbb{R}) \times M_{p,n}(\mathbb{R}) \rightarrow M_{n'}(\mathbb{R}) \times M_{n',m}(\mathbb{R}) \times M_{p,n'}(\mathbb{R})$$

such that, if one denotes by  $\mathcal{T}^{\text{min}}$  the class of measurable switching laws given by

$$\mathcal{T}^{\text{min}} = \{t \mapsto \Pi(A(t), B(t), C(t)) \mid (A(\cdot), B(\cdot), C(\cdot)) \in \mathcal{T}\}$$

taking values in  $\mathcal{M}^{\text{min}} = \Pi(\mathcal{M})$ , then the class  $\mathcal{T}^{\text{min}}$  and the corresponding switched linear control system define a minimal realization for the switched linear control system associated with the class  $\mathcal{T}$  and (17) (i.e., it is  $\mathcal{G}^{\text{min}}$ -controllable and  $\mathcal{G}^{\text{min}}$ -observable in the sense of Definition 17 with  $\mathcal{G}^{\text{min}} = \Pi(\mathcal{G})$ ),  $(\mathcal{T}^{\text{min}}, \mathcal{G}^{\text{min}})$  satisfies Assumption **T2** and

$$\gamma_2(\mathcal{T}) = \gamma_2(\mathcal{T}^{\text{min}}). \quad (23)$$

Moreover, any two such minimal realizations are algebraically similar. Finally, if  $(\mathcal{T}, \mathcal{G})$  satisfies Assumption **A1** then  $(\mathcal{T}^{\text{min}}, \mathcal{G}^{\text{min}})$  does.

*Proof.* This proposition is a simple consequence of the construction of minimal realizations in the case of arbitrary switching, as described in the first part of the section, and of Proposition 19. As for the algebraic similarity, see Remark 18.  $\square$

## 4.2 Finiteness of the $L_2$ -gain

Consider the switched linear control system (17) and the corresponding class  $\mathcal{T}$  of switching laws with values in the bounded set of matrices  $\mathcal{M}$ . Let us introduce the following definition.

**Definition 21.** We say that (17) is uniformly observable (or, equivalently, that  $\mathcal{T}$  is uniformly observable) if there exist  $T, \gamma > 0$  such that, for every  $(A(\cdot), B(\cdot), C(\cdot)) \in \mathcal{T}$  and every  $t \geq 0$ , one has  $W^o(t, t+T) \geq \gamma \text{Id}_n$ , where  $W^o(t, t+T)$  is the observability Gramian in time  $T$  associated with  $(A(t+\cdot), B(t+\cdot), C(t+\cdot))$ .

The following remark will be used in the sequel.

**Remark 22.** If (17) is uniformly observable and  $T, \gamma$  are as in Definition 21, the observability Gramian  $W^o(t, t+T)$  in time  $T$  associated with a switching law belonging to the weak- $\star$  closure of  $\mathcal{T}$  still satisfies  $W^o(t, t+T) \geq \gamma \text{Id}_n$ .

**Remark 23.** Consider the case where  $\mathcal{T}$  contains all the constant  $\mathcal{M}$ -valued switching signals. (Notice that this is the case for all classes of switching signals introduced in Section 2.2 except that of persistently exciting signals.) It is then easy to see that uniform observability implies that  $(A, C)$  is observable for every  $(A, B, C) \in \mathcal{M}$ . In the case where  $\mathcal{M}$  is compact and the class of signals under consideration is  $\mathcal{S}^{d, \tau}(\mathcal{M})$  for some  $\tau > 0$ , one easily shows that the converse is also true, namely, the observability of each pair  $(A, C)$  implies uniform observability. Let us stress that the uniform observability assumption is weaker than the minimal realization assumption (every triple  $(A, B, C) \in \mathcal{M}$  is a minimal realization, i.e.,  $(A, B)$  is controllable and  $(A, C)$  is observable) needed in [8, 10].

We can now state the main result of this section.

**Theorem 24.** Assume that (17) admits a minimal realization defined on  $\mathbb{R}^{n'}$ ,  $n' \leq n$ , which is given by  $\dot{x}^{\min}(t) = A^{\min}(t)x(t) + B^{\min}(t)u(t)$  with output  $y^{\min}(t) = C^{\min}(t)x^{\min}(t)$  and associated with a class  $\mathcal{T}^{\min}$  of switching signals taking values in  $\mathcal{M}^{\min}$ , and a family  $\mathcal{G}^{\min}$  satisfying the following assumptions:

- (a)  $(\mathcal{T}^{\min}, \mathcal{G}^{\min})$  satisfies Assumptions **T1** and **T2**;
- (b) for every subspace  $V$  of  $\mathbb{R}^{n'}$  and every linear projection  $P_V : \mathbb{R}^{n'} \rightarrow V$ , the class of signals  $P_V \mathcal{G}_A^{\min} P_V^\#$  satisfies Assumption **A3**, where we use  $P_V^\#$  to denote the dual map from  $V$  to  $\mathbb{R}^n$  defined by  $x^T P_V y = y^T P_V^\# x$  for every  $x \in V$  and  $y \in \mathbb{R}^n$ .

Then  $\gamma_2(\mathcal{T})$  is finite if  $\rho(\mathcal{T}_A^{\min}) < 1$  and infinite if either  $\rho(\mathcal{T}_A^{\min}) > 1$  or  $\rho(\mathcal{T}_A^{\min}) = 1$  and  $\mathcal{T}^{\min}$  is uniformly observable.

*Proof.* Thanks to Eq. (23), it is enough to treat the case  $\mathcal{T} = \mathcal{T}^{\min}$ .

Assume first that  $\rho(\mathcal{T}_A) < 1$ . Taking into account the definition of  $\rho(\mathcal{T}_A)$  (see (4)) and the boundedness of  $\pi_A(\mathcal{M})$ , one gets the following exponential decay estimate: there exist  $K_1, \lambda > 0$  such that, for every  $A(\cdot) \in \mathcal{T}_A$  and every  $0 \leq s \leq t$ , one has

$$\left\| \overrightarrow{\exp} \int_s^t A(\tau) d\tau \right\| \leq K_1 e^{-\lambda(t-s)}.$$

As a consequence of the above and the boundedness of  $\mathcal{M}$ , one deduces that there exists  $K_2 > 0$  such that, for every  $u \in L_2$ ,  $(A(\cdot), B(\cdot), C(\cdot)) \in \mathcal{T}$  and  $t \geq 0$ , one has

$$\|y_u(t)\| \leq K_2 \int_0^t e^{-\lambda(t-s)} \|u(s)\| ds. \quad (24)$$

If  $\chi_{[0,+\infty)}(\cdot)$  denotes the characteristic function on  $[0, +\infty)$ , the integral function on the right-hand side of (24) can be interpreted as the convolution of

$$f_1(t) = \chi_{[0,+\infty)}(t)e^{-\lambda t}, \quad f_2(t) = \chi_{[0,+\infty)}(t)\|u(t)\|.$$

That yields at once that  $\|y_u\|_2 \leq \frac{K_2}{\lambda}\|u\|_2$ , hence the conclusion.

Assume now that  $\rho(\mathcal{T}_A) \geq 1$ . It is well-known (see, e.g., [4, Proposition 2] for details), that every matrix in  $A \in \mathcal{M}_A$  can be put with a common linear change of coordinates into the upper triangular block form

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & & \\ 0 & A_{22} & A_{23} & \cdots & \\ 0 & 0 & A_{33} & A_{34} & \cdots \\ \vdots & \ddots & \ddots & \ddots & \\ 0 & \cdots & \cdots & 0 & A_{qq} \end{pmatrix}, \quad (25)$$

where, for  $i = 1, \dots, q$ , each  $A_{ii}$  is in  $M_{n_i - n_{i-1}}(\mathbb{R})$ ,  $n_i \in \mathbb{N}$  and the set  $\mathcal{A}_i := \{A_{ii} \mid A \in \mathcal{M}_A\}$  is irreducible (whenever  $\mathcal{M}_A$  is irreducible one has  $q = 1$  and  $\mathcal{A}_1 = \mathcal{M}_A$ ). Define the *subsystems of  $\mathcal{M}_A$*  as the switched systems corresponding to the sets  $\mathcal{A}_i$  and the class of switching signals  $\mathcal{T}_{A,i} := \{A_{ii}(\cdot) \mid A(\cdot) \in \mathcal{T}_A\}$  and  $\mathcal{G}_{A,i} := \{A_{ii}(\cdot) \mid A(\cdot) \in \mathcal{G}_A\}$ . One can then show that  $(\mathcal{T}_{A,1}, \mathcal{G}_{A,1}), \dots, (\mathcal{T}_{A,q}, \mathcal{G}_{A,q})$  satisfy Assumptions **A1–A3**. Indeed, Assumption **A1** follows from Assumption **T1**; Assumption **A2** is a consequence of Assumption **T2** and Proposition 5 while Item (b) yields Assumption **A3**.

Moreover, an induction argument on the number of subsystems and a standard use of the variation of constants formula yields

$$\rho(\mathcal{T}_A) = \max_{1 \leq i \leq q} \rho(\mathcal{T}_{A,i}).$$

Let  $\bar{i} \leq q$  be the largest index such that  $\rho(\mathcal{T}_{A,\bar{i}}) = \rho(\mathcal{T}_A)$ . Since (17) is  $\mathcal{G}$ -controllable, there exists a time  $\bar{t} > 0$ , a switching law  $(A^0(\cdot), B^0(\cdot), C^0(\cdot)) \in \mathcal{G}_{\bar{i}}$  and a measurable control  $\bar{u}(\cdot)$  defined on  $[0, \bar{t}]$  so that the corresponding trajectory  $x_{\bar{u}}(\cdot)$  starting at 0 reaches some point  $\bar{x} = (0, \dots, 0, \bar{x}_{\bar{i}}, 0, \dots, 0)^T \neq 0$  in time  $\bar{t}$ .

Since  $(\mathcal{T}_{A,\bar{i}}, \mathcal{G}_{A,\bar{i}})$  satisfies the hypotheses of Theorem 6, the corresponding semigroup  $\overrightarrow{\text{exp}}(\mathcal{G}_{A,\bar{i}})$  is quasi-Barabanov and admits a quasi-extremal trajectory  $x_{\bar{i}}^{\text{q-ext}}(\cdot)$  starting at  $\bar{x}_{\bar{i}}$  and corresponding to a signal  $A_{\bar{i}}(\cdot)$  belonging to the weak- $\star$  closure of  $(\mathcal{G}_{A,\bar{i}})_{\infty}$ . Let  $\bar{x}(\cdot)$  be a trajectory of  $\dot{x} = A(t)x(t)$  starting at  $\bar{x}$  and corresponding to some signal  $\bar{A}(\cdot)$  in the closure of  $(\mathcal{G}_A)_{\infty}$  so that  $\bar{A}_{\bar{i}}(\cdot) = A_{\bar{i}}(\cdot)$ . Notice that  $\bar{x}_{\bar{i}}(\cdot) = x_{\bar{i}}^{\text{q-ext}}(\cdot)$ .

Let  $\bar{E}(\cdot)$  be the signal defined as the concatenation of  $A^0(\cdot)$  and  $\bar{A}(\cdot)$  and let  $A_{\bar{i}}^0(\cdot)$  and  $\bar{E}_{\bar{i}}(\cdot)$  be the  $(\bar{i}, \bar{i})$ -blocks corresponding to  $A^0(\cdot)$  and  $\bar{E}(\cdot)$  respectively. Let  $(E^l(\cdot))_{l \geq 0}$  (and then  $(E_{\bar{i}}^l(\cdot))_{l \geq 0}$ ) be a sequence in  $(\mathcal{G}_A)_{\infty}$  ( $(\mathcal{G}_{A,\bar{i}})_{\infty}$  respectively) weak- $\star$  converging to  $\bar{A}(\cdot)$  ( $\bar{A}_{\bar{i}}(\cdot)$  respectively). Let  $(F^l(\cdot), G^l(\cdot))_{l \geq 0}$  be such that  $(E^l(\cdot), F^l(\cdot), G^l(\cdot)) \in \mathcal{G}_{\infty}$  for every  $l \geq 0$ . The sequence  $(A^0 * E^l(\cdot))_{l \geq 0}$  ( $(A_{\bar{i}}^0 * E_{\bar{i}}^l(\cdot))_{l \geq 0}$ ) is in  $(\mathcal{G}_A)_{\infty}$  ( $(\mathcal{G}_{A,\bar{i}})_{\infty}$  respectively) and weak- $\star$  converges to  $\bar{E}(\cdot)$  ( $\bar{E}_{\bar{i}}(\cdot)$  respectively). Moreover, for every positive time  $t \geq 0$ , the convergence of  $s \mapsto \overrightarrow{\text{exp}} \int_0^s E^l(\tau) d\tau$  to  $s \mapsto \overrightarrow{\text{exp}} \int_0^s \bar{E}(\tau) d\tau$  is uniform with respect to

$s \in [0, t]$  and similarly for the corresponding  $(\bar{v}, \bar{v})$ -blocks. Thus, for  $l$  large enough and for every  $s \in [0, t]$ , one has

$$\|\overrightarrow{\text{exp}} \int_0^s E^l(\tau) d\tau \bar{x}\| \geq \|\overrightarrow{\text{exp}} \int_0^s E_{\bar{v}}^l(\tau) d\tau \bar{x}_{\bar{v}}\| \geq \frac{1}{2} \|\overrightarrow{\text{exp}} \int_0^s \bar{E}_{\bar{v}}(\tau) d\tau \bar{x}_{\bar{v}}\| \geq \frac{1}{2C_{\text{qx}}} \rho(\mathcal{T}_A)^s \|\bar{x}_{\bar{v}}\|. \quad (26)$$

Denote by  $x^l(\cdot)$  the trajectory of (17) starting at the origin and associated with the control equal to  $\bar{u}(\cdot)$  on  $[0, \bar{t}]$  and zero on  $[\bar{t}, +\infty)$ , and with the switching signal  $(A^0 * E^l(\cdot), B^0 * F^l(\cdot), C^0 * G^l(\cdot))$ . Let  $y^l(\cdot)$  be the corresponding output. The previous inequality exactly says that there exists a positive constant  $K_3$  independent of  $t \geq 0$  such that, for every  $l$  large enough and every  $s \in [\bar{t}, t + \bar{t}]$ , one has

$$\|x^l(s)\| \geq K_3 \rho(\mathcal{T}_A)^{s-\bar{t}}. \quad (27)$$

Assume now that  $\rho(\mathcal{T}_A) = 1$  and  $\mathcal{T}$  is uniformly observable, and let  $T$  be as in Definition 21. Given two positive integers  $l, j$ , let  $W_{l,j}^o$  be the observability Gramian in time  $T$  associated with  $(E^l(jT + \cdot), F^l(jT + \cdot), G^l(jT + \cdot))$ . Applying the uniform observability assumption to each  $(E^l(jT + \cdot), F^l(jT + \cdot), G^l(jT + \cdot))$  and taking into account Eq. (27), we get, for  $J \in \mathbb{N}$  and  $l$  large enough,

$$\begin{aligned} \gamma_2(\mathcal{T}) &\geq \frac{\|y^l\|_2^2}{\|\bar{u}\|_2^2} \geq \frac{\int_0^{JT} \|G^l(t)x^l(\bar{t} + t)\|^2 dt}{\|\bar{u}\|_2^2} = \frac{\sum_{j=0}^{J-1} \int_{jT}^{(j+1)T} \|G^l(t)x^l(\bar{t} + t)\|^2 dt}{\|\bar{u}\|_2^2} \\ &= \frac{\sum_{j=0}^{J-1} x^l(\bar{t} + jT)^T W_{l,j}^o x^l(\bar{t} + jT)}{\|\bar{u}\|_2^2} \geq \gamma \frac{\sum_{j=0}^{J-1} \|x^l(\bar{t} + jT)\|^2}{\|\bar{u}\|_2^2} \geq \gamma J \frac{K_3^2}{\|\bar{u}\|_2^2}. \end{aligned}$$

This implies that the  $L_2$ -gain  $\gamma_2(\mathcal{T})$  is infinite.

Let now  $\rho(\mathcal{T}_A) > 1$ . Let  $\bar{v}$ ,  $\bar{u}(\cdot)$  and  $\bar{x}$  be as above. Since the semigroup  $\overrightarrow{\text{exp}}(\mathcal{G}_{A,\bar{v}})$  is quasi-Barabanov, we can find for every  $l \in \mathbb{N}$  a time  $t_l \geq l$  and a switching law  $(\hat{E}^l(\cdot), \hat{F}^l(\cdot), \hat{G}^l(\cdot)) \in \mathcal{G}_{t_l}$  such that

$$\|\overrightarrow{\text{exp}} \int_0^{t_l} \hat{E}^l(\tau) d\tau \bar{x}\| \geq C_{\text{qb}} \rho(\mathcal{T}_A)^{t_l} \|\bar{x}\| \geq C_{\text{qb}} \rho(\mathcal{T}_A)^l \|\bar{x}\|. \quad (28)$$

According to **T2** and because of the observability counterpart of Proposition 19 there exists a switching law  $(A_*(\cdot), B_*(\cdot), C_*(\cdot)) \in \mathcal{G}_\infty$  and  $\tilde{s} > 0$  such that the observability Gramian  $W^o(\tilde{s})$  in time  $\tilde{s} > 0$  associated with  $\dot{x}(t) = A_*(t)x(t) + B_*(t)u(t)$ ,  $y(t) = C_*(t)x(t)$ , is invertible. For  $l \geq 0$ , consider the sequence of switching signals

$$S^l = (A^0 * \hat{E}^l * A_*(\cdot), B^0 * \hat{F}^l * B_*(\cdot), C^0 * \hat{G}^l * C_*(\cdot)) \in \mathcal{G}_\infty.$$

Denote by  $x^l(\cdot)$  the corresponding trajectory of Eq. (17) starting at the origin and associated with the control equal to  $\bar{u}(\cdot)$  on  $[0, \bar{t}]$  and zero for  $t > \bar{t}$  and let  $y^l(\cdot)$  be the corresponding output. Note that  $x^l(\bar{t} + t_l) = \overrightarrow{\text{exp}} \int_0^{t_l} \hat{E}^l(\tau) d\tau \bar{x}$ . It then follows from (28) that there exists a positive constant  $K_4$  independent of  $l$  so that

$$\gamma_2(\mathcal{T}) \geq \frac{\|y^l\|_{2,t_l}^2}{\|\bar{u}\|_2^2} \geq \frac{\int_{\bar{t}+t_l}^{\bar{t}+t_l+\tilde{s}} \|C_*(s)x^l(s)\|^2 ds}{\|\bar{u}\|_2^2} = \frac{x^l(\bar{t} + t_l)^T W^o(\tilde{s}) x^l(\bar{t} + t_l)}{\|\bar{u}\|_2^2} \geq K_4 \rho(\mathcal{T}_A)^{2l},$$

and the right-hand side clearly tends to infinity as  $l$  tends to infinity.  $\square$

**Remark 25.** Note that the value of  $\rho(\mathcal{T}_A^{\min})$  does not depend on the particular choice of the minimal realization thanks to Eq. (21).

Under the hypotheses of Lemma 4, the classes of switching signals considered in Section 2.2 together with the corresponding families of Section 3.1 satisfy all the hypotheses of Theorem 24. As a consequence, we can now answer some of the questions raised by Hespanha in [7].

**Corollary 26.** Let  $\mathcal{M}$  be a bounded subset of  $M_n(\mathbb{R}) \times M_{n,m}(\mathbb{R}) \times M_{p,n}(\mathbb{R})$  with  $n, m, p$  positive integers and  $\tau \geq 0$ . Consider the switched linear control system  $\dot{x}(t) = A(t)x(t) + B(t)u(t)$ ,  $y(t) = C(t)x(t)$ , where the switching signal  $(A(\cdot), B(\cdot), C(\cdot))$  belongs to the class  $\mathcal{S}^{\text{d},\tau}(\mathcal{M})$  of piecewise constant signals with dwell-time  $\tau$ . Let  $\gamma_2(\tau)$  be the  $L_2$ -gain associated with  $\mathcal{S}^{\text{d},\tau}(\mathcal{M})$ . Consider a minimal realization defined on  $\mathbb{R}^{n'}$ ,  $n' \leq n$ , given by  $\dot{x}^{\min}(t) = A^{\min}(t)x(t) + B^{\min}(t)u(t)$  with output  $y^{\min}(t) = C^{\min}(t)x^{\min}(t)$  and associated with a class  $\mathcal{S}^{\text{d},\tau}(\mathcal{M}^{\min})$  where  $\mathcal{M}^{\min}$  is a bounded subset of  $M_{n'}(\mathbb{R}) \times M_{n',m}(\mathbb{R}) \times M_{p,n'}(\mathbb{R})$ . Assume furthermore that this minimal realization is uniformly observable.

Then,  $\gamma_2(\tau)$  is finite if and only if  $\rho(\mathcal{S}^{\text{d},\tau}(\mathcal{M}_A^{\min})) < 1$  and, if  $\tau_{\min}$  is defined as

$$\tau_{\min} = \inf\{\tau > 0 \mid \gamma_2(\tau) \text{ is finite}\},$$

one has the following characterization:  $\tau_{\min} = \inf\{\tau > 0 \mid \rho(\mathcal{S}^{\text{d},\tau}(\mathcal{M}_A^{\min})) < 1\}$ .

**Remark 27.** One can derive results similar to the previous corollary when one considers variations of certain parameters used in the definition of classes other than  $\mathcal{S}^{\text{d},\tau}(\mathcal{M})$ . For instance, one can characterize the set of  $\mu \in (0, T)$  such that  $\gamma_2(\mathcal{S}^{\text{pe},T,\mu}(\mathcal{M}))$  is finite in terms of the value of  $\rho(\mathcal{S}^{\text{pe},T,\mu}(\mathcal{M}_A^{\min}))$ .

Theorem 24 shows that, under the assumption of uniform observability of a minimal realization, the necessary and sufficient condition for finiteness of the  $L_2$ -gain (i.e., generalized spectral radius smaller than one) is exactly the same as in the unswitched framework. We prove below by means of an example that this is no more the case when the assumption of uniform observability does not hold. For this purpose, we next define a switched linear control system satisfying all the assumptions of Theorem 24 (with  $\mathcal{T}_A = \mathcal{T}_A^{\min} = (\mathcal{G}_A^{\min})_{\infty}$ ) and for which uniform observability does not hold. We have the following proposition.

**Proposition 28.** Assume that  $\mathcal{T}(\alpha) = \mathcal{S}^{\text{arb}}(\mathcal{M}(\alpha))$  where  $\mathcal{M}(\alpha) = \{(A_i, b_i, c_i)\}_{i=1,2,3} \subset M_3(\mathbb{R}) \times \mathbb{R}^3 \times \mathbb{R}^3$ , with

$$A_1 = \begin{pmatrix} -1 & -\alpha & 0 \\ \alpha & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -1 & -\alpha & 0 \\ 1/\alpha & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} -4 & 0 & 1 \\ 0 & -4 & 0 \\ 1 & 0 & -1 \end{pmatrix},$$

$$b_1 = b_2 = 0, \quad b_3 = c_1 = c_2 = c_3 = (0, 0, 1)^T,$$

for  $\alpha > 0$ . We use  $\gamma_2(\alpha)$  to denote the  $L_2$ -gain induced by the switched linear control system given by  $\dot{x} = A(t)x + b(t)u(t)$ ,  $y(t) = c^T(t)x(t)$  and  $(A(\cdot), b(\cdot), c(\cdot)) \in \mathcal{T}(\alpha)$ . Then one has  $\mathcal{T}(\alpha) = \mathcal{T}^{\min}(\alpha)$  for every  $\alpha > 0$  and there exists  $\alpha_*$  (approximately equal to 4.5047) such that  $\rho(\mathcal{T}_A(\alpha_*)) = 1$  and  $\gamma_2(\alpha_*) \leq 4$ .

*Proof.* First of all, note that the assumption of uniform observability does not hold since the observability Gramian in any positive time associated with a switching signal only activating the first two modes contains  $b_3$  in its kernel.

Using the results in [1], one determines the value  $\alpha = \alpha_* \sim 4.5047$  for which the switched system associated with  $\mathcal{M}'_A(\alpha) = \{A_1, A_2\}$  is marginally stable (and reducible). In this case, starting from every point  $(x_1, x_2, 0)$ , there exists a closed (periodic)  $\mathcal{C}^1$  trajectory  $\Gamma_{x_1, x_2}(\cdot)$  of the switched system lying on the plane  $x_3 = 0$  which can be completely determined by explicit computations. In particular, we can pick such a curve  $\Gamma$  in the set  $\{(x_1, x_2, 0) \in \mathbb{R}^3 \mid 1 \leq x_1^2 + x_2^2 \leq 3\}$ . We define the norm  $v(x_1, x_2, 0)$  to the plane  $x_3 = 0$  by setting  $v^{-1}(1) = \Gamma$ . Then  $v$  is a Barabanov norm for the restriction of  $\mathcal{M}'_A(\alpha_*)$  on the plane  $\{x_3 = 0\}$ . We extend  $v$  to  $\mathbb{R}^3$  by setting  $v(x_1, x_2, x_3) = v(x_1, x_2, 0)$  and it follows by explicit computations that  $\|\nabla v(x)\| \leq \sqrt{3}$  and, by homogeneity, that  $v(x) = \nabla v(x)^T(x_1, x_2, 0)^T$  for every  $x \in \mathbb{R}^3$ . Notice, moreover, that

$$v(x) \geq \sqrt{\frac{x_1^2 + x_2^2}{3}}, \quad \forall x \in \mathbb{R}^3.$$

Let us consider the positive definite function  $V(x) = \frac{1}{2}(v(x)^2 + x_3^2)$  and observe that  $\frac{d}{dt}V(x(t)) \leq -x_3(t)^2$  whenever  $A(t) = A_1$  or  $A(t) = A_2$ . If  $A(t) = A_3$ , one deduces from the above properties of  $v$  that

$$\begin{aligned} \frac{d}{dt}V(x(t)) &= \nabla V(x(t))^T(A_3x(t) + u(t)b_3) \\ &= v(x(t))(\nabla v(x(t))^T A_3x(t) + u(t)\nabla v(x(t))^T b_3) + x_3(t)(-x_3(t) + x_1(t) + u(t)) \\ &= -4v(x(t))^2 + v(x(t))\nabla v(x(t))^T(x_3(t), 0, 0)^T + x_3(t)(-x_3(t) + x_1(t) + u(t)) \\ &\leq -4v(x(t))^2 + 2\sqrt{3}v(x(t))|x_3(t)| - x_3(t)^2 + u(t)x_3(t) \\ &\leq -x_3(t)^2/4 + u(t)x_3(t). \end{aligned}$$

Hence  $\frac{d}{dt}V(x(t)) \leq -x_3(t)^2/4 + |u(t)x_3(t)|$  along any trajectory of the switched linear control system. By integrating the previous inequality, using the fact that  $x(0) = 0$  and applying Cauchy–Schwarz inequality, we get

$$0 \leq \liminf_{t \rightarrow \infty} V(x(t)) \leq -\frac{1}{4}\|x_3\|_2^2 + \int_0^\infty |u(s)x_3(s)|ds \leq -\frac{1}{4}\|x_3\|_2^2 + \|u\|_2\|x_3\|_2,$$

so that  $\|x_3\|_2 \leq 4\|u\|_2$ , implying that  $\gamma_2(\alpha_*) \leq 4$ .  $\square$

Notice that the computations above also show that  $V(\cdot)$  is a non-strict Lyapunov function for  $\mathcal{M}_A(\alpha_*)$ , and in particular such a system is marginally stable.

### 4.3 Right-continuity and boundedness of the $L_2$ -gain

In this section, we restrict for simplicity our discussion to the class of signals  $\mathcal{S}^{d, \tau}(\mathcal{M})$ . (For more general considerations, see Remark 31.)

Let  $\mathcal{M}$  be a bounded subset of  $M_n(\mathbb{R}) \times M_{n,m}(\mathbb{R}) \times M_{p,n}(\mathbb{R})$  with  $n, m, p$  positive integers and  $\tau > 0$ . Consider the switched linear control system  $\dot{x}(t) = A(t)x(t) + B(t)u(t)$ ,  $y(t) =$

$C(t)x(t)$ , where the switching law  $(A(\cdot), B(\cdot), C(\cdot))$  belongs to the class  $\mathcal{S}^{d,\tau}(\mathcal{M})$  of piecewise constant signals with *dwell-time*  $\tau \geq 0$ . Let  $\gamma_2(\tau)$  be the  $L_2$ -gain associated with  $\mathcal{S}^{d,\tau}(\mathcal{M})$ .

We can now state our result on the right-continuity of  $\tau \mapsto \gamma_2(\tau)$ , which answers Questions (i) and—partially—(iii) of [7].

**Proposition 29.** *The function  $\gamma_2 : [0, \infty) \rightarrow [0, \infty]$  is right-continuous, i.e., for every  $\bar{\tau} \in [0, \infty)$ ,  $\lim_{\tau \searrow \bar{\tau}} \gamma_2(\tau) = \gamma_2(\bar{\tau})$ .*

*Proof.* For every  $T > 0$  and every  $\tau \in [0, \infty)$ , define the  $L_2$ -gain in time  $T$  as

$$\gamma_2(\tau, T) := \sup \left\{ \frac{\|y_{u,\sigma}\|_{2,T}}{\|u\|_{2,T}} \mid u(\cdot) \in L_2([0, T], \mathbb{R}^m) \setminus \{0\}, \sigma(\cdot) = (A(\cdot), B(\cdot), C(\cdot)) \in \mathcal{S}^{d,\tau}(\mathcal{M}) \right\}. \quad (29)$$

It is immediate to see that  $\gamma_2(\tau, T)$  is finite for every  $(\tau, T) \in [0, \infty) \times (0, \infty)$  and that the maps  $\tau \mapsto \gamma_2(\tau, T)$  (for fixed  $T > 0$ ) and  $T \mapsto \gamma_2(\tau, T)$  (for fixed  $\tau \geq 0$ ) are non-increasing and non-decreasing respectively. Also notice that  $\tau \mapsto \gamma_2(\tau)$  is non-increasing.

We claim that

- (i)  $\lim_{T \rightarrow \infty} \gamma_2(\tau, T) = \gamma_2(\tau)$  for every  $\tau \geq 0$ ;
- (ii) the map  $\tau \mapsto \gamma_2(\tau, T)$  is right-continuous for every  $T > 0$ .

In order to prove property (i) of the claim, notice that, given  $\tau \geq 0$ ,  $0 < T \leq \infty$ , any switching signal  $\sigma(\cdot)$  and nonzero control  $u(\cdot) \in L_2$ , one has

$$\lim_{T' \nearrow T} \frac{\|y_{u,\sigma}\|_{2,T'}}{\|u\|_{2,T'}} = \frac{\|y_{u,\sigma}\|_{2,T}}{\|u\|_{2,T}},$$

since both  $\|y_{u,\sigma}\|_{2,T'}$  and  $\|u\|_{2,T'}$  converge to  $\|y_{u,\sigma}\|_{2,T}$  and  $\|u\|_{2,T}$ , respectively. Property (i) then follows from the definition of  $\gamma_2(\tau)$  and the monotonicity of  $T \mapsto \gamma_2(\tau, T)$ .

Let us now prove point (ii) of the claim. With  $0 \leq \tau < \tau'$ ,  $T > 0$ , and  $\sigma(\cdot) \in \mathcal{S}^{d,\tau}$  we associate  $\sigma'(\cdot) \in L_\infty([0, \infty), \mathcal{M})$  as follows:  $\sigma'(\cdot) := \sigma(\xi \cdot)$  where  $\xi$  is the largest number in  $[0, 1]$  such that the restriction of  $\sigma'(\cdot)$  to  $[0, T]$  has dwell-time  $\tau'$ . Notice that  $\xi$  is larger than or equal to  $\min(\bar{\tau}, \tau')/\tau'$ , where  $\bar{\tau}$  is the largest number such that the restriction of  $\sigma(\cdot)$  to  $[0, T]$  has dwell-time  $\bar{\tau}$ . In particular,  $\xi$  is always positive and converges to 1 as  $\tau' \searrow \tau$ . Note that  $\sigma(\cdot)$  and  $\sigma'(\cdot)$  are equal except on a set of measure upper bounded by  $C(T, \bar{\tau})(\tau' - \tau)$ , where  $C(T, \bar{\tau})$  denotes some positive constant only depending on  $T$  and  $\bar{\tau}$ . As a consequence, for any  $u(\cdot) \in L_2(0, T)$ ,

$$\lim_{\tau' \searrow \tau} \|y_{u,\sigma'}\|_{2,T} = \|y_{u,\sigma}\|_{2,T}. \quad (30)$$

One immediately deduces property (ii) from the definition of  $\gamma_2(\tau, T)$  and the monotonicity of  $\tau' \mapsto \gamma_2(\tau', T)$ .

We can then conclude the proof of the proposition as follows: given  $\tau \geq 0$  and  $\kappa < \gamma_2(\tau)$ , let  $T > 0$  be such that  $\kappa < \gamma_2(\tau, T)$  ( $T$  exists by property (i)). Take then a right-neighbourhood  $[\tau, \tau + \eta)$  of  $\tau$  such that  $\kappa < \gamma_2(\tau', T)$  for every  $\tau' \in [\tau, \tau + \eta)$  (the existence of  $\eta > 0$  follows by property (ii)). We deduce from the monotonicity of each function  $T \mapsto \gamma_2(\tau', T)$  that  $\kappa < \gamma_2(\tau')$  for every  $\tau' \in [\tau, \tau + \eta)$ . Since  $\kappa$  is arbitrary, we conclude that  $\liminf_{\tau' \searrow \tau} \gamma_2(\tau') \geq \gamma_2(\tau)$  and the right-continuity of  $\gamma_2(\cdot)$  is proved thanks to its monotonicity.  $\square$



An immediate consequence of the monotonicity of  $\tau \mapsto \gamma_2(\tau)$  and its right-continuity at  $\tau = 0$  is the following result.

**Corollary 30.** *The function  $\gamma_2 : (0, \infty) \rightarrow [0, \infty]$  is bounded if and only if  $\gamma_2(0) < \infty$ .*

Notice that  $\gamma_2(0)$  is finite if (and only if in the uniformly observable case) the generalized spectral radius of a corresponding minimal realization is smaller than one (see Theorem 24).

**Remark 31.** *One can reason similarly for other classes of switching laws (in particular those introduced Section 3.1), getting similar results to Proposition 29 and Corollary 30. More precisely, if we deal with a one-parameter family  $[0, \bar{\alpha}) \ni \alpha \mapsto \mathcal{S}^\alpha(\mathcal{M})$  of classes of switching laws, then the corresponding parameter-dependent  $L_2$ -gain  $\gamma_2(\alpha)$  is right-continuous and non-increasing with respect to  $\alpha$  provided that  $\mathcal{S}^\alpha(\mathcal{M}) \supset \mathcal{S}^{\alpha'}(\mathcal{M})$  for  $\alpha < \alpha'$  and that, for  $T > 0$ ,  $\alpha \in [0, \bar{\alpha})$ ,  $u(\cdot) \in L_2$ ,  $\sigma(\cdot) \in \mathcal{S}^\alpha(\mathcal{M})$ , and every sequence  $\alpha_n \searrow \alpha$ , there exist a sequence  $(\sigma_n(\cdot))_n$  with  $\sigma_n(\cdot) \in \mathcal{S}^{\alpha_n}(\mathcal{M})$  for every  $n \in \mathbb{N}$  and  $\lim_{n \rightarrow \infty} \|y_{\sigma_n, u}\|_{2, T} = \|y_{\sigma, u}\|_{2, T}$ .*

## 5 Conclusion

In this paper, we address the open Problem 4.1 proposed by J. P. Hespanha in [7], which consists of three questions about the dependence of the  $L_2$ -gain  $\gamma_2(\tau)$  of a switched linear control systems with respect to the dwell-time  $\tau$  of its switching laws. We provide some partial answers to these questions. In particular, we prove that the gain function  $\tau \mapsto \gamma_2(\tau)$  is right-continuous. Moreover, under the assumption of uniform observability (see Definition 21), we show that the infimum  $\tau_{\min}$  of the dwell-times  $\tau$  such that  $\gamma_2(\tau)$  is finite coincides with the largest dwell-time  $\tau$  for which the generalized spectral radius of a corresponding minimal realization is equal to one. Finally, under the uniform observability assumption, if the gain function  $\gamma_2(\cdot)$  is bounded then the generalized spectral radius of a minimal realization corresponding to arbitrary switching is proved to be less than one (the converse holding true in full generality).

In order to deduce these results, we are lead to build an abstract framework, allowing one to address the same  $L_2$ -gain issues for many classes of parameter-dependent switching laws. The main difficulty arises from the fact that most of such classes (e.g., those of switching laws with positive dwell-time) are not closed under concatenation. To overcome this obstacle, we introduce the concepts of quasi-Barabanov semigroup and quasi-extremal trajectory.

A complete answer to the questions asked by J. P. Hespanha remains to be achieved: the continuity of the gain function  $\tau \mapsto \gamma_2(\tau)$  is still open, as well as a new characterization of  $\tau_{\min}$  when the uniform observability assumption does not hold. Indeed, the above characterization of  $\tau_{\min}$  may fail to be true if the uniform observability assumption does not hold (see Proposition 28). Recall that, in the case of finitely many modes, uniform observability is equivalent to the observability of each mode of a minimal realization of the original system.

Another challenging open problem consists of extending the results of [10], which provide an algorithmic approach based on optimal control for the computation of the  $L_2$ -gain, in the abstract framework introduced here.

## References

- [1] M. Balde, U. Boscaïn, and P. Mason. A note on stability conditions for planar switched systems. *Internat. J. Control*, 82(10):1882–1888, 2009.
- [2] N. E. Barabanov. An absolute characteristic exponent of a class of linear nonstationary systems of differential equations. *Sibirsk. Mat. Zh.*, 29(4):12–22, 222, 1988.
- [3] A. Chaillet, Y. Chitour, A. Loria, and M. Sigalotti. Uniform stabilization for linear systems with persistency of excitation: the neutrally stable and the double integrator cases. *Math. Control Signals Systems*, 20(2):135–156, 2008.
- [4] Y. Chitour, P. Mason, and M. Sigalotti. On the marginal instability of linear switched systems. *Systems Control Lett.*, 61(6):747–757, 2012.
- [5] Y. Chitour and M. Sigalotti. On the stabilization of persistently excited linear systems. *SIAM J. Control Optim.*, 48(6):4032–4055, 2010.
- [6] F. Colonius and W. Kliemann. *The dynamics of control*. Springer Science & Business Media, 2012.
- [7] J. P. Hespanha.  $\mathcal{L}_2$ -induced gains of switched linear systems. In V. D. Blondel and A. Megretski, editors, *Unsolved Problems in Mathematical Systems & Control Theory*, pages 131–133. Princeton University Press, Princeton, NJ, 2003.
- [8] J. P. Hespanha. Root-mean-square gains of switched linear systems. *IEEE Trans. on Automat. Contr.*, 48(11), Nov. 2003.
- [9] K. Hirata and J. P. Hespanha.  $L_2$ -induced gains of switched systems and classes of switching signals. In *Proc. of the 49th Conf. on Decision and Contr.*, Dec. 2010.
- [10] M. Margaliot and J. P. Hespanha. Root-mean-square gains of switched linear systems: a variational approach. *Automatica J. IFAC*, 44(9):2398–2402, 2008.
- [11] M. Petreczky. Realization theory for linear switched systems: formal power series approach. *Systems Control Lett.*, 56(9-10):588–595, 2007.
- [12] S. Srikant and M. Akella. Arbitrarily fast exponentially stabilizing controller for multi-input, persistently exciting singular control gain systems. *Automatica J. IFAC*, 54:279–283, 2015.
- [13] Z. Sun, S. S. Ge, and T. H. Lee. Controllability and reachability criteria for switched linear systems. *Automatica J. IFAC*, 38(5):775–786, 2002.
- [14] F. Wirth. The generalized spectral radius and extremal norms. *Linear Algebra Appl.*, 342:17–40, 2002.
- [15] F. Wirth. A converse Lyapunov theorem for linear parameter-varying and linear switching systems. *SIAM J. Control Optim.*, 44(1):210–239, 2005.