



**HAL**  
open science

# Geometrically continuous splines for surfaces of arbitrary topology

Bernard Mourrain, Raimundas Vidunas, Nelly Villamizar

► **To cite this version:**

Bernard Mourrain, Raimundas Vidunas, Nelly Villamizar. Geometrically continuous splines for surfaces of arbitrary topology. 2015. hal-01196996v1

**HAL Id: hal-01196996**

**<https://inria.hal.science/hal-01196996v1>**

Preprint submitted on 10 Sep 2015 (v1), last revised 23 Mar 2016 (v2)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Geometrically continuous splines for surfaces of arbitrary topology

Bernard Mourrain\*, Raimundas Vidunas† and Nelly Villamizar‡

September 10, 2015

## Abstract

We analyze the space of geometrically continuous piecewise polynomial functions or splines for quadrangular and triangular patches with arbitrary topology and general rational transition maps. To define these spaces of  $G^1$  spline functions, we introduce the concept of topological surface with gluing data attached to the edges shared by faces. The framework does not require manifold constructions and is general enough to allow non-orientable surfaces. We describe compatibility conditions on the transition maps so that the space of differentiable functions is ample and show that these conditions are necessary and sufficient to construct ample spline spaces. We determine the dimension of the space of  $G^1$  spline functions which are of degree  $\leq k$  on triangular pieces and of bi-degree  $\leq (k, k)$  on quadrangular pieces, for  $k$  big enough. A separability property on the edges is involved to obtain the dimension formula. An explicit construction of basis functions attached respectively to vertices, edges and faces is proposed and examples of bases of  $G^1$  splines of small degree for topological surfaces with boundary and without boundary are detailed.

## 1 Introduction

The accurate and efficient representation of shapes is a major challenge in geometric modeling. To achieve high order accuracy in the representation of curves, surfaces or functions, piecewise polynomials models are usually employed. Parametric models with prescribed regularity properties are nowadays commonly used in Computer Aided Geometric Design (CAGD) to address these problems. They involve so-called spline functions, which are piecewise polynomial functions on intervals of  $\mathbb{R}$  with continuity and differentiability constraints at some nodes. Extensions of these functions to higher dimension is usually done by taking tensor product spline basis functions. Curves, surfaces or volumes are represented as the image of parametric functions expressed in terms of spline basis functions. For instance, surface patches are described as the image of a piecewise polynomial (or rational) map from a rectangular domain of  $\mathbb{R}^2$  to  $\mathbb{R}^n$ . But to represent objects with complex topology, such maps on rectangular parameter domains are not sufficient. One solution which is commonly used in Computer-Aided Design (CAD) is to trim the B-spline rectangular patches and to “stitch” together the trimmed pieces to create the complete shape representation. This results in complex models, which are not simple to use and to modify, since not structural regularity conditions can easily be imposed along the trimming curve between two trimmed patches.

To allow flexibility in the representation of shapes with complex topology, another technique called geometric continuity has been studied. Rectangular parametric surface patches are glued

---

\*Inria Sophia Antipolis Méditerranée, 2004 route des Lucioles, 06902 Sophia Antipolis, France. E-mail: Bernard.Mourrain@inria.fr

†Department of Mathematical Informatics, University of Tokyo, Bunkyo-ku 113-8656, Japan. E-mail: rvidunas@gmail.com

‡RICAM, Austrian Academy of Sciences, Altenberger straÙe 69, 4040 Linz, Austria. E-mail: nelly.villamizar@oeaw.ac.at

along their common boundary, with continuity constraints on the tangent planes (or on higher osculating spaces). In this way, smooth surfaces can be generated from quadrilateral meshes by gluing several simple parametric surfaces, forming surfaces with the expected smoothness property along the edges.

This approach built on the theory on differential manifolds, in works such as [7], [12], [9]. The idea of using transition maps or reparameterizations in connection with building smooth surfaces had been used for instance by DeRose [7] in CAGD, who gave one of the first general definitions of splines based on fixing a parametrization.

Since these initial developments, several works focused on the construction of such  $G^1$  surfaces [17], [16], [23], [22], [5], [31], [11], [13], [10], [26], [25], [3], ... with polynomial, piecewise polynomial, rational or special functions and on their use in geometric modeling applications such as surface fitting or surface reconstruction [8], [24], [15], ...

The problem of determining minimal degree of polynomial pieces has also been considered [18]. Other research investigate the construction of adapted rational transition maps for a given topological structure [1]. We refer to [19] for a review of these constructions. Constraints that the transition maps must satisfy in order to define regular spline spaces have also been identified [20]. But it has not yet been proved that these constraints are sufficient for the constructions.

The use of  $G^k$  spline functions to approximate functions over computational domains with arbitrary topology received recently a new attention for applications in isogeometric analysis. In this context, describing the space of functions, its dimension and adapted bases is of particular importance. A family of bi-cubic spline functions was recently introduced by Wu et al [30] for isogeometric applications, where constant transition maps are used, which induce singular spline basis functions at extraordinary vertices. Multi-patch representations of computational domains are also used in [4], with constant transition maps at the shared edges of quadrangular faces, using an identification of Locally Refined spline basis functions. In [14],  $G^k$  continuous splines are described and the  $G^1$  condition are transformed into a linear system of relations between the control coefficients. The case of two quadrangular patches, which share an edge is analysed experimentally. In [2], the space of  $G^1$  splines of bi-degree  $\geq 4$  for quadrangular decompositions of planar domains is analyzed. Minimal Determining Sets of points are studied, providing dimension formulae and dual basis for  $G^1$  spline functions over planar quadrangular meshes with linear gluing transition maps.

Our objective is to analyze the space of  $G^1$  spline functions for quadrangular and triangular patches with arbitrary topology and general rational transition maps. We are interested in determining the dimension of the space of  $G^1$  spline functions which are of degree  $\leq k$  on triangular pieces and of bi-degree  $\leq (k, k)$  on quadrangular pieces. To define the space of  $G^1$  spline functions, we introduce the concept of topological surface with gluing data attached to the edges shared by the faces. The framework does not require manifold constructions and is general enough to allow non-orientable surfaces. We describe compatibility conditions on the transition maps so that the space of differentiable functions is ample and show that these conditions are necessary and sufficient to construct ample spline spaces. A separability property is involved to obtain a dimension formula of the  $G^1$  spline spaces of degree  $\leq k$  on such topological surfaces, for  $k$  big enough. This leads to an explicit construction of basis functions attached respectively to vertices, edges and faces.

For the presentation of these results, we structure the paper as follows. The next section introduces the notion of topological surface  $\mathcal{M}$ , differentiable functions on  $\mathcal{M}$  and constraints on the transition maps to have an ample space of differentiable functions. Section 3 deals with the space of spline functions which are piecewise polynomial and differentiable on  $\mathcal{M}$ . Section 4 analyzes the gluing conditions along an edge. Section 5 analyzes the gluing condition around a vertex. In Section 6, we give the dimension formula for the space of spline functions of degree  $\leq k$  over a topological surface  $\mathcal{M}$  and describe explicit basis constructions. Finally, in section 7, we detail an example with boundary edges and another one with no boundary edges. We also provide an appendix with an algorithmic description of the basis construction.

## 2 Differentiable functions on a topological surface

Typically in CAGD, parametric patches are glued into surfaces by splines (i.e., polynomial maps) from polygons in  $\mathbb{R}^2$ . The simplest  $C^r$  construction is with the polygons in  $\mathbb{R}^2$  situated next to each other, so that  $C^r$  continuity across patch edges comes from  $C^r$  continuity of the coordinate functions across the polygon edges. This is called *parametric continuity*. A more general construction to generate a  $C^r$  surface from polygonal patches is called *geometric continuity* [7], [19]. Inspired by differential geometry, attempts have been made [12], [27], [28] to define *geometrically continuous  $G^r$  surfaces* from a collection of polygons in  $\mathbb{R}^2$  with additional data to glue their edges and differentiations. They are defined by parametrization maps from the polygons to  $\mathbb{R}^3$  satisfying geometric regularity conditions along edges.

It is easy to define a  $C^0$  surface from a collection of polygons and homeomorphisms between their edges.

**Definition 2.1.** Given a collection  $\mathcal{M}_2$  of (possibly coinciding) polygons in  $\mathbb{R}^2$ , a *topological surface*  $\mathcal{M}$  is defined by giving a set of homeomorphisms  $\mu : \tau_1 \rightarrow \tau_2$  between pairs of polygonal edges  $\tau_1 \subset \sigma_1, \tau_2 \subset \sigma_2$  ( $\sigma_1, \sigma_2 \in \mathcal{M}_2$ ). Each polygonal edge can be paired with at most one other edge, and it cannot be glued with itself.

A  $G^0$ -continuous function on the topological surface  $\mathcal{M}$  is defined by assigning a continuous function  $f_j$  to each polygon  $\sigma_j$ , such that the restrictions to the polygonal edges are compatible with the homeomorphisms  $\mu$ .

The topological surface  $\mathcal{M}$  is the disjoint union of the polygons, with some points identified to equivalence classes by the homeomorphisms  $\mu$ . The polygons are also called the faces of  $\mathcal{M}$  and their set is denoted  $\mathcal{M}_2$ . Each homeomorphism  $\mu$  identifies the edges  $\tau_1, \tau_2$  of the polygons  $\sigma_1, \sigma_2$  to an *interior edge* of  $\mathcal{M}$ . We say that the edge is shared by the faces  $\sigma_1$  and  $\sigma_2$ . An edge not involved in any homeomorphism  $\mu$  is a *boundary edge* of  $\mathcal{M}$ . The edges of  $\mathcal{M}$  are the equivalent classes of edges of the polygons of  $\mathbb{R}^2$  identified by the homeomorphisms  $\mu$ . Their set is denoted  $\mathcal{M}_1$ . Similarly, let  $\mathcal{M}_0$  denote the set of  $\mathcal{M}$ -vertices, that is, equivalence classes of polygonal vertices. An *interior vertex* is an equivalence class of polygonal vertices  $\gamma_0, \gamma_1, \dots, \gamma_n = \gamma_0$  such that the adjacent vertices  $\gamma_i, \gamma_{i+1}$  are identified by an edge homeomorphism. The set of these equivalent classes of identified vertices of the polygons, or interior vertices of  $\mathcal{M}$ , is denoted  $\mathcal{M}_0^\circ$ .

### 2.1 Gluing data

The definition of a differential surface  $S$  typically requires an atlas of  $S$ , that is a collection  $\{V_p, \psi_p\}_{p \in J}$  such that  $\{V_p\}_{p \in J}$  is an open covering of  $S$  [29]. Each  $\psi_p$  is a homeomorphism  $\psi_p : U_p \rightarrow V_p$ , where  $U_p$  is an open set in  $\mathbb{R}^2$ . For distinct  $p, q \in J$  such that  $V_p \cap V_q \neq \emptyset$ , let  $U_{p,q} := \psi_p^{-1}(V_p \cap V_q)$  and  $V_{q,p} := \psi_q^{-1}(V_p \cap V_q)$ . Then the map  $\psi_q^{-1} \circ \psi_p : U_{p,q} \rightarrow U_{q,p}$  is required to be a  $C^1$ -diffeomorphism. The maps  $\phi_{pq} : \psi_q^{-1} \circ \psi_p$  are called *transition maps*.

A differentiable function  $f$  on  $S$  is a function such that for any open set  $V_p$ , the composition  $f_p = f \circ \psi_p^{-1} : U_p \subset \mathbb{R}^2 \rightarrow \mathbb{R}$  is differentiable.

Our objective is to study the space of differentiable functions that can be constructed on a surface  $S$  associated to the topological surface  $\mathcal{M}$ . Instead of an atlas of  $S$ , we will only consider transition maps. More precisely, we assume that we have

- for each face  $\sigma \in \mathcal{M}_2$  an open set  $U_\sigma$  of  $\mathbb{R}^2$  containing  $\sigma$ ;
- for each edge  $\tau \in \mathcal{M}_1$  of a cell  $\sigma$ , an open set  $U_{\tau,\sigma}$  of  $\mathbb{R}^2$  containing  $\tau$ ;
- for each edge  $\tau \in \mathcal{M}_1$  shared by two faces  $\sigma_1, \sigma_2 \in \mathcal{M}_2$ , the transition map  $\phi_{\sigma_1,\sigma_2} : U_{\tau,\sigma_1} \rightarrow U_{\tau,\sigma_2}$  and its inverse map  $\phi_{\sigma_2,\sigma_1}$ ;
- for each edge  $\tau \in \mathcal{M}_1$  of a cell  $\sigma$ , an identity transition map  $\phi_{\sigma,\tau} = \text{Id}$  between  $U_\sigma$  and  $U_{\tau,\sigma}$ ;

**Definition 2.2.** The set of transition maps  $\phi_{\sigma_1, \sigma_2}$  between two faces  $\sigma_1, \sigma_2$  which share an edge  $\tau$  in  $\mathcal{M}$ , will be called the *gluing data* associated to  $\mathcal{M}$ . We assume that the transition maps are  $C^1$ -diffeomorphism between the open sets  $U_{\tau, \sigma}$ .

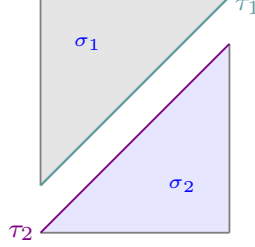


Figure 1: Topological surface constructed from two triangles.

Let  $\tau = (\tau_1, \tau_2)$  be an edge shared by two faces  $\sigma_1, \sigma_2 \in \mathcal{M}_2$  and let  $\gamma = (\gamma_1, \gamma_2)$  be a vertex of  $\tau$  corresponding to  $\gamma_1$  in  $\sigma_1$  and to  $\gamma_2$  in  $\sigma_2$ . We denote by  $\tau'_1$  (resp.  $\tau'_2$ ) the second edge of  $\sigma_1$  (resp.  $\sigma_2$ ) through  $\gamma_1$  (resp.  $\gamma_2$ ). We associate to  $\sigma_1$  and  $\sigma_2$  two coordinate systems  $(u_1, v_1)$  and  $(u_2, v_2)$  such that  $\gamma_1 = (0, 0)$ ,  $\tau_1 = \{(u_1, 0), u_1 \in [0, 1]\}$ ,  $\tau'_1 = \{(0, v_1), v_1 \in [0, 1]\}$  and  $\gamma_2 = (0, 0)$ ,  $\tau_2 = \{(0, v_2), v_2 \in [0, 1]\}$ ,  $\tau'_2 = \{(u_2, 0), u_2 \in [0, 1]\}$ . A transition map from  $U_{\tau, \sigma_1}$  to  $U_{\tau, \sigma_2}$  is then of the form

$$\phi_{\sigma_2, \sigma_1} : (u_1, v_1) \longrightarrow (u_2, v_2) = \begin{pmatrix} v_1 \mathbf{b}_{\tau, \gamma}(u_1) + v_1^2 \rho_1(u_1, v_1) \\ u_1 + v_1 \mathbf{a}_{\tau, \gamma}(u_1) + v_1^2 \rho_2(u_1, v_1) \end{pmatrix} \quad (1)$$

where  $\mathbf{a}_{\tau, \gamma}(u_1), \mathbf{b}_{\tau, \gamma}(u_1), \rho_1(u_1, v_1), \rho_2(u_1, v_1)$  are  $C^1$  functions. We will refer to it as the canonical form of the transition map  $\phi_{\sigma_2, \sigma_1}$  at  $\gamma$  along  $\tau$ . The functions  $[\mathbf{a}_{\tau, \gamma}, \mathbf{b}_{\tau, \gamma}]$  are called the *gluing data* at  $\gamma$  along  $\tau$  on  $\sigma_1$ .

**Definition 2.3.** An edge  $\tau \in \mathcal{M}$  which contains the vertex  $\gamma \in \mathcal{M}$  is called a *crossing edge* at  $\gamma$  if  $\mathbf{a}_{\tau, \gamma}(0) = 0$  where  $[\mathbf{a}_{\tau, \gamma}, \mathbf{b}_{\tau, \gamma}]$  is the gluing data at  $\gamma$  along  $\tau$ . We define  $\mathbf{c}_{\tau}(\gamma) = 1$  if  $\tau$  is a crossing edge at  $\gamma$  and  $\mathbf{c}_{\tau}(\gamma) = 0$  otherwise. If  $\gamma \in \mathcal{M}_0$  is an interior vertex where all adjacent edges are crossing edges at  $\gamma$ , then it is called a *crossing vertex*. Similarly, we define  $\mathbf{c}_+(\gamma) = 1$  if  $\gamma$  is a crossing vertex and  $\mathbf{c}_+(\gamma) = 0$  otherwise.

## 2.2 Differentiable functions on a topological surface

We can now define the notion of differentiable function on  $\mathcal{M}$ :

**Definition 2.4.** A differentiable function  $f$  on the topological surface  $\mathcal{M}$  is a collection  $f = (f_{\sigma})_{\sigma \in \mathcal{M}}$  of differentiable functions  $f_{\sigma} : U_{\sigma} \rightarrow \mathbb{R}$  such that  $\forall \gamma \in \tau = \sigma_1 \cap \sigma_2, \forall \mathbf{u} \in U_{\tau, \sigma_1}$ ,

$$J_{\gamma}(f_{\sigma_1})(\mathbf{u}) = J_{\gamma}(f_{\sigma_2} \circ \phi_{\sigma_2, \sigma_1})(\mathbf{u}) \quad (2)$$

where  $J_{\gamma}$  is the jet or Taylor expansion of order 1 at  $\gamma$ .

If  $f_1, f_2$  are the functions associated to the faces  $\sigma_1, \sigma_2 \in \mathcal{M}_2$  which are glued along the edge  $\tau$  with a transition map of the form (1), the regularity condition (2) leads to the following relations:

- $f_1(u_1, 0) = f_2 \circ \phi_{\sigma_2, \sigma_1}(u_1, 0)$  for  $u_1 \in [0, 1]$ ; that is

$$f_1(u_1, 0) = f_2(0, u_1) \quad (3)$$

- $\frac{\partial f_1}{\partial v_1}(u_1, 0) = \frac{\partial(f_2 \circ \phi)}{\partial v_1}(u_1, 0)$  for  $u_1 \in [0, 1]$ , which translates to

$$\frac{\partial f_1}{\partial v_1}(u_1, 0) = \mathbf{b}_{\tau, \gamma}(u_1) \frac{\partial f_2}{\partial u_2}(0, u_1) + \mathbf{a}_{\tau, \gamma}(u_1) \frac{\partial f_2}{\partial v_2}(0, u_1) \quad (4)$$

for  $u_1 \in [0, 1]$ , with  $\mathbf{a}(u_1) = \frac{\partial \phi_1}{\partial v_1}(u_1, 0)$ ,  $\mathbf{b}(u_1) = \frac{\partial \phi_2}{\partial v_1}(u_1, 0)$ , where  $\phi_1$  and  $\phi_2$  are the components of  $\phi$  in each of the first and the second variable respectively.

A convenient way to describe this regularity condition is to express the relation (4) as a relation between differentials acting on the space of differential functions on the edge  $\tau$ :

$$\mathbf{a}_{\tau, \gamma}(u_1) \partial_{v_2} + \mathbf{b}_{\tau, \gamma}(u_1) \partial_{u_2} - \partial_{v_1} = 0 \quad (5)$$

With this notation, at a crossing vertex  $\gamma$  with 4 edges we have  $\mathbf{b}_{\tau, \gamma}(0) \partial_{u_2} - \partial_{v_1} = 0$ . The differentials along two opposite edges are “aligned”, which explains the terminology of crossing vertex.

**Definition 2.5.** A subspace  $\mathcal{D}$  of the vector space of differentiable functions on  $\mathcal{M}$  is said to be *ample* if at every point  $\gamma$  of a face  $\sigma$  of  $\mathcal{M}$ , the space of values and differentials at  $\gamma$ , namely  $[f(\gamma), \partial_{u_\sigma}(f)(\gamma), \partial_{v_\sigma} f(\gamma)]$  for  $f \in \mathcal{D}$ , is of dimension 3.

This definition does not depend on the choice of the face  $\sigma$  to which  $\gamma$  belongs, since for  $\gamma$  on a shared edge, the value and differentials coincide after transformation by the invertible transition map.

### 2.3 Compatibility condition at a vertex

Giving gluing data on the edges is not sufficient to ensure the existence of an ample space of differentiable functions on  $\mathcal{M}$ . At vertices shared by several edges and faces, additional conditions on the transition maps need to be satisfied. We describe them in this section, and show that they are sufficient to construction an ample space of splines on  $\mathcal{M}$  in the following sections.

For a vertex  $\gamma \in \mathcal{M}_0^\circ$ , (see Fig. 2) which is common to faces  $\sigma_1, \dots, \sigma_F$  glued cyclically around  $\gamma$ , along the edges  $\tau_i = \sigma_{i+1} \cap \sigma_i$  for  $i = 1, \dots, F$  (with  $\sigma_{F+1} = \sigma_1$ ), we impose the following condition:

$$J_\gamma(\phi_{1,F}) \circ \dots \circ J_\gamma(\phi_{3,2}) \circ J_\gamma(\phi_{2,1})(u, v) = (u, v), \quad (6)$$

where  $J_\gamma$  is the jet or Taylor expansion of order 1 at  $\gamma$ .

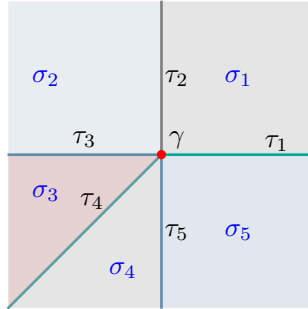


Figure 2: The faces  $\sigma_i$  for  $i = 1, \dots, 5$  are glued cyclically around a vertex  $\gamma$ .

We can assume that for each  $i = 1, \dots, F$ , the edge  $\tau_i$  is defined (linearly) by  $v_i = 0$  in  $\sigma_i$ . It is easy to check that the condition (6) on the Taylor expansion at  $\gamma$  leads to the following:

**Condition 2.6.** *If the vertex  $\gamma$  is on the faces  $\sigma_1, \dots, \sigma_F$  glued cyclically around  $\gamma$ , the gluing  $[\mathbf{a}_i, \mathbf{b}_i]$  at  $\gamma$  on the edges  $\tau_i$  between  $\sigma_{i-1}$  and  $\sigma_i$  satisfy*

$$\prod_{i=1}^F \begin{pmatrix} 0 & 1 \\ \mathbf{b}_i(0) & \mathbf{a}_i(0) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (7)$$

This gives algebraic restrictions on the values  $\mathbf{a}_i(0)$ ,  $\mathbf{b}_i(0)$ . At a crossing vertex  $\gamma$  (see Def. 2.3), the equality (7) amounts to

$$\mathbf{b}_1(0)\mathbf{b}_3(0) = 1, \quad \mathbf{b}_2(0)\mathbf{b}_4(0) = 1. \quad (8)$$

It turns out that Condition 2.6 is not sufficient around crossing vertices for ensuring an ample space of differentiable functions on  $\mathcal{M}$ . An obstruction was noticed in [20] in a setting of rectangular patches. We write this constrain in a general setting:

**Condition 2.7.** *If the vertex  $\gamma$  is a crossing vertex with 4 edges  $\tau_1, \dots, \tau_4$ , the gluing data  $[\mathbf{a}_i, \mathbf{b}_i]$   $i = 1 \dots 4$  on these edges at  $\gamma$  satisfy*

$$\mathbf{a}'_1(0) + \frac{\mathbf{b}'_4(0)}{\mathbf{b}_4(0)} = -\mathbf{b}_1(0) \left( \mathbf{a}'_3(0) + \frac{\mathbf{b}'_2(0)}{\mathbf{b}_2(0)} \right), \quad (9)$$

$$\mathbf{a}'_2(0) + \frac{\mathbf{b}'_1(0)}{\mathbf{b}_1(0)} = -\mathbf{b}_2(0) \left( \mathbf{a}'_4(0) + \frac{\mathbf{b}'_3(0)}{\mathbf{b}_3(0)} \right). \quad (10)$$

**Lemma 2.8.** *If the space of differentiable functions on  $\mathcal{M}$  is ample, the condition 2.7 must be satisfied at every crossing vertex  $\gamma$  with 4 edges.*

*Proof.* The value and first derivatives at every point  $\gamma \in \mathcal{M}$  of all differentiable functions on  $\mathcal{M}$  should span a space of dimension 3.

If  $\gamma$  is a cross vertex, then we have 4 restrictions on the Taylor expansions of a spline components  $(f_1, f_2, f_3, f_4)$ . Let us write the Taylor expansion of  $f_i$  at  $\gamma = (0, 0)$  as  $f_i = p_i + q_i u_i + r_i v_i + s_i u_i v_i + \dots$ . The gluing conditions imply the following. From (3),

$$p_1 = p_2 = p_3 = p_4, \quad q_1 = r_2, \quad q_2 = r_3, \quad q_3 = r_4 \quad \text{and} \quad q_4 = r_1 \quad (11)$$

this together with the condition (4) on the first derivatives imply

$$\mathbf{b}_2(0)q_1 - q_3 = 0, \quad \mathbf{b}_3(0)q_2 - q_4 = 0.$$

When we consider the derivative of  $f_i$  with respect to  $u_i$ , again applying (4), we get the conditions

$$s_1 - \mathbf{b}_1(0)s_4 = \mathbf{b}'_1(0)q_4 + \mathbf{a}'_1(0)q_1,$$

$$s_2 - \mathbf{b}_2(0)s_1 = \mathbf{b}'_2(0)q_1 + \mathbf{a}'_2(0)q_2,$$

$$s_3 - \mathbf{b}_3(0)s_2 = \mathbf{b}'_3(0)q_2 + \mathbf{a}'_3(0)q_3,$$

$$s_4 - \mathbf{b}_4(0)s_3 = \mathbf{b}'_4(0)q_3 + \mathbf{a}'_4(0)q_4.$$

Combining the last four equations respectively with the weights 1,  $\mathbf{b}_4(0)$ ,  $\mathbf{b}_1(0)\mathbf{b}_4(0)$ ,  $\mathbf{b}_1(0)$ , we get

$$\begin{aligned} & q_1 (\mathbf{a}'_1(0) + \mathbf{b}_2(0)\mathbf{b}'_4(0)) + q_3 \mathbf{b}_1(0) (\mathbf{b}_2(0)\mathbf{a}'_3(0) + \mathbf{b}'_2(0)) + \\ & q_2 (\mathbf{b}_1(0)\mathbf{a}'_2(0) + \mathbf{b}'_1(0)) + q_4 \mathbf{b}_2(0) (\mathbf{a}'_4(0) + \mathbf{b}_1(0)\mathbf{b}'_3(0)) = 0. \end{aligned} \quad (12)$$

This relation does not involve the cross derivatives  $s_1, s_2, s_3, s_4$ , but gives an unwanted relation between the first order derivatives. After replacing (11) and (8) in (12), we encounter conditions (9) and (10). Under these conditions, there is no relation between  $q_1, q_2$ , and there is one degree of freedom for  $(s_1, s_2, s_3, s_4)$ .  $\square$

The restrictions (9) and (10) were noticed in [20] in the context of gluing tensor product quadrangular patches with all  $\mathbf{b}_i(0) = -1$ . The restrictions are then simply

$$\mathbf{a}'_1(0) = \mathbf{a}'_3(0), \quad \mathbf{a}'_2(0) = \mathbf{a}'_4(0).$$

## 2.4 Topological restrictions

A guiding principle for the construction of Geometric Continuous functions is that  $G^1$  properties are equivalent to  $C^1$  properties in the plane after an adequate reparameterization of the problem. Gluing two faces along an edge is transformed locally via such reparameterization maps, into gluing two half-planes along a line. Each half-plane is in correspondence with the half-plane determined by one of the faces and the shared edge. A natural gluing is to have the half planes on each side of the line. In this case, the points of one face are mapped by the reparameterizations on one side of the line and the points of the other face on the other side of the line. This implies that the transition maps keep locally the points of a face on the same side of the edge and thus it should have a positive Jacobian at each point of the edge. Therefore the first topological restriction that we ask for each edge  $\tau$ , using the canonical form (1), is the following:

$$\forall u \in [0, 1], \mathbf{b}_\tau(u) < 0.$$

When the function  $\mathbf{b}_\tau$  is positive on the edge; the transition map identifies interiors of the polygons. It corresponds to two patches of surfaces virtually pasted at a sharp edge (i.e., at angle 0) rather than in a proper continuously smooth manner (i.e., at the angle  $\pi$ ). In some CAGD applications, it may be useful to model surfaces with sharp wing-like edges by the  $G^1$  continuity restrictions with  $\mathbf{b}_\tau > 0$ . But typical  $G^1$  continuity applications should require  $\mathbf{b}_\tau < 0$  on the whole edge to prevent this degeneration.

Similarly, gluing the faces around a vertex  $\gamma$  should be equivalent to gluing sectors around a point in the plane, via the reparameterization maps. Such sectors should form a fan around the point. If this fan is defined by vectors  $\mathbf{u}_1, \dots, \mathbf{u}_{F'} \in \mathbb{R}^2$  ( $\mathbf{u}_{i+1}$  is supposed to be outside the union of the sectors defined two consecutive vectors  $\mathbf{u}_{j-1}, \mathbf{u}_j$  for  $2 < j < i$ ), we easily check that the coefficients  $\mathbf{a}_i(0), \mathbf{b}_i(0)$  of the transition map (1) across the edge  $\tau_i$  at  $\gamma$  are such that:

$$\mathbf{u}_{i-1} = \mathbf{a}_i(0)\mathbf{u}_i + \mathbf{b}_i(0)\mathbf{u}_{i+1} \tag{13}$$

or equivalently

$$[\mathbf{u}_i, \mathbf{u}_{i-1}] = \begin{bmatrix} 0 & 1 \\ \mathbf{b}_i(0) & \mathbf{a}_i(0) \end{bmatrix} [\mathbf{u}_{i+1}, \mathbf{u}_i]$$

(see also the construction in the next section 2.5). If the sector angles are less than  $\pi$  (i.e. the sector  $\mathbf{u}_i, \mathbf{u}_{i+1}$  coincides with the cone generated by  $\mathbf{u}_i, \mathbf{u}_{i+1}$ ) the condition that the sectors form a fan and do not overlap translates as follows: the coefficients of the last row of

$$\prod_{i=j}^k \begin{pmatrix} 0 & 1 \\ \mathbf{b}_i(0) & \mathbf{a}_i(0) \end{pmatrix}$$

should not be both non-negative for  $1 < j \leq k < F$ .

A natural way to define transition maps at a vertex  $\gamma$  which satisfy this condition is to choose vectors in the plane that define a fan, as in Figure 2. Then the coefficients  $\mathbf{a}_i(0), \mathbf{b}_i(0)$  are uniquely determined from the relations (13).

The topological constraints could be dropped in some applications, for example, when modeling analytical surfaces with branching points, or surfaces with sharp wing-like interior edges, or with winding-up boundary. In these specific applications, the compatibility condition 2.7 at *crossing vertex* might need to be extended, to allow winding up of 8, 12, etc., crossing edges, and take into account the sharp edges. Apart from this kind of consideration, the topological conditions do not essentially affect our algebraic dimension count.

The framework that we propose is more general than previous approaches used in Geometric Modeling to define  $G^1$  splines (see e.g. [21, §3]) since it allows to define differentiable functions on topological surfaces such as a Möbius strip or a Klein bottle. Moreover, it does not rely on the construction of manifold surfaces and atlas, but only on compatible transition maps.



## 2.5 Example

A simple way to define transition maps is to use a symmetric gluing as proposed in [12]. If  $\tau = (\gamma_0, \gamma_1)$  is the shared edge between  $\sigma_1$  and  $\sigma_2$ , the transition map can be of the form:

$$\phi(u, v) = \begin{pmatrix} -v \\ u + 2v(\mathfrak{d}_0(u) \cos \frac{2\pi}{n_0} - \mathfrak{d}_1(u) \cos \frac{2\pi}{n_1}) \end{pmatrix} \quad (14)$$

where  $n_0$  (resp.  $n_1$ ) is the number of edges at the vertex  $\gamma_0$  (resp.  $\gamma_1$ ). Additionally, if  $\gamma_0$  corresponds to  $u = 0$  and  $\gamma_1$  to  $u = 1$ , the functions  $\mathbf{a}$  and  $\mathbf{b}$  interpolate 0 and 1:  $\mathfrak{d}_0(0) = 1$ ,  $\mathfrak{d}_0(1) = 0$ ,  $\mathfrak{d}_1(0) = 0$ ,  $\mathfrak{d}_1(1) = 1$  and their derivatives of order 1 should vanish at 0, 1. It corresponds to a symmetric gluing, where the angle of two consecutive edges at  $\gamma_i$  is  $\frac{2\pi}{n_i}$ . If  $\mathfrak{d}_0(u)$  and  $\mathfrak{d}_1(u)$  are polynomial functions, their degree must be at least 3. If  $\mathfrak{d}_0(u)$  and  $\mathfrak{d}_1(u)$  are rational functions with the same denominator, the maximal degree of the numerators and denominator must be at least 2. As we will see the dimension of the spline space decreases with this degree. Thus it is important to construct transition maps with low degree numerators and denominators. See e.g. [19, 20] for low degree constructions, which depend on the structure of  $\mathcal{M}$ .

A general construction of gluing data which satisfies the compatibility conditions is as follows.

- (i) For all the vertices  $\gamma \in \mathcal{M}_0$  and for all the edges  $\tau_1, \dots, \tau_F$  of  $\mathcal{M}_1$  that contain  $\gamma$ , choose vectors  $\mathbf{u}_1, \dots, \mathbf{u}_F \in \mathbb{R}^2$  such that the cones generated by  $\mathbf{u}_i, \mathbf{u}_{i+1}$  form a fan of  $\mathbb{R}^2$  and such that the union of these cones is  $\mathbb{R}^2$  when  $\gamma$  is an interior vertex.

Compute the transition map at  $\gamma$  on the edge  $\tau_i$

$$\phi_{\tau_i, \gamma} = S \circ [\mathbf{u}_i, \mathbf{u}_{i+1}]^{-1} \circ [\mathbf{u}_{i-1}, \mathbf{u}_i] \circ S = \begin{bmatrix} 0 & \mathbf{b}_{\tau_i, \gamma} \\ 1 & \mathbf{a}_{\tau_i, \gamma} \end{bmatrix}$$

where  $S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ ,  $[\mathbf{u}_i, \mathbf{u}_{i+1}]$  is the matrix which columns are the vectors  $\mathbf{u}_i, \mathbf{u}_{i+1}$ ,  $|\mathbf{u}_i, \mathbf{u}_j|$  is the determinant of the vectors  $\mathbf{u}_i, \mathbf{u}_j$  and

$$\mathbf{a}_{\tau_i, \gamma} = \frac{|\mathbf{u}_{i-1}, \mathbf{u}_{i+1}|}{|\mathbf{u}_i, \mathbf{u}_{i+1}|}, \quad \mathbf{b}_{\tau_i, \gamma} = -\frac{|\mathbf{u}_{i-1}, \mathbf{u}_i|}{|\mathbf{u}_i, \mathbf{u}_{i+1}|}.$$

- (ii) For all the edges  $\tau \in \mathcal{M}_1$ , define the rational functions  $\mathbf{a}_\tau = \frac{a_\tau}{c_\tau}$ ,  $\mathbf{b}_\tau = \frac{b_\tau}{c_\tau}$  on the edges  $\tau$  by interpolation as follows: if there is no crossing edge in  $\mathcal{M}_1$ , then a linear interpolation of the value at the vertices is sufficient. If  $\gamma_1, \dots, \gamma_n$  is a sequence of crossing vertices, and  $\gamma_0\gamma_1, \gamma_1\gamma_2, \dots, \gamma_n\gamma_{n+1}$  is a sequence of edges passing “straight” through them, we can choose linear gluing data on one edge, and quadratic data on the remaining edges of the sequence so that the constraints (9) and (10) are satisfied.

Therefore, for general meshes, gluing data which satisfy the compatibility condition and the topological condition can be constructed in degree  $\leq 2$ .

## 3 Spline space on a topological surface

The main object of our study is the space of functions on the topological surface  $\mathcal{M}$ , which are differentiable and piecewise polynomial. Such functions are called *spline functions* on  $\mathcal{M}$ . Let  $\mathcal{R}(\sigma) = \mathbb{R}[u_\sigma, v_\sigma]$  be the ring of polynomials in the variables  $(u_\sigma, v_\sigma)$  attached to the face  $\sigma$ . A spline function  $f$  is defined by assigning to each face  $\sigma \in \mathcal{M}_2$  a polynomial  $f_\sigma \in \mathcal{R}(\sigma)$ , and by imposing the regularity conditions across the shared edges.

We also consider rational gluing data on the interior edges  $\tau \in \mathcal{M}_1$ :

$$\mathbf{a}_\tau(u_1) = \frac{a_\tau(u_1)}{c_\tau(u_1)} \quad \text{and} \quad \mathbf{b}_\tau(u_1) = \frac{b_\tau(u_1)}{c_\tau(u_1)} \quad (15)$$

with  $a_\tau(u_1), b_\tau(u_1)$  and  $c_\tau(u_1)$  polynomials in the variable  $u_1$ , where  $b_\tau(u_1)$  and  $c_\tau(u_1)$  do not vanish on  $\tau$  (i.e. for  $u_1 \in [0, 1]$ ). As  $b(u_1), c(u_1)$  do not vanish on  $\tau$ , the transition map  $\phi_{\sigma_2, \sigma_1}$  is a  $C^1$ -diffeomorphism in a neighborhood of the edge  $\tau = (\tau_1, \tau_2)$  between  $\sigma_1$  and  $\sigma_2$ . The polynomial vector  $[a_\tau, b_\tau, c_\tau]$  is also called hereafter the gluing data of the edge  $\tau$ . We assume hereafter that *the transition maps satisfy Conditions 2.6, 2.7 and all crossing vertices of  $\mathcal{M}$  have 4 edges.*

We can now define the space of splines on  $\mathcal{M}$ :

**Definition 3.1.** We denote by  $\mathcal{S}_k^1(\mathcal{M})$  the ( $\mathbb{R}$ -linear) space of differentiable functions on the topological surface  $\mathcal{M}$  which are defined by assigning polynomials to the faces  $\sigma \in \mathcal{M}_2$  satisfying the  $G^1$  constrains (2). More precisely,

$$\mathcal{S}_k^1(\mathcal{M}) := \{f \in \oplus_{\sigma \in \mathcal{M}_2} \mathcal{R}(\sigma)_\sigma \mid J_\gamma(f_{\sigma_1}) = J_\gamma(f_{\sigma_2} \circ \phi_{\sigma_2, \sigma_1}) \forall \gamma \in \tau = \sigma_1 \cap \sigma_2\},$$

where  $J_\gamma$  is the jet or Taylor expansion of order 1.

A spline  $f \in \mathcal{S}_k^1(\mathcal{M})$  gives a piecewise-polynomial map, defined on every face of  $\mathcal{M}$ , and the jets of order 1 coincide on the shared edges. This definition implies that a spline function  $f \in \mathcal{S}_k^1(\mathcal{M})$  is  $C^1$  on a neighborhood of a shared edge  $\tau = \sigma_1 \cap \sigma_2$  if we use the re-parametrization  $f_{\sigma_2} \circ \phi_{\sigma_2, \sigma_1}$ .

Definition 3.1 can be directly extended to splines of any order  $r$  but in this paper we only consider  $r = 1$ .

### 3.1 Polynomials on faces

On each face  $\sigma \in \mathcal{M}_2$ , we consider polynomials of degree bounded by  $k \in \mathbb{N}$ .

If  $\sigma \in \mathcal{M}_2$  is a triangle  $(P, Q, R)$ , we denote by  $\mathcal{R}_k(\sigma)$  the finite dimensional vector space of polynomials in  $\mathbb{R}[u_\sigma, v_\sigma]$  with total degree bounded by  $k$ .

After a change of coordinates, we may assume that the coordinate function  $u = u_\sigma$  satisfies  $u_\sigma(PR) = 0$  and  $u_\sigma(Q) = 1$ , while  $v = v_\sigma$  satisfies  $v_\sigma(PQ) = 0$  and  $v_\sigma(R) = 1$ . Introducing  $w = 1 - u - v$ , we can express any polynomial in  $\mathcal{R}_k(\sigma)$  as a homogeneous polynomial of degree  $\leq k$  in the *barycentric* coordinates  $u, v, w$  using the *Bernstein-Bézier basis*:

$$b_{i,j}^\Delta(u, v, w) = \frac{k!}{i!j!(k-i-j)!} u^i v^j w^{k-i-j},$$

for  $0 \leq i + j \leq k$ . We verify directly that for a function

$$f(u, v) = \sum_{0 \leq i+j \leq k} c_{i,j} b_{i,j}^\Delta(u, v, 1-u-v)$$

expressed in this basis, we have

$$\begin{aligned} f(0, 0) &= c_{0,0}, \\ \partial_u f(0, 0) &= k(c_{1,0} - c_{0,0}), \quad \partial_v f(0, 0) = k(c_{0,1} - c_{0,0}), \\ \partial_u \partial_v f(0, 0) &= k(k-1)(c_{1,1} - c_{1,0} - c_{0,1} + c_{0,0}). \end{aligned} \tag{16}$$

If  $\sigma \in \mathcal{M}_2$  is a rectangle  $(P, Q, R, S)$ , we will denote by  $\mathcal{R}_k(\sigma)$  the finite dimensional vector space of polynomials in  $\mathbb{R}[u_\sigma, v_\sigma]$  with partial degree in  $u_\sigma$  and  $v_\sigma$  bounded by  $k$ , where  $u = u_\sigma$  is chosen such that  $u_\sigma(PS) = 0$ ,  $u_\sigma(QR) = 1$ , and  $v = v_\sigma$  is chosen such that  $v_\sigma(PQ) = 0$ ,  $v_\sigma(RS) = 1$ . Introducing  $\tilde{u} = 1 - u$ ,  $\tilde{v} = 1 - v$ , we can express any polynomial function of  $\mathcal{R}_k(\sigma)$  as a bi-homogeneous polynomial of degree  $k$  in  $u, \tilde{u}$  and degree  $k$  in  $v, \tilde{v}$ , using the *tensor product Bernstein-Bézier basis*

$$b_{i,j}^\square(u, \tilde{u}, v, \tilde{v}) = \frac{k!k!}{i!j!(k-i)!(k-j)!} u^i \tilde{u}^{k-i} v^j \tilde{v}^{k-j}.$$

for  $0 \leq i \leq k$ ,  $0 \leq j \leq k$ . We verify directly that for a function  $f = \sum_{0 \leq i, j \leq k} c_{i, j} b_{i, j}^\square$  expressed in this basis, we have

$$\begin{aligned} f(0, 0) &= c_{0,0}, \\ \partial_u f(0, 0) &= k(c_{1,0} - c_{0,0}), \quad \partial_v f(0, 0) = k(c_{0,1} - c_{0,0}), \\ \partial_u \partial_v f(0, 0) &= k^2(c_{1,1} - c_{1,0} - c_{0,1} + c_{0,0}). \end{aligned} \tag{17}$$

### 3.2 Taylor maps

An important tool that we are going to use intensively is the Taylor map associated to a vertex or to an edge of  $\mathcal{M}$ .

Let  $\gamma \in \mathcal{M}_0$  be a vertex on a face  $\sigma \in \mathcal{M}_2$  belonging to two edges  $\tau, \tau' \in \mathcal{M}_1$  of  $\sigma$ . We define the *ring of  $\gamma$  on  $\sigma$*  by  $\mathcal{R}^\sigma(\gamma) = \mathcal{R}(\sigma)/(\ell_\tau^2, \ell_{\tau'}^2)$  where  $(\ell_\tau^2, \ell_{\tau'}^2)$  is the ideal generated by the squares of  $\ell_\tau$  and  $\ell_{\tau'}$ , the equations  $\ell_\tau(u, v) = 0$  and  $\ell_{\tau'}(u, v) = 0$  are respectively the equations of  $\tau$  and  $\tau'$  in  $\mathcal{R}(\sigma) = \mathbb{R}[u, v]$ .

The *Taylor expansion at  $\gamma$  on  $\sigma$*  is the map

$$T_\gamma^\sigma : f \in \mathcal{R}(\sigma) \mapsto f \pmod{(\ell_\tau^2, \ell_{\tau'}^2)} \text{ in } \mathcal{R}^\sigma(\gamma).$$

Choosing an adapted basis of  $\mathcal{R}^\sigma(\gamma)$ ,  $T_\gamma^\sigma$  can also be defined by

$$T_\gamma^\sigma(f) = [f(\gamma), \partial_u f(\gamma), \partial_v f(\gamma), \partial_u \partial_v f(\gamma)].$$

The map  $T_\gamma^\sigma$  can also be defined in another basis of  $\mathcal{R}^\sigma(\gamma)$  in terms of the Bernstein coefficients by

$$T_\gamma^\sigma(f) = [c_{0,0}(f), c_{1,0}(f), c_{0,1}(f), c_{1,1}(f)]$$

where  $c_{0,0}, c_{1,0}, c_{0,1}, c_{1,1}$  are the first Bernstein coefficients associated to  $\gamma = (0, 0)$ .

We define the Taylor map  $T_\gamma$  on all the faces  $\sigma$  that contain  $\gamma$ ,

$$T_\gamma : f = (f_\sigma) \in \oplus_\sigma \mathcal{R}(\sigma) \rightarrow (T_\gamma^\sigma(f_\sigma)) \in \oplus_{\sigma \supset \gamma} \mathcal{R}^\sigma(\gamma).$$

Similarly, we define  $T_0$  as the Taylor map at all the vertices on all the faces of  $\mathcal{M}$ .

For an edge  $\tau \in \mathcal{M}_1$  on a face  $\sigma \in \mathcal{M}_2$ , we define the *ring of  $\tau$  on  $\sigma$*  by  $\mathcal{R}^\sigma(\tau) = \mathcal{R}(\sigma)/(\ell_\tau^2)$  where  $\ell_\tau(u, v) = 0$  is the equation of  $\tau$  in  $\mathcal{R}(\sigma) = \mathbb{R}[u, v]$ . The *Taylor expansion along  $\tau$  on  $\sigma$*  is defined by

$$T_\tau^\sigma : f \in \mathcal{R}(\sigma) \mapsto f \pmod{(\ell_\tau^2)} \text{ in } \mathcal{R}^\sigma(\tau),$$

and the Taylor map on all the faces  $\sigma$  that contain  $\tau$  is given by

$$T_\tau : f = (f_\sigma) \in \oplus_\sigma \mathcal{R}(\sigma) \rightarrow (T_\tau^\sigma(f_\sigma)) \in \oplus_{\sigma \supset \tau} \mathcal{R}^\sigma(\tau).$$

Similarly, we define  $T_1$  as the Taylor map along all the edges on all the faces of  $\mathcal{M}$ .

## 4 $G^1$ splines along an edge

To analyze the constraints imposed by gluing data along an edge, we consider first a simple topological surface  $\mathcal{N}$  composed of two faces  $\sigma_1, \sigma_2$  glued along an edge  $\tau$ .

A spline function  $f \in S_k^1(\mathcal{N})$  on  $\mathcal{N}$  is represented by a pair of polynomials  $f = (f_1, f_2)$  with  $f_i \in \mathcal{R}(\sigma_i) = \mathbb{R}[u_i, v_i]$  for  $i = 1, 2$ .

By a change of coordinates, we assume that the edge  $\tau$  is defined by  $v_1 = 0$  and  $u_1 \in [0, 1]$  in  $\sigma_1$  and by  $u_2 = 0$  and  $v_2 \in [0, 1]$  in  $\sigma_2$ .

## 4.1 Splines and syzygies

With the transition map  $\phi_{\sigma_2, \sigma_1}$  defined by the rational functions  $\mathbf{a} = \frac{a_\tau}{c_\tau}$  and  $\mathbf{b} = \frac{b_\tau}{c_\tau}$  as in (15), the differentiability condition (4) along the interior edge  $\tau$  becomes

$$a(u_1)A(u_1) + b(u_1)B(u_1) + c(u_1)C(u_1) = 0,$$

where

$$A(u_1) = \frac{\partial f_2}{\partial v_2}(0, u_1), \quad B(u_1) = \frac{\partial f_2}{\partial u_2}(0, u_1), \quad C(u_1) = -\frac{\partial f_1}{\partial v_1}(u_1, 0).$$

Thus, the  $G^1$ -smoothness condition along an interior edge is equivalent to the condition on  $(A, B, C)$  of being a *syzygy* of the polynomials  $a(u_1), b(u_1), c(u_1)$ .

More precisely, a  $G^1$  spline  $(f_1, f_2)$  on  $\mathcal{N}$  is constructed from a syzygy  $(A, B, C)$  of  $a, b, c$  by defining:

$$f_1(u_1, v_1) = c_0 + \int_0^{u_1} A(t)dt - v_1 C(u_1) + v_1^2 E_1(u_1, v_1), \quad (18)$$

$$f_2(u_2, v_2) = c_0 + \int_0^{u_2} A(t)dt + u_2 B(v_2) + u_2^2 E_2(u_2, v_2), \quad (19)$$

where  $c_0 \in \mathbb{R}$  is any constant, and  $E_1, E_2$  are (any) polynomials in  $\mathbb{R}[u_i, v_i]$  for  $i = 1, 2$ , respectively.

We will use this representation for the splines on  $\mathcal{N}$  to compute the dimension of the space of  $G^1$  splines  $\mathcal{S}_k^1(\mathcal{N})$ , see Proposition 4.6 below. Before, we introduce some notation, both for the proof and the dimension formula.

The module of syzygies of  $a(u_1), b(u_1), c(u_1)$  over the ring  $\mathbb{R}[u_1]$  is denoted by  $Z = \text{Syz}(a, b, c)$ . For  $(A, B, C) \in Z$ , the maximum of the degrees,  $\max(\deg A, \deg B, \deg C)$  is called the *coefficient degree* of the syzygy.

Each of the faces  $\sigma_1$  and  $\sigma_2$  in  $\mathcal{N}$  can be a triangle or a quadrangle. Let us denote by  $F_\square$  the number of quadrangles and by  $F_\Delta$  the number of triangles in  $\mathcal{N}$ .

**Definition 4.1.** As before, let  $\sigma_1, \sigma_2$  be the faces of  $\mathcal{N}$ . We define

$$m = \min(F_\Delta(\sigma_1), F_\Delta(\sigma_2)),$$

where  $F_\Delta(\sigma_i) = 1$  if  $\sigma_i$  is a triangle and 0 otherwise. For the polynomials  $a, b, c \in \mathbb{R}[u_1]$  defining the gluing data along the edge  $\tau$ , let  $n = \max(\deg(a), \deg(b), \deg(c))$ ,

$$d_a = n + 1, \quad d_b = n + F_\Delta(\sigma_2), \quad \text{and} \quad d_c = n + F_\Delta(\sigma_1),$$

and

$$e = \begin{cases} 0, & \text{if } \min(d_a - \deg(a), d_b - \deg(b), d_c - \deg(c)) = 0 \text{ and} \\ 1, & \text{otherwise.} \end{cases}$$

By the formulas (18) and (19) representing a spline  $(f_1, f_2) \in \mathcal{S}_k^1(\mathcal{N})$ , let us notice that we need to consider syzygies  $(A, B, C)$  of  $a, b, c$  such that  $\deg(A) \leq k - 1$ ,  $\deg(B) \leq k - F_\Delta(\sigma_2)$ , and  $\deg(C) \leq k - F_\Delta(\sigma_1)$ .

**Definition 4.2.** For  $k \geq 0$ , we will denote by  $Z_k$  the vector subspace of  $Z$  of syzygies of  $(a, b, c)$  defined as the set

$$Z_k = \{(A, B, C) \in Z : \deg(A) \leq k - 1, \deg(B) \leq k - F_\Delta(\sigma_2), \\ \text{and } \deg(C) \leq k - F_\Delta(\sigma_1)\}.$$

Let us consider the map

$$\begin{aligned} \Theta : Z &\rightarrow \mathcal{S}^1(\mathcal{N}) \\ (A, B, C) &\mapsto \left( \int_0^{u_1} A(t)dt - v_1 C(u_1), \int_0^{v_2} A(t)dt + u_2 B(v_2) \right). \end{aligned} \quad (20)$$

By construction, we have  $\Theta(Z_k) \subset \mathcal{S}_k^1(\mathcal{N})$ .

The dimension of  $Z_k$ , as a vector space over  $\mathbb{R}$ , will be deduced from classical results on graded modules over  $S = \mathbb{R}[u_0, u_1]$ . We will study the module  $\text{Syz}(\bar{a}, \bar{b}, \bar{c})$ , where  $\bar{a}, \bar{b}, \bar{c} \in S$  are the homogenization of  $a, b$  and  $c$  in degree  $d_a, d_b$  and  $d_c$  respectively. The elements in  $\text{Syz}(\bar{a}, \bar{b}, \bar{c})$  in degree  $n + k$  will precisely lead to the syzygies in  $Z_k$ .

**Lemma 4.3.** *For polynomials  $a, b, c \in \mathbb{R}[u_1]$ , with  $b, c \neq 0$ ,  $\gcd(a, b, c) = 1$  and  $Z = \text{Syz}(a, b, c)$  as defined above,*

- (i)  $Z$  is a free  $\mathbb{R}[u_1]$ -module of rank 2.
- (ii) The module  $Z$  is generated by vectors  $(A_1, B_1, C_1), (A_2, B_2, C_2)$  of coefficient degree  $\mu$  and  $\nu = n - \mu + 1 + F_\Delta - e - 2m$  where  $\mu$  is the smallest possible coefficient degree.
- (iii) For  $k \in \mathbb{N}$ , the dimension of  $Z_k$  as vector space over  $\mathbb{R}$  is given by

$$\dim Z_k = (k - \mu - m + 1)_+ + (k - n + \mu + m - F_\Delta + e)_+$$

- (iv) The generators  $(A_1, B_1, C_1), (A_2, B_2, C_2)$  of  $Z$  can be chosen so that

$$(a, b, c) = (B_1 C_2 - B_2 C_1, C_1 A_2 - C_2 A_1, A_1 B_2 - A_2 B_1).$$

*Proof.* We study the syzygy module  $Z = \text{Syz}(a, b, c)$  using results on graded resolutions. For this purpose, we homogenize  $a, b$  and  $c$  in degree  $d_a = n + 1$ ,  $d_b = n + F_\Delta(\sigma_2)$ , and  $d_c = n + F_\Delta(\sigma_1)$ , respectively, where  $F_\Delta(\sigma_i)$  is as in Definition 4.1. Let  $u_0, u_1$  be the homogeneous coordinates, and  $\bar{a}, \bar{b}, \bar{c}$  the corresponding homogenizations of  $a, b$ , and  $c$ . We consider the module of homogeneous syzygies  $\text{Syz}(\bar{a}, \bar{b}, \bar{c})$  over the polynomial ring  $S = \mathbb{R}[u_0, u_1]$ .

**Claim 4.4.** *For any  $k \geq 0$ , the elements in  $Z_k$  are exactly the syzygies of degree  $n + k$  in  $\text{Syz}(\bar{a}, \bar{b}, \bar{c})$  after dehomogenization by setting  $u_0 = 1$ .*

*Proof.* It is clear that if  $\bar{A}\bar{a} + \bar{B}\bar{b} + \bar{C}\bar{c} = 0$ , then by dehomogenization taking  $u_0 = 1$ , we get a syzygy  $(A, B, C)$  of  $(a, b, c)$ . Moreover, if  $\deg(\bar{A}\bar{a}) = \deg(\bar{B}\bar{b}) = \deg(\bar{C}\bar{c}) = n + k$ , then  $\deg(\bar{A}) = k - 1$ ,  $\deg(\bar{B}) = k - F_\Delta(\sigma_2)$  and  $\deg(\bar{C}) = k - F_\Delta(\sigma_1)$ . It follows that  $(A, B, C) \in Z_k$ .

On the other hand, any syzygy  $(A, B, C) \in Z_k$  is given by polynomials that satisfy the conditions in Definition 4.2. Thus  $\max\{\deg A, \deg B, \deg C\} \leq k$ , and since  $n = \max\{\deg a, \deg b, \deg c\}$  then we may consider the homogenization of the polynomial  $Aa + Bb + Cc$  in degree  $n + k$ . These polynomials satisfy

$$\begin{aligned} 0 &= u_0^{k+n}(Aa + Bb + Cc)(u_1/u_0) \\ &= u_0^{k-1} \cdot u_0^{n+1} Aa(u_1/u_0) + u_0^{k-F_\Delta(\sigma_2)} \cdot u_0^{n+F_\Delta(\sigma_2)} Bb(u_1/u_0) \\ &\quad + u_0^{k-F_\Delta(\sigma_1)} \cdot u_0^{n+F_\Delta(\sigma_1)} Cc(u_1/u_0). \end{aligned}$$

It is easy to check that

$$\bar{A} = u_0^{k-1} A(u_1/u_0), \quad \bar{B} = u_0^{k-F_\Delta(\sigma_2)} B(u_1/u_0), \quad \bar{C} = u_0^{k+F_\Delta(\sigma_1)} C(u_1/u_0)$$

are all polynomials in  $\mathbb{R}[u_1, u_0]$ , and define a syzygy of  $\bar{a}, \bar{b}, \bar{c}$  of degree  $n + k$ . Let us also notice that the polynomials

$$\bar{a} = u_0^{n+1} a(u_1/u_0), \quad \bar{b} = u_0^{n+F_\Delta(\sigma_2)} b(u_1/u_0), \quad \text{and} \quad \bar{c} = u_0^{n+F_\Delta(\sigma_1)} c(u_1/u_0)$$

are precisely the homogenization of  $a, b, c$  in degree  $d_a, d_b, d_c$ , respectively.  $\square$

As  $\gcd(a, b, c) = 1$ , we have  $\gcd(\bar{a}, \bar{b}, \bar{c}) = u_0$  if  $e = 1$ , and  $\gcd(\bar{a}, \bar{b}, \bar{c}) = 1$  otherwise.

Let  $I = (\bar{a}, \bar{b}, \bar{c})$  be the ideal generated by  $\bar{a}, \bar{b}, \bar{c}$  in  $S$ . If  $\gcd(\bar{a}, \bar{b}, \bar{c}) = 1$  then there exists  $t_0 \in \mathbb{N}$  such that  $\forall t \geq t_0$ ,  $I_t = (u_0, u_1)^t$  and in that case,  $\dim_{\mathbb{R}}(S/I)_t = 0$  for  $t$  sufficiently large. It follows that the Hilbert polynomial  $HP_{S/I}$  of  $S/I$  is the zero polynomial.

For the second case, namely if  $\gcd(\bar{a}, \bar{b}, \bar{c}) = u_0$ , since  $\gcd(a, b, c) = 1$  then the polynomials  $\bar{a}/u_0$ ,  $\bar{b}/u_0$  and  $\bar{c}/u_0$  have gcd equal to 1. Hence there exists  $t_0 \in \mathbb{N}$  such that  $\forall t \geq t_0$ ,  $I_t = u_0(u_0, u_1)^{t-1}$ . In this case  $\dim_{\mathbb{R}}(S/I)_t = 1$  for  $t$  sufficiently large, and it follows that the Hilbert polynomial  $HP_{S/I}$  is the constant polynomial equal to 1.

Then the exact sequence

$$0 \rightarrow I \rightarrow S \rightarrow S/I \rightarrow 0$$

implies that

$$HP_I(t) = HP_S(t) - HP_{S/I}(t) = \binom{t+1}{1} - e, \quad (21)$$

where  $HP_M$  is the Hilbert polynomial of the module  $M$ .

By the Graded Hilbert Syzygy Theorem, we get a resolution of the form

$$0 \rightarrow S(-d_1) \oplus \cdots \oplus S(-d_L) \xrightarrow{\lambda} S(-d_a) \oplus S(-d_b) \oplus S(-d_c) \rightarrow I \rightarrow 0.$$

Since this is an exact sequence, then the Hilbert polynomial of the middle term is the sum of the other two Hilbert polynomials, and applying (21) we get

$$3t - (d_a + d_b + d_c) + 3 = (t - d_1 + 1) + \cdots + (t - d_L + 1) + (t + 1) - e.$$

It follows that  $L = 2$  which proves (i). Furthermore, we have that the degrees  $d_1$  and  $d_2$  of the syzygies satisfy  $d_1 + d_2 = d_a + d_b + d_c - e$ .

The matrix  $\Lambda$  representing  $\lambda$  is a  $3 \times 2$  matrix

$$\begin{pmatrix} \bar{A}_1 & \bar{A}_2 \\ \bar{B}_1 & \bar{B}_2 \\ \bar{C}_1 & \bar{C}_2 \end{pmatrix}$$

the first column corresponding to the generator of degree  $d_1$  and the second of degree  $d_2$ . These two syzygies correspond to vectors of polynomial coefficients of degree  $\mu = d_1 - \min(d_a, d_b, d_c)$  and  $\nu = d_2 - \min(d_a, d_b, d_c)$ . By Definition 4.1,  $\min(d_a, d_b, d_c) = n + \min(1, F_{\Delta}(\sigma_1), F_{\Delta}(\sigma_2)) = n + m$ , and also  $d_a + d_b + d_c = 3n + F_{\Delta} + 1$ . Let us assume that  $d_1 \leq d_2$ , then  $\mu$  is the smallest degree of the coefficient vector of a syzygy of  $(\bar{a}, \bar{b}, \bar{c})$ , and  $\nu = n - \mu + 1 + F_{\Delta} - e - 2m$ .

By exactness, the two columns of  $\Lambda$  generate  $\text{Syz}(\bar{a}, \bar{b}, \bar{c})$ . The dehomogenization (by setting  $u_0 = 1$ ) of the syzygies in  $\text{Syz}(\bar{a}, \bar{b}, \bar{c})$  leads to syzygies of  $(a, b, c)$  over  $\mathbb{R}[u_1]$ . In particular, it is straightforward to show that the dehomogenization  $(A_i, B_i, C_i)$  of  $(\bar{A}_i, \bar{B}_i, \bar{C}_i)$  for  $i = 1, 2$  generate  $Z = \text{Syz}(a, b, c)$  as a module over  $\mathbb{R}[u_1]$ . This proves (ii).

By Claim 4.4, the space  $Z_k$  is in correspondence with the space of homogeneous syzygies of degree  $n+k$ , which is spanned by the multiples of degree  $n+k$  of  $(\bar{A}_1, \bar{B}_1, \bar{C}_1)$  and  $(\bar{A}_2, \bar{B}_2, \bar{C}_2)$ . Therefore,

$$\begin{aligned} \dim Z_k &= (n+k-d_1+1)_+ + (n+k-d_2+1)_+ \\ &= (k-\mu-m+1)_+ + (k-\nu-m+1)_+ \end{aligned}$$

with  $\nu = n - \mu + 1 + F_{\Delta} - e - 2m$ . This proves (iii).

The point (iv) is a consequence of Hilbert-Burch theorem. More details on this proof can be found in [6, Chapter 6, § 4.17].  $\square$

**Definition 4.5.** For an interior edge  $\tau$  in the topological surface  $\mathcal{M}$  shared by the faces  $\sigma_1, \sigma_2$  with gluing data  $[a_{\tau}, b_{\tau}, c_{\tau}]$ , we denote by  $\mu_{\tau}$  the smallest coefficient degree among the two generators of the module  $Z = \text{Syz}(a_{\tau}, b_{\tau}, c_{\tau})$ . Let  $\nu_{\tau} = n_{\tau} - \mu_{\tau} + 1 + F_{\Delta}(\tau) - e_{\tau} - 2m_{\tau}$  denote the complementary degree, where  $n_{\tau} = \max(\deg(a_{\tau}), \deg(b_{\tau}), \deg(c_{\tau}))$ ,  $m_{\tau} = \min(F_{\Delta}(\sigma_1), F_{\Delta}(\sigma_2))$ ,  $e_{\tau} = \min(n_{\tau} + 1 - \deg(a_{\tau}), n_{\tau} + F_{\Delta}(\sigma_2) - \deg(b_{\tau}), n_{\tau} + F_{\Delta}(\sigma_1) - \deg(c_{\tau}))$ ,  $F_{\Delta}(\tau) = F_{\Delta}(\sigma_1) + F_{\Delta}(\sigma_2)$ . The corresponding basis of the syzygy module  $Z$  of  $[a_{\tau}, b_{\tau}, c_{\tau}]$  is called the  $\mu_{\tau}$ -basis.

This construction allows us to determine the dimension of  $S_k^1(\mathcal{N})$ .

**Proposition 4.6.** For  $F_\square$  (resp.  $F_\Delta$ ) the number of quadrangles (resp. triangles) of  $\mathcal{N}$ ,

$$\begin{aligned} \dim \mathcal{S}_k(\mathcal{N}_\tau) &= 1 + (k^2 - 1)F_\square + \frac{1}{2}(k^2 - k)F_\Delta + d_\tau(k) \\ \text{where } d_\tau(k) &= (k - \mu_\tau - m_\tau + 1)_+ + (k - n_\tau + \mu_\tau + m_\tau - F_\Delta + e_\tau)_+ \end{aligned}$$

*Proof.* The number of linearly independent splines on  $\mathcal{N}$  can be easily counted by using (18) and (19), and the linearly independent terms in the Bernstein-Bézier representation of the polynomials  $f_1$  and  $f_2$  that conform a spline.

The gluing data and the smoothness along the edge  $\tau$  impose conditions on the terms in  $f_1$  and  $f_2$  which are linear in  $v_1$  and  $v_2$ , respectively. Thus, the dimension of the space of splines on  $\mathcal{N}$  of degree exactly 1 in  $v_1$  and  $v_2$ , is given by  $\dim Z_k = d_\tau(k)$ . The formula for  $d_\tau$  follows from Lemma 4.3, by considering  $Z = \text{Syz}(a_\tau, b_\tau, c_\tau)$ , where  $a_\tau, b_\tau, c_\tau$  define the gluing data along  $\tau$ .  $\square$

## 4.2 Separation of vertices

We analyze now the separability of the spline functions on an edge, that is when the Taylor map at the vertices separate the spline functions.

Let  $f = (f_1, f_2) \in \mathcal{R}(\sigma_1) \oplus \mathcal{R}(\sigma_2)$  of the form  $f_i(u_i, v_i) = p_i + q_i u_i + \tilde{q}_i v_i + s_i u_i v_i + r_i u_i^2 + \tilde{r}_i v_i^2 + \dots$ . Then

$$T_\gamma(f) = [p_1, q_1, \tilde{q}_1, s_1, p_2, q_2, \tilde{q}_2, s_2].$$

If  $f = (f_1, f_2) \in \mathcal{S}_k^1(\mathcal{N})$ , then the gluing conditions (3), (4) imply that

$$\begin{aligned} \tilde{q}_1 + s_1 u_1 &= (\mathbf{a}(0) + \mathbf{a}'(0)u_1 + \dots) (\tilde{q}_2 + 2\tilde{r}_2 u_1 + \dots) \\ &\quad + (\mathbf{b}(0) + \mathbf{b}'(0)u_1 + \dots) (q_2 + s_2 u_1 + \dots) \end{aligned} \quad (22)$$

that is

$$\begin{aligned} p_1 &= p_2 \\ q_1 &= \tilde{q}_2 \\ r_1 &= \tilde{r}_2 \\ \tilde{q}_1 &= \mathbf{a}(0) \tilde{q}_2 + \mathbf{b}(0) q_2 \\ s_1 &= 2 \mathbf{a}(0) r_2 + \mathbf{b}(0) s_2 + \mathbf{a}'(0) \tilde{q}_2 + \mathbf{b}'(0) q_2. \end{aligned}$$

Let  $\mathcal{H}(\gamma)$  be the linear space spanned by the vectors  $[p_1, q_1, \tilde{q}_1, s_1, p_2, q_2, \tilde{q}_2, s_2]$ , which are solution of these equations.

If  $\mathbf{a}(0) \neq 0$ , it is a space of dimension 5 otherwise its dimension is 4. Thus  $\dim \mathcal{H}(\gamma) = 5 - \mathbf{c}_\tau(\gamma)$ .

**Proposition 4.7.** For  $k \geq \nu_\tau + m_\tau + 1$ ,  $T_\gamma(\mathcal{S}_k^1(\mathcal{N})) = \mathcal{H}(\gamma)$ . Its dimension is  $\dim T_\gamma(\mathcal{S}_k^1(\mathcal{N})) = 5 - \mathbf{c}_\tau(\gamma)$ .

*Proof.* Let  $G(\gamma) = T_\gamma(\mathcal{S}_k^1(\mathcal{N}))$ . By construction  $G(\gamma) \subset \mathcal{H}(\gamma)$ . We are going to prove that for  $k \geq \nu_\tau + m_\tau + 1$ ,  $G(\gamma)$  and  $\mathcal{H}(\gamma)$  have the same dimension and thus are equal.

By the decompositions (18) and (19), the elements of  $T_\gamma(\mathcal{S}_k^1(\mathcal{N}))$  are of the form

$$[c_0, A(0), -C(0), -C'(0), c_0, B(0), A(0), B'(0)]$$

where  $c_0 \in \mathbb{R}$  and  $[A, B, C] \in Z_k$ . By Lemma 4.3, an element of  $Z_k$  is of the form  $[A, B, C] = P[A_1, B_1, C_1] + Q[A_2, B_2, C_2]$  with  $P, Q \in \mathbb{R}[u]$ ,  $\deg(P) \leq k - \mu_\tau - m_\tau$  and  $\deg(Q) \leq k - \nu_\tau - m_\tau$ . By removing the repeated columns, reordering and changing some signs, we see that  $G(\gamma) = T_\gamma(\mathcal{S}_k^1(\mathcal{N}))$  is isomorphic to the space spanned by the elements

$$\begin{bmatrix} f_1(\gamma) \\ \partial_{u_1} f_1(\gamma) \\ \partial_{u_2} f_2(\gamma) \\ -\partial_{v_1} f_1(\gamma) \\ \partial_{u_2} \partial_{v_2} f_2(\gamma) \\ -\partial_{u_1} \partial_{v_1} f_1(\gamma) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & A_1(0) & A_2(0) & 0 & 0 \\ 0 & B_1(0) & B_2(0) & 0 & 0 \\ 0 & C_1(0) & C_2(0) & 0 & 0 \\ 0 & B'_1(0) & B'_2(0) & B_1(0) & B_2(0) \\ 0 & C'_1(0) & C'_2(0) & C_1(0) & C_2(0) \end{bmatrix} \begin{bmatrix} c_0 \\ P(0) \\ Q(0) \\ P'(0) \\ Q'(0) \end{bmatrix} \quad (23)$$

for  $P, Q \in \mathbb{R}[u]$  with  $\deg(P) \leq k - \mu_\tau - m_\tau$  and  $\deg(Q) \leq k - \nu_\tau - m_\tau$ . Let us assume that  $k \geq \nu_\tau + m_\tau + 1$  so that  $k - \mu_\tau - m_\tau \geq k - \nu_\tau - m_\tau \geq 1$ .

As  $A_1 B_2 - A_2 B_1 = c$  and  $A_1(0)B_2(0) - A_2(0)B_1(0) = c(0) \neq 0$ , we deduce that  $[B_1(0), B_2(0)] \neq [0, 0]$  and that  $\dim G(\gamma) \geq 4$ .

If  $\mathfrak{c}_\tau(\gamma) = 0$ , then  $a(0) = B_1(0)C_2(0) - B_2(0)C_1(0) \neq 0$  and  $\dim G(\gamma) = 5 = 5 - \mathfrak{c}_\tau(\gamma) = \dim \mathcal{H}(\gamma)$ .

If  $\mathfrak{c}_\tau(\gamma) = 1$ , then  $a(0) = B_1(0)C_2(0) - B_2(0)C_1(0) = 0$  and  $\dim G(\gamma) = 4 = 5 - \mathfrak{c}_\tau(\gamma) = \dim \mathcal{H}(\gamma)$ .

In both cases, we have  $\dim G(\gamma) = \dim \mathcal{H}(\gamma)$ , which implies that  $G(\gamma) = \mathcal{H}(\gamma)$ . This completes the proof of the proposition.  $\square$

If  $\gamma'$  is the other end point of  $\tau$ , we have a Taylor map for each  $\gamma$  and  $\gamma'$ , that we join together. Let

$$\begin{aligned} T_{\gamma, \gamma'} : \mathcal{R}(\sigma_1) \oplus \mathcal{R}(\sigma_2) &\rightarrow \mathcal{R}^{\sigma_1}(\tau) \oplus \mathcal{R}^{\sigma_2}(\tau) \oplus \mathcal{R}^{\sigma_1}(\tau) \oplus \mathcal{R}^{\sigma_2}(\tau) \\ f = (f_1, f_2) &\mapsto (T_\gamma(f), T_{\gamma'}(f)) \end{aligned} \quad (24)$$

and let  $G(\tau) = T_{\gamma, \gamma'}(\mathcal{S}_k^1(\mathcal{N}))$ .

**Proposition 4.8.** *For  $k \geq \nu_\tau + m_\tau + 4$ , we have  $T_{\gamma, \gamma'}(\mathcal{S}_k^1(\mathcal{N})) = (\mathcal{H}(\gamma), \mathcal{H}(\gamma'))$  and*

$$\dim T_{\gamma, \gamma'}(\mathcal{S}_k^1(\mathcal{N})) = 10 - \mathfrak{c}_\tau(\gamma) - \mathfrak{c}_\tau(\gamma').$$

*Proof.* By a change of coordinates, we can assume that the coordinates of  $\gamma$  (resp.  $\gamma'$ ) are  $(0, 0)$  (resp.  $(1, 0)$ ) in  $\sigma_1$  and  $(0, 0)$  (resp.  $(0, 1)$ ) in  $\sigma_2$ .

Similarly to the proof of the previous proposition,  $T_{\gamma'}(\mathcal{S}_k^1(\mathcal{N}))$  is spanned by the vectors

$$\left[ c_0 + \int_0^1 A(u)du, A(1), -C(1), C'(1), c_0 + \int_0^1 A(u)du, B(1), A(1), B'(1) \right]$$

for  $c_0 \in \mathbb{R}$  and  $[A, B, C] = P[A_1, B_1, C_1] + Q[A_2, B_2, C_2] \in Z_k$  with  $\deg(P) \leq k - \mu_\tau - m_\tau$  and  $\deg(Q) \leq k - \nu_\tau - m_\tau$ .

For  $k \geq \nu_\tau + m_\tau + 4$ , we can find polynomials  $P = p_0(1 - 3u^2 + 2u^3) + p_1(u - 2u^2 - u^3) + p_2u^2(1 - u)^2$ ,  $Q = q_0(1 - 3u^2 + 2u^3) + q_1(u - 2u^2 - u^3) + q_2u^2(1 - u)^2$ , of degree  $\leq 4$  such that  $P(0) = p_0, P'(0) = p_1, Q(0) = q_0, Q'(0) = q_1, P(1) = 0, P'(1) = 0, Q(1) = 0, Q'(1) = 0$  and  $\int_0^1 (PA_1 + QA_2)(u)du = -c_0$ .

This implies that  $(\mathcal{H}(\gamma), 0) \subset G(\tau)$ . By symmetry, we also have  $(0, \mathcal{H}(\gamma')) \subset G(\tau)$ . By construction  $G(\tau) \subset (\mathcal{H}(\gamma), \mathcal{H}(\gamma'))$ , therefore we have  $G(\tau) = (\mathcal{H}(\gamma), \mathcal{H}(\gamma'))$  and  $\dim G(\tau) = \dim \mathcal{H}(\gamma) + \dim \mathcal{H}(\gamma')$ . We deduce the dimension formula from Proposition 4.7.  $\square$

**Definition 4.9.** The separability  $\mathfrak{s}(\tau)$  of the edge  $\tau$  is the minimal  $k$  such that  $T_{\gamma, \gamma'}(\mathcal{S}_k^1(\mathcal{N})) = (T_\gamma(\mathcal{S}_k^1(\mathcal{N})), T_{\gamma'}(\mathcal{S}_k^1(\mathcal{N})))$ .

**Remark 4.10.** The bound  $\nu_\tau + m_\tau + 4 \geq \mathfrak{s}(\tau)$  is not necessarily the minimal degree of separability. Separability can be attained as soon as  $d_\tau(k) \geq 9 - \mathfrak{c}_\tau(\gamma) - \mathfrak{c}_\tau(\gamma')$ .

### 4.3 Decompositions and dimension

Let  $K_1 = (v_1^2) \cap \mathcal{R}_k(\sigma_1)$  and  $K_2 = (u_2^2) \cap \mathcal{R}_k(\sigma_2)$  be the polynomials of  $\mathcal{R}_k(\sigma_1)$  (resp.  $\mathcal{R}_k(\sigma_2)$ ) divisible by  $v_1^2$  (resp.  $u_2^2$ ). The elements of  $(K_1, K_2)$  are obviously in  $\mathcal{S}_k^1(\mathcal{N})$  since they vanish at the order 1 along  $\tau$ . Let  $W_k(\tau) = \Theta_\tau(Z_k)$  where  $\Theta_\tau$  is defined in (20). Then, from the relations (18) and (19), we deduce the following decomposition:

$$\mathcal{S}_k^1(\mathcal{N}) = (K_1, 0) \oplus (0, K_2) \oplus \mathbb{R}(1, 1) \oplus W_k(\tau). \quad (25)$$

The sum of these spaces is direct, since the supports of the functions of each space do not intersect.



The map  $T_{\gamma, \gamma'}$  defined in (24) induces the exact sequence

$$0 \rightarrow \mathcal{K}_k(\tau) \rightarrow \mathcal{S}_k^1(\mathcal{N}) \xrightarrow{T_{\gamma, \gamma'}} G(\tau) \rightarrow 0$$

where  $\mathcal{K}_k(\tau) = \ker T_{\gamma, \gamma'}$  and  $G(\tau) = T_{\gamma, \gamma'}(\mathcal{S}_k^1(\mathcal{N}))$ . It is clear that  $(K_1, K_2) \subset \mathcal{K}_k(\tau)$ .

**Definition 4.11.** Let  $\mathcal{E}_k(\tau) = \ker(T_{\gamma, \gamma'}) \cap W_k(\tau)$ . The elements in  $\mathcal{E}_k(\tau)$  are the splines of  $\mathcal{S}_k^1(\mathcal{N})$  with support in  $\mathcal{R}^{\sigma_1}(\tau) \oplus \mathcal{R}^{\sigma_2}(\tau)$ , which Taylor expansions at  $\gamma$  and  $\gamma'$  vanish. For a boundary edge  $\tau'$ , which belongs to a face  $\sigma$ , we also define  $\mathcal{E}_k(\tau)$  as the elements of  $\ker(T_{\gamma}^{\sigma}) \cap \ker(T_{\gamma'}^{\sigma})$  with support in  $\mathcal{R}^{\sigma}(\tau)$ .

**Lemma 4.12.** *We have  $\mathcal{K}_k(\tau) = (K_1, 0) \oplus (0, K_2) \oplus \mathcal{E}_k(\tau)$ .*

*Proof.* As  $(K_1, 0), (0, K_2) \subset \ker T_{\gamma, \gamma'} = \mathcal{K}_k(\tau)$  and  $\mathcal{K}_k(\tau) \cap (W_k(\tau) \oplus \mathbb{R}(1, 1)) = \mathcal{K}_k(\tau) \cap W_k(\tau) = \mathcal{E}_k(\tau)$ , we have

$$\begin{aligned} \mathcal{K}_k(\tau) &= (K_1, 0) \oplus (0, K_2) \oplus ((W_k(\tau) \oplus \mathbb{R}(1, 1)) \cap \mathcal{K}_k(\tau)) \\ &= (K_1, 0) \oplus (0, K_2) \oplus \mathcal{E}_k(\tau). \end{aligned}$$

□

**Corollary 4.13.** *For  $k \geq \mathfrak{s}(\tau)$ , the dimension of  $\mathcal{E}_k(\tau)$  is*

$$\dim \mathcal{E}_k(\tau) = d_{\tau}(k) - 9 + \mathfrak{c}_{\tau}(\gamma) + \mathfrak{c}_{\tau}(\gamma').$$

*Proof.* By Lemma 4.12, we have

$$\dim \mathcal{E}_k(\tau) = \dim \mathcal{K}_k(\tau) - \dim K_1 - \dim K_2.$$

As  $\mathcal{K}_k(\tau)$  is the kernel of  $T_{\gamma, \gamma'}$  and  $G(\tau)$  is its image, we have

$$\dim \mathcal{K}_k(\tau) = \dim \mathcal{S}_k^1(\mathcal{N}) - \dim G(\tau).$$

From the decomposition (25), we deduce that  $\dim \mathcal{S}_k^1(\mathcal{N}) = 1 + d_{\tau}(k) + \dim K_1 + \dim K_2$ . Using Proposition 4.8,  $G(\tau) = (\mathcal{H}(\gamma), \mathcal{H}(\gamma'))$  and we obtain

$$\begin{aligned} \dim \mathcal{E}_k(\tau) &= \dim \mathcal{S}_k^1(\mathcal{N}) - \dim G(\tau) - \dim K_1 - \dim K_2 \\ &= d_{\tau}(k) - 9 + \mathfrak{c}_{\tau}(\gamma) + \mathfrak{c}_{\tau}(\gamma'). \end{aligned}$$

□

## 5 $G^1$ splines around a vertex

We consider now a topological surface  $\mathcal{O}$  composed of faces  $\sigma_1, \dots, \sigma_F \in \mathcal{O}_2$  sharing a single vertex  $\gamma$ , and such that  $\sigma_i$  and  $\sigma_{i+1}$  share the edge  $\tau_{i+1} = (\gamma, \delta_{i+1})$ . In particular  $\tau_i, \tau_{i+1}$  are the two edges of  $\sigma_i$  containing the vertex  $\gamma$ . The number of edges containing  $\gamma$  is denoted  $F'$ . All the vertices of  $\mathcal{O}$  different from  $\gamma$  are boundary vertices. The vertex  $\gamma$  is an interior vertex, iff  $\sigma_F$  and  $\sigma_1$  share the edge  $\tau_1$ . In this case, we identify the indices modulo  $F$  and we have  $F' = F$ , otherwise we have  $F' = F + 1$ . The gluing data for the interior edge  $\tau_i$  is  $\mathfrak{a}_i = \frac{a_i}{c_i}, \mathfrak{b}_i = \frac{b_i}{c_i}$ .

The coordinates in the ring  $\mathcal{R}(\sigma_i)$  are chosen so that the coordinates of  $\gamma$  are  $(0, 0)$  and  $\tau_i$  is defined by  $v_i = 0, u_i \in [0, 1]$  and by  $u_{i-1} = 0, v_{i-1} \in [0, 1]$  in  $\mathcal{R}(\sigma_{i-1})$ . The canonical form of the transition map at  $\gamma$  across the edge  $\tau_i$  is then

$$\phi_{\tau_i} : (u, v) \longrightarrow \begin{pmatrix} v_i \mathfrak{b}_i(u_i) \\ u_i + v_i \mathfrak{a}_i(u_i) \end{pmatrix}$$

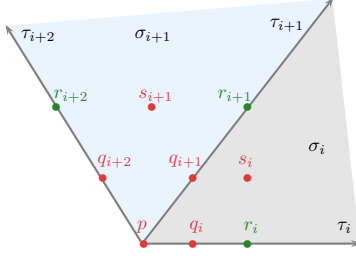


Figure 3: Taylor coefficients around a vertex.

Let  $f = (f_i)_{i=1, \dots, F} \in \mathcal{S}^1(\mathcal{O})$ . The gluing condition (4) implies that the Taylor expansion of  $f_i$  at  $\gamma$  is of the form

$$f_i(u_i, v_i) = p + q_i u_i + q_{i+1} v_i + s_i u_i v_i + r_i u_i^2 + r_{i+1} v_i^2 + \dots$$

for  $p, q_i, s_i, r_i \in \mathbb{R}$ ,  $i = 1, \dots, F$  (see Fig. 3).

By a computation similar to (22), Condition (4) implies that

$$q_{i+1} = \mathbf{a}_i(0) q_i + \mathbf{b}_i(0) q_{i-1} \quad (i = 2, \dots, F) \quad (26)$$

$$s_i = 2 \mathbf{a}_i(0) r_i + \mathbf{b}_i(0) s_{i-1} + \mathbf{a}'_i(0) q_i + \mathbf{b}'_i(0) q_{i-1} \quad (i = 2, \dots, F) \quad (27)$$

Let  $\mathcal{H}(\gamma)$  be the vector space spanned by the vectors  $\mathbf{h} = [p, q_1, \dots, q_{F'}, s_1, \dots, s_F]$  for  $\mathbf{h}' = [p, q_1, \dots, q_{F'}, s_1, \dots, s_F, r_1, \dots, r_{F'}]$  a solution of the linear system (26), (27).

**Proposition 5.1.**

$$\dim \mathcal{H}(\gamma) = 3 + F(\gamma) - \sum_{\tau \ni \gamma} \mathbf{c}_\tau(\gamma) + \mathbf{c}_+(\gamma)$$

where  $F = F(\gamma)$  is the number of faces around the vertex  $\gamma$ .

*Proof.* Notice that  $\mathcal{H}(\gamma)$  is isomorphic to the projection of the solution set of system (26), (27) on the space of the variables  $[p, \mathbf{q}, \mathbf{s}] = [p, q_1, \dots, q_{F'}, s_1, \dots, s_F]$ .

The solutions in  $\mathbf{q} = (q_1, \dots, q_{F'})$  of the first set of equations satisfy the induction relations

$$\begin{pmatrix} q_i \\ q_{i+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \mathbf{b}_i(0) & \mathbf{a}_i(0) \end{pmatrix} \begin{pmatrix} q_{i-1} \\ q_i \end{pmatrix} \text{ for } i = 2, \dots, F.$$

As we have the compatibility condition 2.6 at an interior vertex, the solutions of (26) span a linear space of dimension 2, parametrized for instance by  $q_1, q_2$ .

The system (27) is formed by linearly independent equations which involve  $r_k$  and  $q_i, s_j$  when  $\mathbf{a}_k(0) \neq 0$  and by equations which only involve  $s_i, s_{i-1}$  and  $q_j$  when  $\mathbf{a}_i(0) = 0$ .

Therefore the projection of the solution set of (27) on the space corresponding to the variables  $[p, \mathbf{q}, \mathbf{s}]$  is defined by the equations which only involve  $s_i, s_{i-1}$  and  $q_i, q_{i-1}$  when  $\mathbf{a}_i(0) = 0$ .

If one of the edges around  $\gamma$  is not a crossing edge, then the codimension of this space is  $\sum_{\tau \ni \gamma} \mathbf{c}_\tau(\gamma)$ .

If all the edges around  $\gamma$  are crossing edges, then the compatibility conditions 2.7 at  $\gamma$  imply that one of these equations is dependent from the other. Therefore, the codimension of this space is  $\sum_{\tau \ni \gamma} \mathbf{c}_\tau(\gamma) - \mathbf{c}_+(\gamma)$ .

By intersecting it with the solution space of (26), we deduce that the dimension of  $\mathcal{H}(\gamma)$  is

$$1 + 2 + F - \sum_{\tau \ni \gamma} \mathbf{c}_\tau(\gamma) + \mathbf{c}_+(\gamma).$$

□

Let  $T_\gamma = \prod_{\sigma \ni \gamma} T_\gamma^\sigma$  be the Taylor map at  $\gamma$  on  $\mathcal{O}$  and let  $T_{\partial\mathcal{O}} = \prod_{\tau \not\ni \gamma} T_\tau^\sigma$  be the Taylor map along all the boundary edges which do not contain  $\gamma$ .

For  $k \in \mathbb{N}$ , we define  $\mathcal{V}_k(\gamma) = \ker T_{\partial\mathcal{O}} \cap \mathcal{S}_k^1(\mathcal{O})$  the set of  $G^1$  spline functions on  $\mathcal{O}$  which vanish at the first order along the boundary edges (which do not contain  $\gamma$ ).

**Proposition 5.2.** For  $k \geq \max_{i=1, \dots, F} (\mathfrak{s}(\tau_i))$ ,  $T_\gamma(\mathcal{V}_k(\gamma)) = \mathcal{H}(\gamma)$ .

*Proof.* By construction, the elements of  $\mathcal{V}_k(\gamma)$  satisfy the equations (26), (27). This implies that  $T_\gamma(\mathcal{V}_k(\gamma)) \subset \mathcal{H}(\gamma)$ .

Consider an element  $\mathbf{h} \in \mathcal{H}(\gamma)$ . By Proposition 4.8, for  $k \geq \mathfrak{s}(\tau_i)$ , there exists  $(f_i, \tilde{f}_i) \in \mathcal{S}_k^1(\mathcal{N}_{\tau_i})$  supported on  $\mathcal{R}^{\sigma_i}(\tau_i) \oplus \mathcal{R}^{\sigma_{i-1}}(\tau_i)$  such that  $T_\gamma(f_i, \tilde{f}_i) = (h_i, h_{i-1})$  and  $T_{\delta_i}(f_i, \tilde{f}_i) = 0$ .

Let  $g_i \in \mathcal{R}_k(\sigma_i)$  be the element supported on  $\mathcal{R}^{\sigma_i}(\tau_i) + \mathcal{R}^{\sigma_i}(\tau_{i+1})$  such that  $T_{\tau_i}^{\sigma_i}(g) = f_i$ ,  $T_{\tau_{i+1}}^{\sigma_i}(g) = \tilde{f}_{i+1}$ . Let  $\mathbf{g} = [g_1, \dots, g_F]$ .

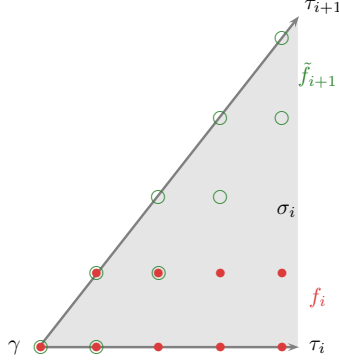


Figure 4: Lifting the edge functions.

By construction,  $\mathbf{g}$  vanishes at the first order along all the boundary edges of  $\mathcal{O}$ , which do not contain  $\gamma$ . Moreover,  $T_{\tau_i}(\mathbf{g}) = (f_i, \tilde{f}_i) \in \mathcal{S}_k^1(\mathcal{N}_{\tau_i})$ , thus  $\mathbf{g}$  satisfies the gluing conditions along the edge  $\tau_i$ . Then  $\mathbf{g}$  satisfies the gluing conditions along all edges. We deduce that  $\mathbf{g} \in \mathcal{V}_k(\gamma)$ . By construction,  $T_\gamma(\mathbf{g}) = \mathbf{h}$ . This shows that  $\mathcal{H}(\gamma) \subset T_\gamma(\mathcal{V}_k(\gamma))$  and concludes the proof.  $\square$

**Remark 5.3.** When  $\gamma$  is boundary vertex which thus belongs to two boundary edges  $\tau_1, \tau_{F'}$ , we observe that attaching virtually a new quadrangular face at these boundary edges, with constant gluing data does not change the dimension count at the vertex and around the boundary edge. Therefore, in the dimension analysis we will assume that the boundary edges  $\tau$  are coming with gluing data such that the  $\mu$ -basis is of degree  $\mu_\tau = \nu_\tau = 0$  and  $d_\tau(k) = 2(k+1)$  and  $\mathfrak{s}(\tau) = 3$ .

## 6 $G^1$ splines on a general mesh

We consider now a general mesh  $\mathcal{M}$  with an arbitrary number of faces, possibly with boundary edges.

We denote by  $T_0 = \prod_{\gamma \in \mathcal{M}_0} T_\gamma$  the Taylor map at all the vertices of  $\mathcal{M}$  and  $\mathcal{H} = T_0(\mathcal{S}_k^1(\mathcal{M}))$ . We have the following exact sequence:

$$0 \rightarrow \mathcal{K}_k \rightarrow \mathcal{S}_k^1(\mathcal{M}) \xrightarrow{T_0} G \rightarrow 0$$

where  $\mathcal{K}_k = \ker T_0 \cap \mathcal{S}_k^1(\mathcal{M})$  and  $G = T_0(\mathcal{S}_k^1(\mathcal{M}))$ . Let  $\mathfrak{s}^* = \max\{\mathfrak{s}(\tau) \mid \tau \in \mathcal{M}_1\}$ . We have  $\mathfrak{s}^* \leq \max\{\nu_\tau + m_\tau + 4 \mid \tau \in \mathcal{M}_1\}$

### 6.1 Splines at a vertex

Let  $\gamma \in \mathcal{M}_0$  be a vertex of  $\mathcal{M}$  and let  $\mathcal{O}_\gamma$  be the sub-mesh associated to the faces of  $\mathcal{M}$  which contain  $\gamma$ . Let  $\mathcal{V}_k(\gamma)$  be the set of spline functions in  $\mathcal{S}_k^1(\mathcal{M})$  supported on the faces of  $\mathcal{O}_\gamma$ , which vanish at the first order along the edges that do not contain  $\gamma$ .

**Proposition 6.1.** For  $k \geq \nu^* + 4$ ,  $T_0(\mathcal{S}_k^1(\mathcal{M})) = \prod_{\gamma} \mathcal{H}(\gamma)$  and

$$\dim T_0(\mathcal{S}_k^1(\mathcal{M})) = \sum_{\gamma \in \mathcal{M}_0} (F(\gamma) + 3) - \sum_{\gamma \in \mathcal{M}_0} \sum_{\tau \ni \gamma} \mathfrak{c}_{\tau}(\gamma) + \sum_{\gamma \in \mathcal{M}_0} \mathfrak{c}_{+}(\gamma),$$

where  $F(\gamma)$  is the number of faces of  $\mathcal{M}$  that contain the vertex  $\gamma \in \mathcal{M}_0$ .

*Proof.* By proposition 5.2, for  $k \geq \mathfrak{s}^*$  the image of  $\mathcal{V}_k(\gamma)$  by  $T_0$  is  $G(\gamma)$ , and  $T_{\gamma'}(\mathcal{V}_k(\gamma)) = 0$  for any other vertex  $\gamma' \neq \gamma$ .

This shows that  $T_0(\mathcal{S}_k^1(\mathcal{M})) = \prod_{\gamma} G(\gamma) = \prod_{\gamma} \mathcal{H}(\gamma)$ . We deduce the dimension formula from Proposition 5.1.  $\square$

## 6.2 Splines on edges

For an interior edge  $\tau = (\gamma, \gamma') \in \mathcal{M}_1$ , let  $\mathcal{N}_{\tau}$  be the sub-mesh made of the faces  $\sigma_1, \sigma_2$  of  $\mathcal{M}$  containing  $\tau$ . Let  $\mathcal{E}_k(\tau) = \ker T_{\gamma, \gamma'} \cap \Theta_{\tau}(Z_k)$  (see Definition 4.11). The elements of  $\mathcal{E}_k(\tau)$  correspond to splines of  $\mathcal{S}_k^1(\mathcal{N}_{\tau})$ , which are in the kernel of  $T_{\gamma, \gamma'}$  and with a support in  $\mathcal{R}^{\sigma_1}(\tau) \oplus \mathcal{R}^{\sigma_2}(\tau)$ . Thus they are also elements of  $\mathcal{S}_k^1(\mathcal{M}) \cap \ker T_0 = \mathcal{K}_k$ .

As the support of the elements of  $\mathcal{E}_k(\tau)$  and  $\mathcal{E}_k(\tau')$  for two distinct edges  $\tau, \tau'$  do not intersect, their sum is direct. Let  $\mathcal{E} = \bigoplus_{\tau \in \mathcal{M}_1} \mathcal{E}_k(\tau)$ .

Let  $\mathcal{F}_k = \ker T_1 \cap \mathcal{S}_k^1(\mathcal{M})$  be the set of spline functions, which Taylor expansions along all edges vanish.

**Proposition 6.2.**

$$\mathcal{K}_k = \mathcal{F}_k \oplus \mathcal{E}_k$$

*Proof.* Let  $f \in \mathcal{K}_k$  and take an interior edge  $\tau \in \mathcal{M}_1$ . Let  $\sigma_1, \sigma_2$  be the two faces of  $\mathcal{N}_{\tau}$ .

Then  $(f_{\sigma_1}, f_{\sigma_2}) \in \mathcal{S}_k^1(\mathcal{N}_{\tau}) \cap \ker T_0 = \mathcal{K}_k(\tau)$ . By Lemma 4.12,  $(f_{\sigma_1}, f_{\sigma_2}) = s_{\tau} + (k_1, k_2)$  with  $s_{\tau} \in \mathcal{E}_k(\tau)$  and  $(k_1, k_2) \in (K_1, K_2)$ . As  $s_{\tau}$  lifts to a spline  $\in \mathcal{S}_k^1(\mathcal{M})$ ,  $f - s_{\tau}$  is an element of  $\mathcal{S}_k^1(\mathcal{M})$ , which image by the Taylor expansion  $T_{\tau}$  along the edge  $\tau$  vanishes.

If  $\tau$  is a boundary edge of  $\mathcal{M}$ , which belongs to the face  $\sigma_1$ , we have a similar decomposition  $f_1 = s_{\tau} + k_1$  with  $s_{\tau} \in \mathcal{E}_k(\tau)$  and  $k_1 \in K_1$ , using the convention of Remark 5.3. Similarly  $s_{\tau}$  lifts to a spline  $\in \mathcal{S}_k^1(\mathcal{M})$ ,  $f - s_{\tau}$  is an element of  $\mathcal{S}_k^1(\mathcal{M})$  in the kernel of  $T_{\tau}$ .

Repeating this process for all edges  $\tau \in \mathcal{M}_1$ , we can construct an element  $\tilde{f} = f - \sum_{\tau \in \mathcal{M}_1} s_{\tau}$  such that  $\forall \tau \in \mathcal{M}_1, T_{\tau}(\tilde{f}) = 0$ , i.e.  $\tilde{f}$  belongs to  $\ker T_1 = \mathcal{F}_k$ . This shows that  $\mathcal{K}_k \subset \mathcal{F}_k + \sum_{\tau \in \mathcal{M}_1} \mathcal{E}_k(\tau)$ . By construction, we have  $\mathcal{F}_k \subset \mathcal{K}_k$  and  $\mathcal{E}_k = \bigoplus_{\tau \in \mathcal{M}_1} \mathcal{E}_k(\tau) \subset \mathcal{K}_k$ . Considering the support of the functions in  $\mathcal{F}_k$  and  $\mathcal{E}_k$ , we deduce that their sum is direct and equal to  $\mathcal{K}_k$ .  $\square$

## 6.3 The dimension formula

We can now determine the dimension of  $\mathcal{S}_k^1(\mathcal{M})$ .

**Theorem 6.3.** Let  $\mathfrak{s}^* = \max\{\mathfrak{s}(\tau) \mid \tau \in \mathcal{M}_1\}$ . Then, for  $k \geq \mathfrak{s}^*$ ,

$$\begin{aligned} \dim \mathcal{S}_k^1(\mathcal{M}) &= (k-3)^2 F_{\square} + \frac{1}{2}(k-5)(k-4)F_{\Delta} \\ &\quad + \sum_{\tau \in \mathcal{M}_1} d_{\tau}(k) + 4F_{\square} + 3F_{\Delta} - 9F_1 + 3F_0 + F_+ \end{aligned}$$

where

- $d_{\tau}(k)$  is the dimension of the syzygies of the gluing data along  $\tau$  in degree  $\leq k$ ,
- $F_{\square}$  is the number of quadrangular faces,  $F_{\Delta}$  is the number of triangular faces,
- $F_1$  is the number of edges,
- $F_0$  (resp.  $F_+$ ) is the number of (resp. crossing) vertices,

*Proof.* By definition, we have

$$\dim \mathcal{S}_k^1(\mathcal{M}) = \dim \mathcal{H} + \dim \mathcal{K}_k.$$

By Proposition 6.2, we have

$$\begin{aligned} \dim \mathcal{K}_k &= \dim \mathcal{F}_k + \dim \mathcal{E}_k = \dim \mathcal{F}_k + \sum_{\tau \in \mathcal{M}_1} \dim \mathcal{E}_k(\tau) \\ &= (k-3)^2 F_{\square} + \frac{1}{2}(k-5)(k-4)F_{\Delta} + \sum_{\tau \in \mathcal{M}_1} (d_{\tau}(k) - 9 + \mathbf{c}_{\tau}(\gamma) + \mathbf{c}_{\tau'}(\gamma')) \end{aligned}$$

From Proposition 6.1, we deduce that

$$\begin{aligned} \dim \mathcal{S}_k^1(\mathcal{M}) &= \dim \mathcal{K}_k + \dim \mathcal{H} \\ &= (k-3)^2 F_{\square} + \frac{1}{2}(k-5)(k-4)F_{\Delta} \\ &\quad + \sum_{\tau=(\gamma, \gamma') \in \mathcal{M}_1} (d_{\tau}(k) - 9 + \mathbf{c}_{\tau}(\gamma) + \mathbf{c}_{\tau'}(\gamma')) \\ &\quad + \sum_{\gamma \in \mathcal{M}_0} (F(\gamma) + 3) - \sum_{\gamma \in \mathcal{M}_0} \sum_{\tau \ni \gamma} \mathbf{c}_{\tau}(\gamma) + \sum_{\gamma \in \mathcal{M}_0} \mathbf{c}_{+}(\gamma) \\ &= F_{\square}(k-3)^2 + F_{\Delta} \frac{1}{2}(k-5)(k-4) + \sum_{\tau \in \mathcal{M}_1} d_{\tau}(k) - 9F_1 \\ &\quad + 4F_{\square} + 3F_{\Delta} + 3F_0 + F_+ \end{aligned}$$

since  $\sum_{\tau=(\gamma, \gamma') \in \mathcal{M}_1} (\mathbf{c}_{\tau}(\gamma) + \mathbf{c}_{\tau'}(\gamma')) = \sum_{\gamma \in \mathcal{M}_0} \sum_{\tau \ni \gamma} \mathbf{c}_{\tau}(\gamma)$  and  $\sum_{\gamma \in \mathcal{M}_0} F(\gamma) = 4F_{\square} + 3F_{\Delta}$ .  $\square$

As a direct corollary, we obtain the following result:

**Corollary 6.4.** *If  $\mathcal{M}$  is a topological surface with gluing data satisfying the compatibility conditions 2.6-2.7 and if all crossing vertices of  $\mathcal{M}$  have 4 edges, then  $\mathcal{S}_k^1(\mathcal{M})$  is an ample space of differentiable functions on  $\mathcal{M}$  for  $k \geq \mathfrak{s}^*$ .*

## 6.4 Basis

We are going now to describe an explicit construction of spline functions which form a basis of  $\mathcal{S}_k^1(\mathcal{M})$ . An algorithmic description of the computation of the Bernstein coefficients of these basis functions is provided in Appendix A.

We assume that  $k$  is bigger than the separability  $\mathfrak{s}^*$  of all edges.

### 6.4.1 Basis functions associated to a vertex

Let  $\gamma \in \mathcal{M}_0$  be a vertex and  $\sigma_1, \dots, \sigma_F$  be the faces of  $\mathcal{M}_2$  adjacent to  $\gamma$ . We also assume that  $\sigma_i$  and  $\sigma_{i-1}$  share the edge  $\tau_i \in \mathcal{M}_1$  and that  $\tau_1$  is not a crossing edge at  $\gamma$  if such an edge exists.

To compute the basis functions attached to  $\gamma$ , we compute first the Taylor coefficients of  $f_{\sigma_i} = p + q_i u_i + q_{i+1} v_i + s_i u_i v_i + \dots$  solutions of the system (26)-(27) and then lift these Taylor coefficients to define a spline function with support in  $\mathcal{O}_{\gamma}$ . This leads to the following type of basis functions:

- 1 basis function attached to the value at  $\gamma$ :  $p = 1, q_i = 0, s_i = 0$
- 2 basis functions attached to the derivatives at  $\gamma$ :  $p = 0, [q_1, q_2] \in \{[1, 0], [0, 1]\}$  and  $s_i = 0$  if  $\tau_i$  is not a crossing edge at  $\gamma$  and determined by the relations (26)-(27) if  $\tau_i$  is a crossing edge at  $\gamma$ .
- $F(\gamma) - \sum_{i=1}^{F'} \mathbf{c}_{\tau_i}(\gamma) + \mathbf{c}_{+}(\gamma)$  basis functions attached to the free cross derivatives, with  $p = 0, q_i = 0$  and  $s_i \in \{0, 1\}$  if  $\tau_i$  is not a crossing edge and determined by the relations (26)-(27) if  $\tau_i$  is a crossing edge at  $\gamma$ .

### 6.4.2 Basis functions associated to an edge

Let  $\tau$  be an edge of  $\mathcal{M}_1$  shared by two faces  $\sigma_1, \sigma_2$  with vertices  $\gamma, \gamma'$ . Let us assume that the coordinates of these points in the face  $\sigma_1$  are  $\gamma = (0, 0)$  and  $\gamma' = (1, 0)$ .

The elements of  $\mathcal{E}_k(\tau)$  are the image by  $\Theta_\tau$  of the elements of  $Z_k$  of the form  $P[A_1, B_1, C_1] + Q[A_2, B_2, C_2]$  with degree  $\deg(P) \leq k - \mu_\tau - m_\tau$ ,  $\deg(Q) \leq k - \nu_\tau - m_\tau$  which are in the kernel of  $T_\gamma$  and  $T_{\gamma'}$ .

From the relation (23), we deduce that  $P(0) = 0$ ,  $Q(0) = 0$ . That is,  $P$  and  $Q$  are divisible by  $u$ .

- If  $\mathbf{c}_\tau(\gamma) = 0$ , i.e.  $\gamma$  is not a crossing vertex, we have  $B_1(0)C_2(0) - B_2(0)C_1(0) = a(0) \neq 0$  and the relation (23) implies that  $P'(0) = 0$ ,  $Q'(0) = 0$ . That is  $P = u^2\tilde{P}$ ,  $Q = u^2\tilde{Q}$ .
- If  $\mathbf{c}_\tau(\gamma) = 1$ , then the kernel of  $T_\gamma$  is generated by polynomials such that  $P(0) = 0$ ,  $Q(0) = 0$ ,  $P'(0) = \lambda C_2(0)$ ,  $Q'(0) = -\lambda C_1(0)$ . That is  $P = u(\lambda C_2(0) + u\tilde{P})$ ,  $Q = u(-\lambda C_1(0) + u\tilde{Q})$ .

That is

$$P = u \left( \lambda \mathbf{c}_\tau(\gamma) C_2(0) + u \tilde{P} \right), Q = u \left( -\lambda \mathbf{c}_\tau(\gamma) C_1(0) + u \tilde{Q} \right).$$

By symmetry at  $\gamma'$ , we see that  $P$  and  $Q$  are of the form:

$$\begin{aligned} P &= u(1-u) \left( \lambda \mathbf{c}_\tau(\gamma) C_2(0) (1-u) + \lambda' \mathbf{c}_\tau(\gamma') C_2(1) u + u(1-u) \tilde{P} \right), \\ Q &= -u(1-u) \left( \lambda \mathbf{c}_\tau(\gamma) C_1(0) (1-u) + \lambda' \mathbf{c}_\tau(\gamma') C_1(1) u + u(1-u) \tilde{Q} \right), \end{aligned}$$

with  $\lambda, \lambda' \in \mathbb{R}$ ,  $\deg(\tilde{P}) \leq k - \mu - m - 4$ ,  $\deg(\tilde{Q}) \leq k - \nu - m - 4$ .

We construct a basis of  $\mathcal{E}_k(\tau)$  by taking the image by  $\Theta_\tau$  of a maximal set of linearly independent elements of this form (see Section 4.3). This yields  $d_\tau(k) - 9 + \mathbf{c}_\tau(\gamma) + \mathbf{c}_\tau(\gamma')$  spline basis functions.

### 6.4.3 Basis functions associated to a face

Finally, we define the basis functions attached to a face  $\sigma \in \mathcal{M}_2$  as the 2-interior Bernstein basis functions in degree  $\leq k$ . There are  $(k-3)^2$  such basis spline functions for a quadrangular face and  $\binom{k-4}{2}$  for a triangular face.

## 7 Examples

### 7.1 A round corner

We consider a mesh  $\mathcal{M}$  composed of 3 quadrangles  $\sigma_1, \sigma_2, \sigma_3$  glued around an interior vertex  $\gamma$ , along the 3 interior edges  $\tau_1, \tau_2, \tau_3$ . There are 6 boundary edges and 6 boundary vertices. We take symmetric gluing data at  $\gamma$  and at the crossing boundary vertices  $\delta_i$ . The transition map across the interior edge  $\tau_i$  is given by the polynomials:  $[a, b, c] = [(u-1), -1, 1]$  where  $\gamma$  is the end point with  $u = 0$  and  $\delta_i$  is the end point with  $u = 1$ . The generating syzygies are

$$S_1 = [0, 1, 1], S_2 = [1, u, 1].$$

For the interior edges  $\tau_i$ , we have  $n = 1$ ,  $m = 0$ ,  $\mu = 0$ ,  $\nu = 1$  and  $d_{\tau_i}(k) = k + 1 + k = 2k + 1$ . For the boundary edges  $\tau'$ , we have  $n = 0$ ,  $m = 0$ ,  $\mu = 0$ ,  $\nu = 0$  and  $d_{\tau'}(k) = 2(k + 1)$ .

As  $a(0) = -1$  (resp.  $a(1) = 0$ ),  $\gamma$  is not a crossing vertex ( $\mathbf{c}_{\tau_i}(\gamma) = 0$ ) and  $\delta_i$  is a crossing vertex of  $\tau_i$  ( $\mathbf{c}_{\tau_i}(\delta_i) = 1$ ).

We check that the separability of all the interior edges is 4. For  $k = 4$ , the dimension of  $S_k^1(\mathcal{M})$  is

$$3 \times (4-3)^2 + 3 \times (2 \times 4 + 1) + 6 \times (2 \times 4 + 2) + 4 \times 3 - 9 \times 9 + 3 \times 7 + 6 = 48.$$

The basis functions are constructed as in Section 6.4, using the algorithms of Appendix A.

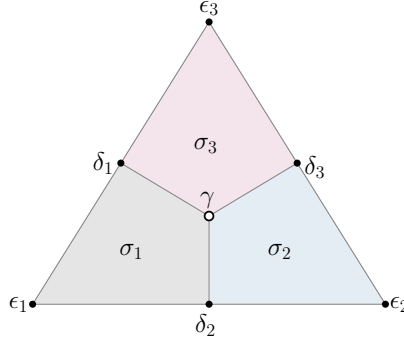


Figure 5: Smooth corner.

- The number of basis functions attached to  $\gamma$  is  $6 = 1 + 2 + 3$ .
  - The basis function associated to the value at  $\gamma$  is

$$[b_{0,0} + b_{1,0} + b_{0,1} + b_{1,1}, b_{0,0} + b_{1,0} + b_{0,1} + b_{1,1}, b_{0,0} + b_{1,0} + b_{0,1} + b_{1,1}]$$

- The two basis functions associated to the derivatives at  $\gamma$  are

$$\begin{aligned} & [\frac{1}{4} b_{1,0} + \frac{1}{4} b_{1,1} + \frac{7}{12} b_{2,0} + \frac{7}{12} b_{2,1} + \frac{1}{8} b_{1,2}, \\ & \quad -\frac{1}{4} b_{0,1} - \frac{1}{4} b_{1,1} - \frac{1}{8} b_{2,1} - \frac{7}{12} b_{0,2} - b_{1,2}, \\ & \quad \frac{1}{4} b_{0,1} - \frac{1}{4} b_{1,0} - \frac{7}{12} b_{2,0} - \frac{11}{24} b_{2,1} + \frac{7}{12} b_{0,2} + \frac{7}{8} b_{1,2}] \\ & [\frac{1}{4} b_{0,1} + \frac{1}{4} b_{1,1} + \frac{1}{8} b_{2,1} + \frac{7}{12} b_{0,2} + b_{1,2}, \\ & \quad -\frac{1}{4} b_{0,1} + \frac{1}{4} b_{1,0} + \frac{7}{12} b_{2,0} + \frac{11}{24} b_{2,1} - \frac{7}{12} b_{0,2} - \frac{7}{8} b_{1,2}, \\ & \quad -\frac{1}{4} b_{1,0} - \frac{1}{4} b_{1,1} - \frac{7}{12} b_{2,0} - \frac{7}{12} b_{2,1} - \frac{1}{8} b_{1,2}] \end{aligned}$$

- The three basis functions associated to the cross derivatives at  $\gamma$  are

$$\begin{aligned} & [\frac{1}{16} b_{1,1} - \frac{1}{12} b_{2,0} - \frac{1}{24} b_{2,1} - \frac{1}{12} b_{0,2} - \frac{1}{8} b_{1,2}, \\ & \quad -\frac{1}{12} b_{2,0} - \frac{1}{12} b_{2,1}, -\frac{1}{12} b_{0,2} - \frac{1}{6} b_{1,2}] \\ & [-\frac{1}{12} b_{0,2} - \frac{1}{6} b_{1,2}, \\ & \quad \frac{1}{16} b_{1,1} - \frac{1}{12} b_{2,0} - \frac{1}{24} b_{2,1} - \frac{1}{12} b_{0,2} - \frac{1}{8} b_{1,2}, -\frac{1}{12} b_{2,0} - \frac{1}{12} b_{2,1}] \\ & [-\frac{1}{12} b_{2,0} - \frac{1}{12} b_{2,1}, \\ & \quad -\frac{1}{12} b_{0,2} - \frac{1}{6} b_{1,2}, \frac{1}{16} b_{1,1} - \frac{1}{12} b_{2,0} - \frac{1}{24} b_{2,1} - \frac{1}{12} b_{0,2} - \frac{1}{8} b_{1,2}] \end{aligned}$$

- The number of basis functions attached to  $\delta_i$  is  $4 = 1 + 2 + 2 - 1$ . Here are the 4 basis functions associated to  $\delta_1$ :

$$\begin{aligned} & [b_{3,0} + b_{3,1} + b_{4,0} + b_{4,1}, 0, b_{0,3} + b_{1,3} + b_{0,4} + b_{1,4}], \\ & [b_{3,0} + b_{3,1}, 0, b_{0,3} + b_{1,3}], \\ & [b_{3,1} + b_{4,1}, 0, -b_{1,3} - b_{1,4}], \\ & [b_{3,1}, 0, -b_{1,3}]. \end{aligned}$$

The basis functions associated to the other boundary points  $\delta_2, \delta_3$  are obtained by cyclic permutation.

- The number of basis functions attached to the remaining boundary points is  $4 = 1 + 2 + 1$ . For  $\epsilon_1$ , the 4 basis functions are

$$[b_{3,3} + b_{3,4} + b_{4,3} + b_{4,4}, 0, 0], [b_{3,3} + b_{4,3}, 0, 0], [b_{3,3} + b_{3,4}, 0, 0], [b_{3,3}, 0, 0]$$

The basis functions associated to the other boundary points are obtained by cyclic permutation.

- The number of basis functions attached to the edge  $\tau_i$  is  $2 \times 4 - 7 = 1$ . For the edge  $\tau_1$ , it is

$$[b_{2,1}, 0, -b_{1,2}].$$

The basis functions associated to the other interior edges are obtained by cyclic permutation.

- The number of basis functions attached to the boundary edges  $2(4 - 3) = 2$ . For the boundary edge  $(\delta_1, )$ , the two basis functions are

$$[b_{3,2}, 0, 0], [b_{4,2}, 0, 0].$$

- The number of basis functions attached to a face  $\sigma_i$  is  $(4 - 3)^2 = 1$ . The basis function associated to  $\sigma_1$  is

$$[b_{2,2}, 0, 0]$$

and the two other ones are obtained by cyclic permutation.

## 7.2 A pruned octahedron

We consider a mesh  $\mathcal{M}$  with 6 triangular faces  $AEF$ ,  $CEF$ ,  $ABE$ ,  $BCE$ ,  $ADF$ ,  $CDF$  and one quadrangular face  $ABCD$ , depicted in Figure 6 as the *Schlegel diagram* of a convex polyhedron in  $\mathbb{R}^3$ . It is an octahedron where an edge  $BD$  is removed and two triangular faces are merged into a quadrangular face (see [28] for the complete octahedron, which involves only triangular faces).

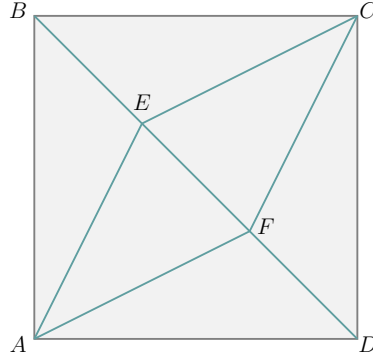


Figure 6: A pruned octahedron

We are going to use the following notation for the variables on the faces of  $\mathcal{M}$ . For  $X, Y \in \mathbb{R}^2$  two vertices defining an edge  $XY$  of a face  $\sigma$ , let  $u_X^Y : \mathbb{R}^2 \rightarrow \mathbb{R}$  be the linear function with  $u_X^Y(X) = 0$ ,  $u_X^Y(Y) = 1$  and  $u_X^Y(Z) = 0$  for all the points  $Z$  on the edge of  $\sigma$  through  $X$  and distinct from  $XY$ . We will use these linear functions  $u_E^B$ ,  $u_F^A$ , etc., as variables on the different faces. As the restriction on a share edge  $XY$  of the two functions defined on the faces adjacent to  $XY$  coincide, there is no ambiguity in evaluating these linear functions on  $\mathcal{M}$ . For a triangular face  $XYZ$ , we have  $u_X^Y + u_Y^Z + u_Z^X = 1$ . For a quadrangular face  $XYZW$ , we have  $u_X^Y = u_Z^W$  and  $u_X^W = u_Y^Z$ . We denote by  $\partial_{XY}$  the derivative with respect to the variable  $u_X^Y$ . It is such that  $\partial_{XY}(u_X^Y) = 1$ . On a triangular face  $XYZ$ , we have  $\partial_{XY} + \partial_{YZ} + \partial_{ZX} = 0$ .

We use a symmetric gluing at all the vertices and therefore the vertices  $A, C, E, F$  are crossing vertices. Let us describe how we construct the gluing data on the edges by interpolation at the vertices, in a smaller degree than the degree associated to the gluing (14) proposed in [12].



In terms of differentials (see relation (5)), the symmetric gluing at the vertices translates as

$$\begin{aligned}
\partial_{EA} + \partial_{EC} &= 0, & \partial_{EB} + \partial_{EF} &= 0 & \text{at } E, \\
\partial_{FA} + \partial_{FC} &= 0, & \partial_{FD} + \partial_{FE} &= 0 & \text{at } F, \\
\partial_{AB} + \partial_{AF} &= 0, & \partial_{AE} + \partial_{AD} &= 0 & \text{at } A, \\
\partial_{CB} + \partial_{CF} &= 0, & \partial_{CE} + \partial_{CD} &= 0 & \text{at } C,
\end{aligned} \tag{28}$$

and around the vertices of order 3:

$$\partial_{BA} + \partial_{BC} + \partial_{BE} = 0 \quad \text{at } B, \quad \partial_{DA} + \partial_{DC} + \partial_{DF} = 0 \quad \text{at } D. \tag{29}$$

For gluing the triangles  $EFA$  and  $EFC$  along  $EF$ , we interpolate the following relations between the derivatives:

$$\partial_{EA} + \partial_{EC} = 0 \text{ at } E, \quad \partial_{EA} + \partial_{EC} = 2\partial_{EF} \text{ at } F, \tag{30}$$

where the second expression is  $\partial_{FA} + \partial_{FC} = 0$  rewritten using  $\partial_{FA} = \partial_{EA} - \partial_{EF}$ ,  $\partial_{FC} = \partial_{EC} - \partial_{EF}$ . We choose the linear interpolation

$$\partial_{EA} + \partial_{EC} = 2u_E^F \partial_{EF}. \tag{31}$$

Thereby we have  $\mathbf{a}_{EF} = 2u_E^F$ ,  $\mathbf{b}_{EF} = -1$  and the gluing data for the edge  $EF$  is  $[2u_E^F, -1, 1]$ .

For the edge  $EB$  between the triangles  $EBA$  and  $EBC$ , we interpolate the following relations:

$$\partial_{EA} + \partial_{EC} = 0 \text{ at } E, \quad \partial_{EA} + \partial_{EC} = 3\partial_{EB} \text{ at } B,$$

where the latter relation is  $\partial_{BA} + \partial_{BC} + \partial_{BE} = 0$  rewritten using  $\partial_{BA} = \partial_{EA} - \partial_{EB}$ ,  $\partial_{BC} = \partial_{EC} - \partial_{EB}$ ,  $\partial_{BE} = -\partial_{EB}$ . Additionally, we have to take into account the compatibility conditions (9)-(10) since  $E$  is a crossing vertex. It translates as  $\partial_{EB}(\mathbf{a}_{EB}) = \partial_{EF}(\mathbf{a}_{EF})$  and  $\partial_{EC}(\mathbf{a}_{EC}) = \partial_{EA}(\mathbf{a}_{EA})$  at the vertex  $E$ . This leads to the following gluing data on the edge  $EB$ :

$$EB : [2u_E^B + (u_E^B)^2, -1, 1].$$

Similarly, the gluing data of the edge  $FD$  is

$$FD : [2u_F^D + (u_F^D)^2, -1, 1].$$

The edges  $EA$ ,  $EC$ ,  $FA$ ,  $FC$  connect cross vertices just as  $EF$  and yield linear gluing data

$$\begin{aligned}
EF : & [2u_E^A, -1, 1], & EC : & [2u_E^C, -1, 1], \\
FA : & [2u_F^A, -1, 1], & FC : & [2u_F^C, -1, 1]
\end{aligned}$$

We check that the compatibility conditions (9)-(10) are satisfied across  $EA, EC$  and  $FA, FC$ . The gluing data along  $AB, AD, CB, CD$  looks the same:

$$\begin{aligned}
AB : & [2u_A^B, -1, 1], & AD : & [2u_A^D, -1, 1], \\
CB : & [2u_C^B, -1, 1], & CD : & [2u_C^D, -1, 1].
\end{aligned} \tag{32}$$

We have linear gluing data everywhere except on the edges  $EB$  and  $FD$ . Let us analyze the syzygies associated this data.

- For the edges  $EB$  and  $FD$  with one crossing vertex, the gluing data is of the form  $[2u + u^2, -1, 1]$ . We have  $n = 2$  and  $m = 1$  since the edge is connecting two triangles,  $\mu = 0$  and  $\nu = 2$  and  $d(k) = 2k - 2$ . The  $\mu$ -basis is  $[0, 1, 1]$ ,  $[-1, -2u - u^2, 0]$ . The separability is achieved in degree  $k \geq 6$  and not 5 as it could be expected ( $d(5) \geq 8$ ).
- For the edges  $EA, EC, FA, FC, EF$  between triangular faces, with two crossing vertices, the linear gluing data is of the form  $[2u, 1, -1]$ . We have  $n = 1$  and  $m = 1$ ,  $\mu = 0$ ,  $\nu = 1$  and  $d(k) = 2k - 1$ . The  $\mu$ -basis is  $[0, 1, 1]$ ,  $[-1, -2u, 0]$ . The separability is achieved in degree  $k \geq 4$ .

- For the edges  $AB, AD, CB, CD$  between a triangular face and a quadrangular face, with one crossing vertices, the linear gluing data is of the form  $[2u, 1, -1]$ . We have  $n = 1$  and  $m = 0, \mu = 1$  since the degree of the homogeneization  $[d_a, d_b, d_c]$  (see Definition 4.1) is  $[3, 2, 1]$  or  $[3, 1, 2]$ ,  $\nu = 1$  and  $d(k) = 2k$ . The  $\mu$ -basis is  $[0, 1, 1], [-1, -2u, 0]$ . The separability is also achieved in degree  $k \geq 4$ .

Now we count how many splines do we have in degree  $k \geq 6$ :

- For the four crossing vertices  $A, C, E, F$  we have  $1+2+1 = 4$  dimensions and  $1+2+3 = 6$  dimensions for  $B$  and for  $D$ . In total we have  $4 \cdot 4 + 2 \cdot 6 = 28$  degrees of freedom around the vertices of  $\mathcal{M}$ .
- For the edges  $EB$  and  $FD$ , we have  $2(k-2) - 8 = 2k - 12$  dimensions. For the edges  $EA, EA, FA, FC, EF$ , we have  $2k - 1 - 7 = 2k - 8$  dimensions. For the edges  $AB, AD, CB, CD$ , we have  $2k - 8$ .
- For the 6 triangular faces, we have  $\binom{k-4}{2}$  dimensions and for the quadrangular face  $(k-3)^2$ .

The dimension formula in degree  $k \geq 6$  is then

$$28 - 4 + 11 \cdot 2(k-4) + 6 \binom{k-4}{2} + (k-3)^2 = (2k-3)^2 + k - 4. \quad (33)$$

For  $k = 6$ , the dimension is 83. It turns out that this formula also holds for degree  $k = 4, k = 5$ .

The construction of basis functions can be done as described in Section 6.4. Let us give the basis functions associated to the value and first derivatives at the point  $A$ . Here are the Bernstein coefficients in degree 4 of the basis function for the value at  $A$  with the vertex  $A$  represented in the center and the edges represented by horizontal and vertical central lines (in bold):

$$\begin{array}{cccccc}
 & & & \mathbf{0} & & \\
 & \cdot \cdot & 0 & \frac{1}{2} & 0 & \cdot \cdot \\
 & 0 & 1 & \mathbf{1} & 1 & 0 \\
 \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \frac{1}{2} & \mathbf{0} \\
 0 & -1 & 1 & \mathbf{1} & 1 & 0 \\
 \vdots & 0 & -1 & \mathbf{0} & 0 & \cdot \cdot \\
 \dots & 0 & \mathbf{0} & & & 
 \end{array}
 \quad
 \begin{array}{c}
 \uparrow E \\
 \leftarrow B \quad A \quad \rightarrow F \\
 \downarrow D
 \end{array}
 \quad (34)$$

This gives the following specializations to the polygons  $ABE, AEF, AFD, ABCD$  (respectively), selectively de-homogenized:

$$\begin{aligned}
 & (u_E^A)^2 (1 + 3u_A^B)(1 + 2u_A^E - u_A^B), \\
 & (u_E^A)^2 (1 + 2u_A^F + 2u_A^E + 6u_A^F u_A^E), \\
 & (u_F^A)^2 (1 + 3u_A^D)(1 + 2u_A^F - u_A^D), \\
 & (u_B^A u_D^A)^2 (u_B^A u_D^A (1 + 3u_A^D)(1 + 3u_A^B) - 24u_A^D u_A^B (u_{AB}^D u_B^A + u_A^B u_D^A)),
 \end{aligned}$$

and 0 on the other faces. The basis function associated to the first derivative in one of the directions at the cross vertices is:

$$\begin{array}{cccccc}
 & & & \mathbf{0} & & \\
 & \cdot \cdot & 0 & \frac{1}{6} & 0 & \cdot \cdot \\
 & 0 & \frac{1}{3} & \frac{1}{4} & \frac{1}{3} & 0 \\
 \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
 0 & \frac{7}{24} & -\frac{5}{16} & -\frac{1}{4} & -\frac{1}{3} & 0 \\
 \vdots & 0 & -\frac{1}{6} & \mathbf{0} & 0 & \cdot \cdot \\
 \dots & 0 & \mathbf{0} & & & 
 \end{array}
 \quad (35)$$

The non-zero specializations to  $ABE$ ,  $AEF$ ,  $AFD$ ,  $ABCD$  are, respectively:

$$(u_E^A)^2 u_A^E (1 + 3u_A^B), (u_E^A)^2 u_A^E (1 + 3u_A^F), -(u_F^A)^2 u_A^D (u_F^A + 4u_A^F),$$

$$(u_B^A u_D^A)^2 u_A^D (-u_D^A (1 + 3u_A^B) + 7u_A^B u_B^A u_A^D).$$

The basis function for the derivative in the other direction is obtained by a mirror image of (35).

The basis function corresponding to the cross derivatives is realized by

$$\begin{array}{ccccccc} & & & & \mathbf{0} & & \\ & & & & & & \\ & \cdot\cdot & 0 & \mathbf{0} & 0 & \cdot\cdot & \\ & 0 & -\frac{1}{12} & \mathbf{0} & \frac{1}{12} & 0 & \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 0 & \frac{1}{24} & \frac{1}{16} & \mathbf{0} & -\frac{1}{12} & 0 & \\ \vdots & 0 & \frac{1}{24} & \mathbf{0} & 0 & \cdot\cdot & \\ \dots & & 0 & \mathbf{0} & & & \end{array} \quad (36)$$

In this spline, we could modify the 0 entry next to two  $\frac{1}{24}$  entries to  $\frac{1}{36}$ , so to lower the degree of the specialization to the rectangle. After the modification, the 4 non-zero specializations would be

$$-(u_E^A)^2 u_A^E u_A^B, (u_E^A)^2 u_A^E u_A^F, -(u_F^A)^2 u_A^D u_A^F, (u_B^A u_D^A)^2 u_A^D u_A^B.$$

The local splines around  $C$  look the same. The local splines around other vertices involve the edges  $EB$  and  $FD$ , and we would need degree 6 splines.

## References

- [1] C. Beccari, D. Gonsor, and M. Neamtu. RAGS: Rational geometric splines for surfaces of arbitrary topology. *Comput. Aided Geom. Des.*, 31(2):97–110, 2014.
- [2] M. Bercovier and T. Matskewich. Smooth Bézier surfaces over arbitrary quadrilateral meshes. *arXiv:1412.1125*, 2015.
- [3] G.-P. Bonneau and S. Hahmann. Flexible  $G^1$  interpolation of quad meshes. *Graphical Models*, 76(6):669–681, 2014.
- [4] F. Buchegger, B. Jüttler, and A. Mantzaflaris. Adaptively refined multi-patch B-splines with enhanced smoothness. *NFN report No. 26*, 2015.
- [5] E. Catmull and J. Clark. Seminal graphics. chapter Recursively Generated B-spline Surfaces on Arbitrary Topological Meshes, pages 183–188. ACM, New York, NY, USA, 1998.
- [6] D. A. Cox, J. Little, and D. O’Shea. *Using algebraic geometry*, volume 185 of *Graduate Texts in Mathematics*. Springer, 2 edition, 2005.
- [7] T. DeRose. Geometric continuity: a parametrization independent measure of continuity for computer aided geometric design. *PhD Thesis, University of California at Berkeley*, 1985.
- [8] M. Eck and H. Hoppe. Automatic reconstruction of b-spline surfaces of arbitrary topological type. In *Proceedings of the 23rd Annual Conference on Computer Graphics and Interactive Techniques*, SIGGRAPH ’96, pages 325–334, New York, NY, USA, 1996. ACM.
- [9] J. A. Gregory. Geometric continuity. In L. L. Schumaker T. Lyche, editor, *Mathematical models in Computer Aided Geometric Design*, pages 353–371. Academic Press, 1989.

- [10] X. Gu, Y. He, M. Jin, F. Luo, H. Qin, and S.-T. Yau. Manifold splines with a single extraordinary point. *Computer-Aided Design*, 40(6):676–690, 2008.
- [11] X. Gu, Y. He, and H. Qin. Manifold splines. In *Graphical Models*, pages 27–38, 2005.
- [12] J. Hahn. Geometric continuous patch complexes. *Comput. Aided Geom. Des.*, 6(1):55–67, 1989.
- [13] Y. He, K. Wang, H. Wang, X. Gu, and H. Qin. Manifold t-spline. In *In Proceedings of Geometric Modeling and Processing*, pages 409–422, 2006.
- [14] M. Kapl, V. Vitrih, and B. Jüttler. Isogeometric analysis with geometrically continuous functions. *NFN report No. 22*, 2014.
- [15] H. Lin, W. Chen, and H. Bao. Adaptive patch-based mesh fitting for reverse engineering. *Comput. Aided Des.*, 39(12):1134–1142, December 2007.
- [16] C. Loop. Smooth spline surfaces over irregular meshes. In *Proceedings of the 21st Annual Conference on Computer Graphics and Interactive Techniques*, SIGGRAPH '94, pages 303–310, New York, NY, USA, 1994. ACM.
- [17] J. Peters. Constructing  $C^1$  surface of arbitrary topology using biquadratic and bicubic splines. SIAM, pages 277–293, 1994.
- [18] J. Peters.  $C^2$  free-form surfaces of degree (3, 5). *Comput. Aided Geom. Des.*, 19(2):113–126, 2002.
- [19] J. Peters. Geometric continuity. In *Handbook of computer aided geometric design*, pages 193–227. North-Holland, Amsterdam, 2002.
- [20] J. Peters and J. Fan. On the complexity of smooth spline surfaces from quad meshes. *Comput. Aided Geom. Des.*, 27(1):96–105, 2010.
- [21] J. Peters and U. Reif. *Subdivision surfaces*, volume 3 of *Geometry and Computing*. Springer, 2008.
- [22] H. Prautzsch. Freeform splines. *Comput. Aided Geom. Des.*, 14(3):201–206, April 1997.
- [23] U. Reif. Biquadratic G-spline surfaces. *Comput. Aided Geom. Des.*, 12(2):193–205, March 1995.
- [24] X. Shi, T. Wang, P. Wu, and F. Liu. Reconstruction of convergent  $G^1$  smooth b-spline surfaces. *Comput. Aided Geom. Des.*, 21(9):893–913, November 2004.
- [25] E. Tosun and D. Zorin. Manifold-based surfaces with boundaries. *Comput. Aided Geom. Des.*, 28(1):1–22, January 2011.
- [26] G. Della Vecchia, B. Jüttler, and M.-S. Kim. A construction of rational manifold surfaces of arbitrary topology and smoothness from triangular meshes. *Computer Aided Geometric Design*, 25(9):801–815, 2008.
- [27] R. Vidunas. Aspects of algorithmic algebra: Differential equations and splines. *PhD Thesis, Rijksuniversiteit Groningen*, pages 103–150, 1999. Chapter 6: Geometrically continuous surface complexes, available at <http://irs.ub.rug.nl/ppn/185863523>.
- [28] R. Vidunas. Geometrically continuous octahedron. In R. Krasauskas R. Goldman, editor, *Topics in Algebraic Geometry and Geometric Modeling*, volume 334 of *Contemporary Mathematics*, pages 37–52. AMS, 2003.
- [29] F. W. Warner. *Foundations of Differentiable Manifolds and Lie Groups*. Number 94 in Graduate Texts in Mathematics. Springer Verlag, 1983.

- [30] M. Wu, B. Mourrain, A. Galligo, and B. Nkonga. Spline Spaces over Quadrangle Meshes with Complex Topologies. Available at <https://hal.inria.fr/hal-00952455>, February 2014.
- [31] L. Ying and D. Zorin. A simple manifold-based construction of surfaces of arbitrary smoothness. *ACM Trans. Graph.*, 23(3):271–275, August 2004.

## A Algorithms for the basis construction

Our input data is the topological surface  $\mathcal{M}$  and the gluing data. For each edge  $\tau$  of  $\mathcal{M}$ , we are given the  $\mu_\tau$ -basis of  $Z_k$ :

$$S_1^\tau = [A_1^\tau, B_1^\tau, C_1^\tau], \quad S_2^\tau = [A_2^\tau, B_2^\tau, C_2^\tau].$$

The rational map is then described by

$$\mathbf{a}_\tau = \frac{a_\tau}{c_\tau}, \quad \mathbf{b}_\tau = \frac{b_\tau}{c_\tau}$$

with  $a_\tau = B_1^\tau C_2^\tau - B_2^\tau C_1^\tau$ ,  $b_\tau = A_2^\tau C_1^\tau - A_1^\tau C_2^\tau$ ,  $c_\tau = A_1^\tau B_2^\tau - A_2^\tau B_1^\tau$ .

The spline basis functions  $f = (f_\sigma)$  are represented on each face  $\sigma$  by their coefficients in the Bernstein basis of the face in degree  $k$ :

$$f_i = \sum_{l,m} c_{l,m}^i b_{l,m}^\sigma(u_i, v_i)$$

Let  $e_\sigma(k) = k^2$  if  $\sigma$  is a quadrangular face and  $e_\sigma(k) = k(k-1)$  if  $\sigma$  is a triangular face.

### A.1 Vertex basis functions

Let  $\gamma$  be a vertex of  $\mathcal{M}$  shared by the faces  $\sigma_1, \dots, \sigma_F$  and such that  $\sigma_i$  and  $\sigma_{i+1}$  share the edge  $\tau_{i+1}$ .

We compute the Bernstein coefficients  $\mathbf{c}^i = [c_{0,0}^i, c_{1,0}^i, c_{0,1}^i, c_{1,1}^i, \dots]$  of the basis functions attached to a vertex  $\sigma_i$ , using the equations (26), (27) and the relation between the Bernstein coefficients and the Taylor coefficients of the function at  $(0,0)$  (see (16), (17)).

If  $c_{0,0}^i$  corresponds to the point  $\gamma$ , with coordinates are  $(0,0)$  in the face  $\sigma_i$  and  $f_{\sigma_i} = p + q_i u_i + q_{i+1} v_i + s_i u_i v_i + \dots$  are  $\mathbf{c}^i$  then we use the relations  $p = c_{0,0}^i$ ,  $q_i = k c_{1,0}^i$ ,  $q_{i+1} = k c_{0,1}^i$ ,

$$s_i = e_k (c_{1,1}^i - c_{1,0}^i - c_{0,1}^i).$$

BASIS FUNCTION FOR THE VALUE AT VERTEX  $\gamma$

```

for  $i$  in  $[1, F]$  do
  | let  $c_{0,0}^i := 1, c_{1,0}^i := 1, c_{0,1}^i := 1, c_{1,1}^i := 1$  and  $c_{l,m}^i := 0$  for
  |  $(l, m) \notin \{(0, 0), (1, 0), (0, 1), (1, 1)\}$ ;
end

```

BASIS FUNCTIONS FOR THE DERIVATIVES AT VERTEX  $\gamma$

```

for  $[c_{0,0}^1, c_{0,1}^1]$  in  $\{[1, 0], [0, 1]\}$  do
  | for  $i$  in  $[2, F]$  do
    | 
$$\begin{bmatrix} c_{1,0}^i \\ c_{0,1}^i \end{bmatrix} = \begin{pmatrix} 0 & 1 \\ \mathbf{b}_{\tau_i}(0) & \mathbf{a}_{\tau_i}(0) \end{pmatrix} \begin{bmatrix} c_{1,0}^{i-1} \\ c_{0,1}^{i-1} \end{bmatrix}$$

    | end
    | if all edges  $\tau_i$  are crossing edges at  $\gamma$  then
      | let  $c_{1,1}^i := 0$ ;
    | end
    | for  $i$  in  $[2, F]$  do
      | if  $\tau_i$  is a crossing edge at  $\gamma$  then
        | 
$$\begin{aligned} c_{1,1}^i &= c_{1,0}^i + c_{0,1}^i \\ &\quad + \frac{1}{e_{\sigma_i}(k)} (e_{\sigma_{i-1}}(k) \mathbf{b}_{\tau_i}(0) (c_{1,1}^{i-1} - c_{1,0}^{i-1} - c_{0,1}^{i-1}) \\ &\quad + k \mathbf{a}'_{\tau_i}(0) c_{0,1}^{i-1} + k \mathbf{b}'_{\tau_i}(0) c_{1,0}^{i-1}) \end{aligned}$$

        | else
          |  $c_{1,1}^i := 0$ ;
        | end
      | end
      | for  $i$  in  $[1, F']$  do
        |  $\text{lift}(\mathbf{c}_{i-1}, \mathbf{c}_i, \tau_i)$ 
      | end
    | end
  | end

```

BASIS FUNCTIONS FOR THE CROSS DERIVATIVES AROUND  $\gamma$

```

for  $i$  in  $[1, F]$  do
  | let  $c_{0,0}^i := 0, c_{1,0}^i := 0, c_{0,1}^i := 0$ ;
end
if all edges  $\tau_i$  are crossing edges at  $\gamma$  then
  | let  $c_{1,1}^1 := 1$ ;
  | for  $i$  in  $[2, F]$  do
    |  $c_{1,1}^i = \frac{e_{\sigma_{i-1}}}{e_{\sigma_i}(k)} \mathbf{b}_{\tau_i}(0) c_{1,1}^{i-1}$ ;
  | end
  | for  $i$  in  $[1, L]$  do
    |  $\text{lift}(\mathbf{c}_i, \mathbf{c}_{i-1}, \tau_i)$ ;
  | end
else
  | for  $j$  in  $[1, F]$  such that  $\tau_j$  is not a crossing edge at  $\gamma$  do
    | let  $c_{1,1}^j = 1$  and  $c_{1,1}^l = 0$  for  $l \neq j$ ;
    | for  $i$  in  $[1, F']$  do
      | if  $\tau_i$  is a crossing edge then
        |  $c_{1,1}^i = \frac{e_{\sigma_{i-1}}}{e_{\sigma_i}(k)} \mathbf{b}_{\tau_i}(0) c_{1,1}^{i-1}$ 
        | end
      | end
      | for  $i$  in  $[1, L]$  do
        |  $\text{lift}(\mathbf{c}_i, \mathbf{c}_{i-1}, \tau_i)$ ;
      | end
    | end
  | end

```

end

The function  $\text{lift}(\mathbf{c}_i, \mathbf{c}_{i-1}, \tau_i)$  used in the algorithm consists in computing the coefficient of a spline function with support along the edge  $\tau_i$ , from its first Taylor coefficients on the faces  $\sigma_{i-1}, \sigma_i$ .

**LIFT**( $\mathbf{c}_i, \mathbf{c}_{i-1}, \tau_i$ )

**for**  $i$  in  $[1, F]$  **do**

    solve the systems:

$$\begin{bmatrix} k c_{1,0}^i \\ k c_{1,0}^{i-1} \end{bmatrix} = \begin{bmatrix} A_1(0) & A_2(0) \\ B_1(0) & B_2(0) \end{bmatrix} \begin{bmatrix} p_0^i \\ q_0^i \end{bmatrix}$$

    and

$$\begin{bmatrix} e_{\sigma_{i-1}}(k)(c_{1,1}^{i-1} - c_{1,0}^{i-1} - c_{0,1}^{i-1}) \\ -e_{\sigma_i}(k)(c_{1,1}^i - c_{1,0}^i - c_{0,1}^i) \end{bmatrix} - \begin{bmatrix} B_1'(0) & B_2'(0) \\ C_1'(0) & C_2'(0) \end{bmatrix} \begin{bmatrix} p_0^i \\ q_0^i \end{bmatrix} \\ = \begin{bmatrix} B_1(0) & B_2(0) \\ C_1(0) & C_2(0) \end{bmatrix} \begin{bmatrix} p_1^i \\ q_1^i \end{bmatrix}.$$

    compute  $P^i := p_0^i(1 - 3u_i^2 + 2u_i^3) + p_1^i(u - 2u_i^2 + u_i^2)$ ,

$Q^i := q_0^i(1 - 3u_i^2 + 2u_i^3) + q_1^i(u - 2u_i^2 + u_i^2)$ ;

    compute the image  $(g_i, \tilde{g}_i)$  of  $P^i S_1^i + Q^i S_2^i$  by  $\Theta_{\tau_i}$  and update the coefficients of

$\mathbf{c}_{i-1}, \mathbf{c}_i$ ;

**end**

As  $A_1(0)B_2(0) - A_2(0)B_1(0) = c(0) \neq 0$ , the first system has a unique solution. When  $B_1(0)C_2(0) - B_2(0)C_1(0) = a(0) \neq 0$  (i.e. when  $\tau_i$  is not a crossing edge at  $\gamma$ ), the second system has a unique solution. When  $a(0) = 0$  (i.e. when  $\tau_i$  is a crossing edge at  $\gamma$ ), the second system is degenerate, but it still has a (least square) solution.

The polynomials  $P^i$  (resp.  $Q^i$ ) are constructed so that  $P^i(0) = p_0^i, P^{i'}(0) = p_1^i, P^i(1) = 0, P^{i'}(1) = 0$  (resp.  $Q^i(0) = q_0^i, Q^{i'}(0) = q_1^i, Q^i(1) = 0, Q^{i'}(1) = 0$ ).

By construction, the Taylor expansions of their image by  $\Theta_{\tau_i}$  vanish at  $\gamma'$  and coincide with  $[c_{0,0}^{i-1}, c_{1,0}^{i-1}, c_{0,1}^{i-1}, e_{\sigma_{i-1}}(k)(c_{1,1}^{i-1} - c_{1,0}^{i-1} - c_{0,1}^{i-1})], [c_{0,0}^i, c_{1,0}^i, c_{0,1}^i, e_{\sigma_i}(k)(c_{1,1}^i - c_{1,0}^i - c_{0,1}^i)]$  at  $\gamma$  respectively on  $\sigma_{i-1}$  and  $\sigma_i$ .

## A.2 Edge basis functions

BASIS FUNCTIONS FOR THE EDGE  $\tau$

**Input:**  $[A_1^\tau, B_1^\tau, C_1^\tau], [A_2^\tau, B_2^\tau, C_2^\tau]$  the  $\mu$ -basis of the syzygy module  $Z(\tau)$ ;

**if**  $\mathbf{c}_\tau(\gamma) = 1$  **then**

    | compute the image by  $\Theta_\tau$  of  $u(1-u)^2(C_2^\tau(0)[A_1^\tau, B_1^\tau, C_1^\tau] - C_1^\tau(0)[A_2^\tau, B_2^\tau, C_2^\tau])$

**end**

**if**  $\mathbf{c}_\tau(\gamma') = 1$  **then**

    | compute the image by  $\Theta_\tau$  of  $u^2(1-u)(C_2^\tau(1)[A_1^\tau, B_1^\tau, C_1^\tau] - C_1^\tau(1)[A_2^\tau, B_2^\tau, C_2^\tau])$

**end**

Let  $\Delta = u^2(1-u)^2$ ;

**for**  $i$  in  $[0, k - \mu - m - 4]$  **do**

    | compute the image by  $\Theta_\tau$  of  $u^i \Delta [A_1^\tau, B_1^\tau, C_1^\tau]$ .

**end**

**for**  $i$  in  $[0, k - \nu - m - 4]$  **do**

    | compute the image by  $\Theta_\tau$  of  $u^i \Delta [A_2^\tau, B_2^\tau, C_2^\tau]$ .

**end**

## A.3 Face basis functions

BASIS FUNCTIONS FOR THE FACE  $\sigma$

**for**  $2 \leq i \leq k-2, 2 \leq j \leq k-2$  (and  $i+j \leq k-2$  if  $\sigma$  is a triangle) **do**

    | let  $c_{i,j} := 1$  and  $c_{i',j'} = 0$  for  $i' \neq i$  or  $j' \neq j$ .

**end**