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A Conjecture on the Number of Hamiltonian Cycles on Thin Grid Cylinder Graphs

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We study the enumeration of Hamiltonian cycles on the thin grid cylinder graph $C_m \times P_{n+1}$. We distinguish two types of Hamiltonian cycles depending on their contractibility (as Jordan curves) and denote their numbers $h_m^{nc}(n)$ and $h_m^c(n)$. For fixed m , both of them satisfy linear homogeneous recurrence relations with constant coefficients. We derive their generating functions and other related results for $m \leq 10$. The computational data we gathered suggests that $h_m^{nc}(n) \sim h_m^c(n)$ when m is even.

Keywords: Hamiltonian cycles, generating functions, thin grid cylinder, contractible curves.

1 Introduction

A Hamiltonian path of a simple graph is a path that visits each vertex exactly once. A closed Hamiltonian path is called a Hamiltonian cycle or Hamiltonian circuit, which we shall abbreviate as HC. The enumeration of Hamiltonian cycles on rectangular grid graphs $P_m \times P_n$ had been studied extensively in, among others, [2, 4, 9, 15, 10, 13, 14, 17, 19, 20]. In contrast, little work [2, 9, 11, 17] was devoted to enumerate Hamiltonian cycles on rectangular grid cylinders $C_m \times P_n$.

In this paper we investigate, for each fixed $m \geq 2$, the generation and enumeration of Hamiltonian cycles on $C_m \times P_{n+1}$, where $n \geq 1$. Since n grows while m is fixed, such graphs are called *thin grid cylinders* in the literature. In [2], vertices were encoded. We adopt a different approach by coding the cells or squares on the cylindrical surface, along with the so-called k-SIST equivalence relation. This equivalence relation was formerly called k-SISET, and was first used in [4] to enumerate Hamiltonian cycles on $P_m \times P_n$. A very similar approach for the same enumeration was implemented in [19] using the language of finite automata.

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We distinguish two different types of HCs. In the sense of homotopy: one type of HCs are contractible (as Jordan curves) to a point, and the other type of HCs are not. We denote them HC^c and HC^{nc} , respectively. Simply put, a HC^{nc} is one that “perches” on or wraps around the cylinder like a bracelet on an arm, and a HC^c can be “pasted” on the cylindrical surface. Let $h_m^{nc}(n)$ and $h_m^c(n)$ be the number of HC^{nc} s and HC^c s, respectively, on $C_m \times P_{n+1}$. Our objective is to determine, for each fixed m , the sequences $h_n^{nc} = \{h_m^{nc}(n)\}_{n \geq 1}$ and $h_n^c = \{h_m^c(n)\}_{n \geq 1}$. It is obvious that the number of HCs on $C_m \times P_{n+1}$ is given by $h_m(n) = h_m^{nc}(n) + h_m^c(n)$.

We characterize both types of HCs, and use it to define, for each fixed $m \geq 2$, a digraph \mathcal{D}_m . The original enumeration problem is equivalent to counting oriented walks of length $n - 1$ in this digraph with first and last vertices from two special sets. Using the transfer matrix method [5, 18], we obtain the generating functions for the sequences h_m^{nc} and h_m^c , thereby proving that they both satisfy some linear homogeneous recurrence relations with constant coefficients.

For each fixed m , these two generating functions share the same denominator, hence the same recurrence relation. We used Pascal programs and Mathematica 6 to carry out the computation. Our results agree with those reported in [2, 11], which used a different approach. The computational data from $m = 2, 4, 6, 8, 10$ suggest that $h_m^{nc}(n)$ and $h_m^c(n)$ have the same number of digits and start with the same sequence of digits. For example,

$$\begin{aligned} h_{10}^{nc}(100) &= 106189661997982901262641694866260787081353490654045349773784 \\ &\quad 008483411988691035247114502475722767402987233190282387756909 \\ &\quad 3701143503070291097245473763298031619982266082, \\ h_{10}^c(100) &= 106189661997133629777153967627991207437193145571362259096752 \\ &\quad 805056007992463634686046052605540587643324294617040045670714 \\ &\quad 1143497346647742593316608877569233239238111440. \end{aligned}$$

Both numbers have 166 digits, and their first 12 digits are identical. Why is this happening?

2 Preliminaries

The graph $C_m \times P_{n+1}$ can be drawn on a cylindrical surface in such a way that no edges cross each other, see Figure 1. There are mn squares (4-cycles) called **windows**. Label the vertices (i, j) and the windows $w_{i,j}$, where $1 \leq i \leq m$, $1 \leq j \leq n + 1$ for vertices, and $1 \leq j \leq n$ for windows, as shown in Figure 1. Construct a **window lattice graph** $W_{m,n}$ with vertices representing the windows of $C_m \times P_{n+1}$, and two vertices are adjacent if and only if their corresponding windows in $C_m \times P_{n+1}$ share a common edge. It should be clear that $W_{m,n}$ is isomorphic to $C_m \times P_n$.

We distinguish two types of closed Jordan curves on a cylindrical surface: those that divide the surface into two infinite regions (image the cylinder being extended indefinitely in both directions to the left and to the right), see the curve \mathcal{K}^{nc} in Figure 2, and those that divide the surface into one finite and one infinite region, see the curve \mathcal{K}^c in Figure 2. The first type (non-contractible HC) wraps around the cylindrical surface, hence divides the cylindrical surface into the left half and the right half, it resembles a bracelet around an arm. The second type (contractible HC) encloses a finite region and leaves an infinite region on the outside. One could imagine it being pasted onto the cylindrical surface.

We abbreviate these two types of Hamiltonian cycles as HC^{nc} and HC^c , respectively. We use the following convention to name the two regions separated by a HC:

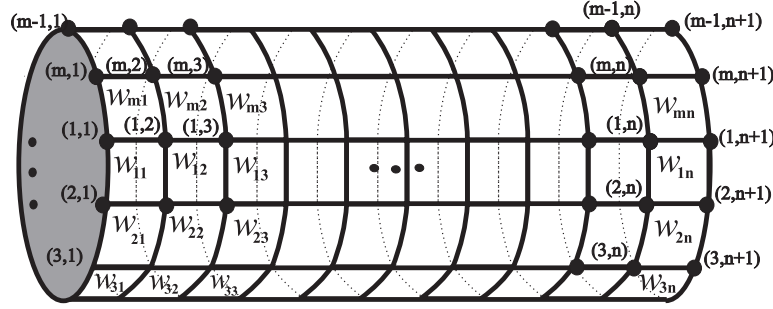


Fig. 1: The labeled graph $C_m \times P_{n+1}$ and its windows.

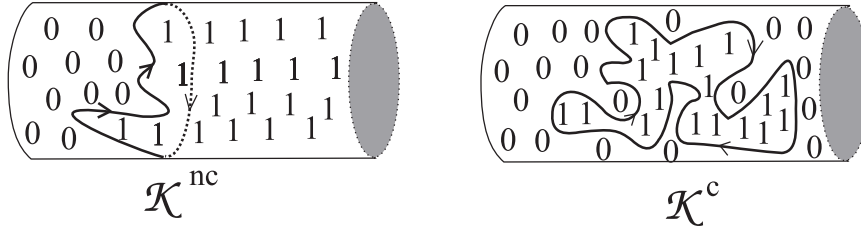


Fig. 2: Two types of closed Jordan curves on a cylindrical surface.

- For a non-contractible HC: all edges that connect two adjacent vertices from $\{(i, 1) \mid 1 \leq i \leq m\}$, but do not lie on the HC, belong to the same region. We call this region (on the left of the HC) the **zero region**, and the other region (on the right of the HC) the **positive region**.
- For a contractible HC: the windows within the bounded region are marked with 1s, hence the bounded region is the **positive region**, which makes the exterior unbounded region the **zero region**.

Alternatively, the orientation of the HC is chosen such that the zero region is always on our left as we traverse through the HC (see Figure 3). For HC^c this orientation is in the clockwise direction.

We use $h_m^{nc}(n)$ and $h_m^c(n)$ to indicate the number of HC^{nc} s and HC^c s. Their respective generating functions are written as $\mathcal{H}_m^{nc}(x)$ and $\mathcal{H}_m^c(x)$. Using a standard parity argument (likes the one used on a checkerboard), it is easy to tell which thin grid cylinders have a Hamiltonian cycle.

Theorem 2.1 For $m \geq 2$ and $n \geq 1$, we have $h_m^{nc}(n) = 0$ if and only if both m and n are odd, and $h_m^c(n) = 0$ if and only if m is odd and n is even.

Proof: It is straightforward to construct a HC^{nc} for even m or even n , and a HC^c for even m or odd n (see Figure 4). It remains to establish the condition under which no HC exists.

Consider the “vertical” edges joining vertices (m, i) to $(1, i)$ for $1 \leq i \leq n + 1$, see Figure 3. Any HC may contain some of these vertical edges, and the number of such edges is odd for a HC^{nc} , and even for a HC^c .

As we travel along a non-contractible Hamiltonian cycle, the number of steps “to the left” and “to the right” must be equal, while the difference between the “up” and “down” steps is m . Since the HC contains

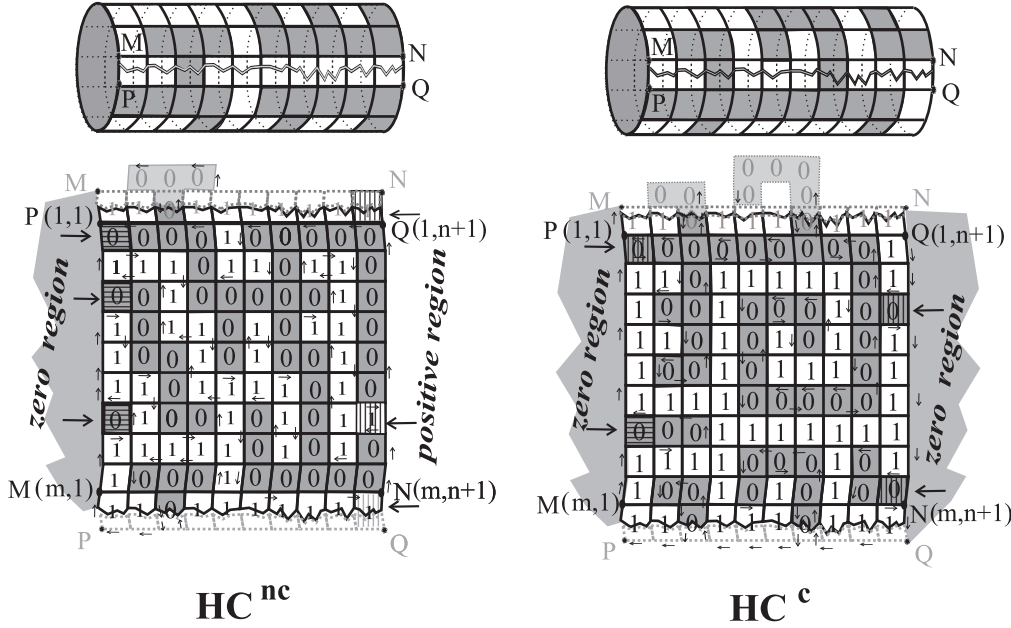


Fig. 3: Two types of Hamiltonian cycles.

$m(n+1)$ edges, we deduce that $m(n+1) \equiv m \pmod{2}$. Thus, a HC^{nc} does not exist if both m and n are odd.

Similarly, if there exists a contractible Hamiltonian cycle, then $m(n+1)$ must be even, because there is an equal number of left and right steps, and an equal number of up and down steps. Hence, there is no HC^c if m is odd and n is even. \square

Hamiltonicity of a graph has both a local (every vertex is visited exactly once) and a global (the subgraph is connected) aspect. For a HC^{nc} , the windows belonging to any one of the two regions induce a forest in the window lattice graph $W_{m,n}$. We call the trees in these forests **zero trees** (abbreviated ZTs) or **positive trees** (abbreviated PTs) depending on which region they belong to. Accordingly, their respective windows are called **zero windows** or **positive windows**. Every zero tree contains exactly one window on the first column of $W_{m,n}$ from the set $\{w_{i,1} \mid 1 \leq i \leq m\}$ called the **left root**, and every positive tree contains exactly one window on the last column of $W_{m,n}$ from the set $\{w_{i,n} \mid 1 \leq i \leq m\}$ called the **right root**. For example, the HC^{nc} in Figure 3 has three zero trees with left roots $w_{1,1}$, $w_{3,1}$, and $w_{7,1}$ (striped), and two positive trees with right roots $w_{7,10}$, and $w_{10,10}$ (striped).

For a HC^c , the interior windows (they are marked with 1s in the HC^c in Figure 3) form a tree in $W_{m,n}$, but the exterior windows form a forest of **exterior trees** (abbreviated ETs). Note that only one ET from this forest contains exactly one window on the first column of $W_{m,n}$ (the **left root**), and also exactly one window on the last column of $W_{m,n}$ (the **right root**). We call this ET the **split tree** of the HC. Any ET different from the split tree contains either exactly one left root or exactly one right root, but not both. For example, the HC^c in Figure 3 has a split tree with the left root $w_{1,1}$ and the right root $w_{3,10}$, one ET with the left root $w_{7,1}$, and one ET with the right root $w_{9,10}$. For the purpose of this study, interior tree

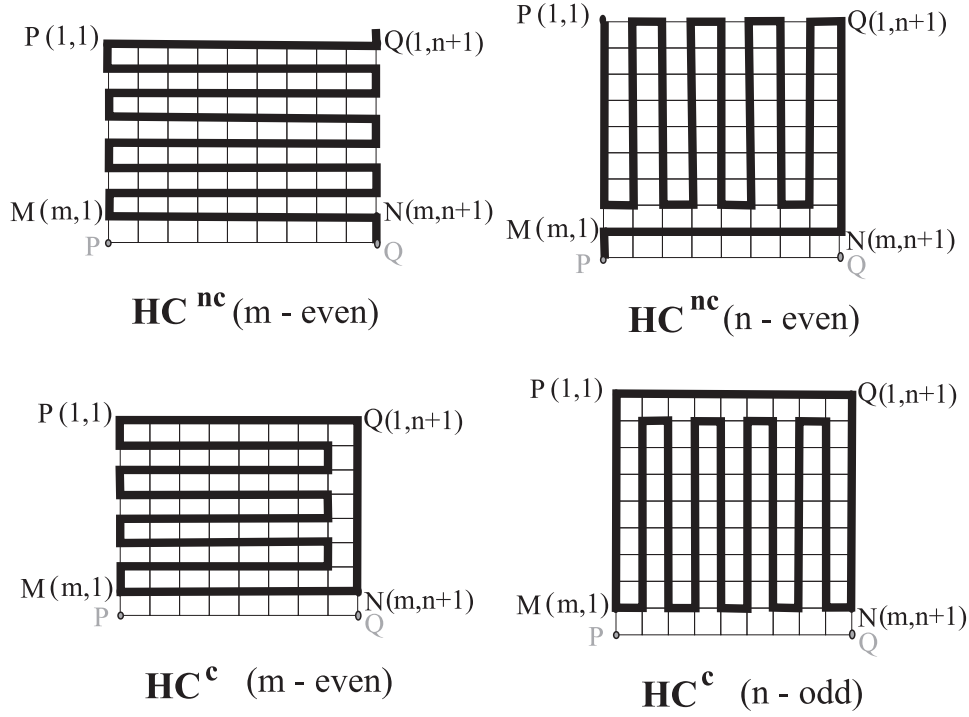


Fig. 4: The construction of the two types of Hamiltonian cycles.

and exterior trees are also called *positive tree* and *zero trees*, and their windows are labeled by 1 and 0, respectively.

We need a few additional definitions to facilitate our discussion.

Definition 1 Given a nonnegative integer word $d_1 d_2 \dots d_m$, its **support** is defined as the binary word $\bar{d}_1 \bar{d}_2 \dots \bar{d}_m$, where

$$\bar{d}_i = \begin{cases} 1 & \text{if } d_i > 0, \\ 0 & \text{if } d_i = 0. \end{cases}$$

The **support** of a nonnegative integer matrix $[d_{i,j}]$ is defined in a similar manner.

Definition 2 The factor u of a word v is called a **b-factor** if it is a block of consecutive letters all of which equal to b . A b -factor of v is said to be **maximal** if it is not a proper factor of another b -factor of v .

The approach described in the next section allows us to simultaneously analyze both types of Hamiltonian cycles.

3 First Characterization of HC

We associate with each Hamiltonian cycle of $C_m \times P_{n+1}$, for both types, a binary matrix $[a_{i,j}]_{m \times n}$, denoted A^{nc} for HC^{nc} , and A^c for HC^c , according to

$$a_{i,j} = \begin{cases} 1 & \text{if } w_{i,j} \text{ is a positive window,} \\ 0 & \text{otherwise.} \end{cases}$$

This matrix satisfies the following necessary conditions which are easy to verify (we adopt the convention that $a_{m+1,j} = a_{1,j}$, for $1 \leq j \leq n$).

Theorem 3.1 *The matrix $[a_{i,j}]_{m \times n}$ satisfies the following conditions:*

[A1] (**First column condition**): *The cyclic word $a_{1,1}a_{2,1} \dots a_{m,1} \in \{0,1\}^m$ has at least one 0 and does not contain the factor 00.*

[A2] (**Adjacency condition**): *For each i and j with $1 \leq i \leq m$ and $1 \leq j \leq n-1$,*

$$(a_{i,j}, a_{i+1,j}, a_{i,j+1}, a_{i+1,j+1}) \notin \{(1,1,1,1), (0,0,0,0), (0,1,1,0), (1,0,0,1)\}.$$

[A3] (**Root condition**): *Each connected component of the subgraph of $W_{m,n}$ induced by the 1-windows has a tree structure, and*

- *For HC^{nc} , every positive tree has exactly one square from the last column of $W_{m,n}$.*
- *For HC^c , there is exactly one positive tree.*

[A4] (**Last column condition**): *The cyclic word $a_{1,n}a_{2,n} \dots a_{m,n} \in \{0,1\}^m$ has*

- *For HC^{nc} , at least one 1, and does not contain the factor 11.*
- *For HC^c , at least one 0, and does not contain the factor 00.*

It is clear that every HC^{nc} (HC^c , resp.) yields exactly one matrix A^{nc} (A^c resp.) that satisfies conditions [A1]–[A4]. The converse is also true.

Theorem 3.2 *Every matrix $[a_{i,j}]_{m \times n}$ with entries from $\{0,1\}$ that satisfies conditions [A1]–[A4] determines a unique HC^{nc} (or HC^c) on $C_m \times P_{n+1}$.*

Proof: The entries in the matrix A can be used to label the windows of $C_m \times P_{n+1}$ with 0 and 1. Construct a subgraph on $C_m \times P_{n+1}$ by forming its edges as follows. Any edge neighboring a 0-window and a 1-window is selected. For A^{nc} , a left edge that joins the vertices $(m,1)$ and $(1,1)$, or the vertices $(i,1)$ and $(i+1,1)$, for $1 \leq i \leq m-1$, is selected if it is adjacent to a 1-window, and a right edge that joins $(m,n+1)$ to $(1,n+1)$ or $(i,n+1)$ to $(i+1,n+1)$, for $1 \leq i \leq m-1$, is selected if it is adjacent to a 0-window. For A^c , an edge on the left or right boundary is selected if it adjacent to a 1-window. For example, for the matrices in Figure 3, the edge between the vertices $(3,n+1)$ and $(4,n+1)$ is selected for A^{nc} but not for A^c .

The conditions [A1], [A2] and [A4] imply that this subgraph of $C_m \times P_{n+1}$ is a 2-factor. The global aspect of Hamiltonicity is provided by condition [A3]. The boundary of the positive region determines the uniqueness of the HC. \square

We note that every possible first column in both A^{nc} and A^c and last column in A^c is a circular binary words of length m with no consecutive 0's, and is different from the word 1^m . Likewise, every possible last column in A^{nc} is a circular binary words of length m with no consecutive 1's, and is different from the word 0^m . It is well-known that the number of such binary words is $L_m - 1$, where L_m is the m th Lucas numbers with $L_0 = 2$, $L_1 = 1$, and $L_{k+1} = L_{k+1} + L_k$ for $k \geq 0$. See, for example, [1].

4 Second Characterization of HC

In this section, we propose an alternate characterization of the HCs on $C_m \times P_{n+1}$. Although it is more complicated, it leads to an effective way to compute the generating functions $\mathcal{H}_m^{nc}(x)$ and $\mathcal{H}_m^c(x)$. In the following discussion, A denotes either A^{nc} or A^c .

Definition 3 Given a fixed positive integer k , two windows $w_{i,l}$ and $w_{j,s}$ that satisfy $a_{i,l} = a_{j,s} = 1$ (from either A^{nc} or A^c) and $l, s \leq k$ are said to be k -SIST (surely in the same tree looking from the k -th column) if and only if they belong to the same component in the subgraph of $W_{m,n}$ induced by $\{w_{p,t} \mid a_{p,t} = 1 \text{ and } t \leq k\}$.

For fixed k , being k -SIST is an equivalence relation on the set $\{w_{i,k} \mid a_{i,k} = 1 \text{ and } 1 \leq i \leq m\}$ and it has at most $\lfloor m/2 \rfloor$ equivalence classes. It is possible that two different classes eventually belong to the same positive tree of a Hamiltonian cycle on the entire cylindrical surface of $C_m \times P_{n+1}$. In other words, two windows that are not k -SIST could become ℓ -SIST for some integer $\ell > k$. However, we cannot tell whether it is true just from the first k columns of the matrix A .

Let $C^+ = \{2, 3, \dots, \lfloor m/2 \rfloor + 1\}$. For any HC^{nc} or HC^c , we associate to the matrix A^{nc} or A^c from the first characterization a second matrix $[b_{i,j}]_{m \times n}$, denoted B^{nc} or B^c , where $b_{i,j} \in C^+ \cup \{0\}$, in the following way (see Figure 5). For each j :

- If $a_{i,j} = 0$, then $b_{i,j} = 0$.
- Partition the positive windows in the j th column into j -SIST equivalence classes, label all the windows within each equivalence class 2, 3, \dots , according to the order in which the equivalence classes first appear within the j th column, from top to bottom.

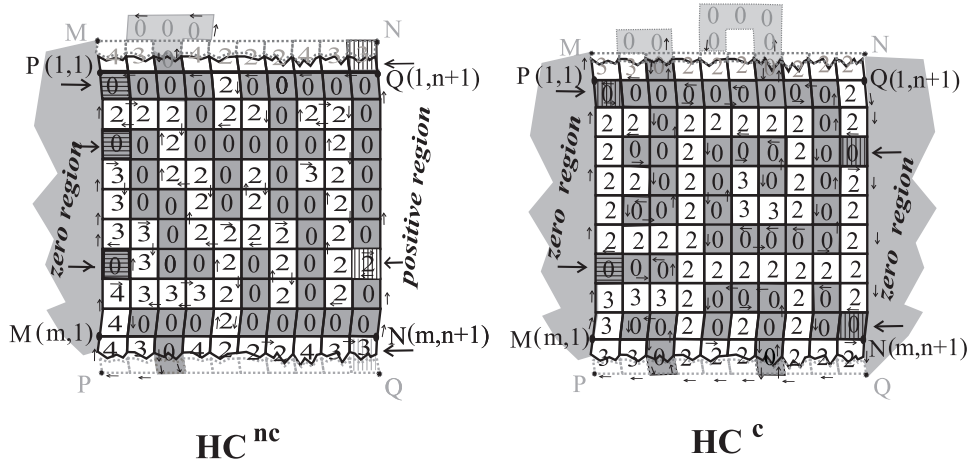


Fig. 5: The labeling of the windows of a HC^{nc} on $C_{10} \times P_{11}$, and a HC^c on $C_{10} \times P_{11}$.

Theorem 4.1 The matrix $B = [b_{i,j}]_{m \times n}$ (either B^{nc} or B^c) satisfies the following properties (we adopt the convention $b_{m+1,j} = b_{1,j}$, and $b_{0,j} = b_{m,j}$, for $1 \leq j \leq n$):

[B1] The first column $b_{1,1}b_{2,1} \dots b_{m,1}$ is either

$$02^{d_1}03^{d_2}04^{d_3} \dots 0(p+1)^{d_p}, \quad p + \sum_{i=1}^p d_i = m,$$

or

$$2^{d_1}03^{d_2}04^{d_3} \dots 0(p+1)^{d_p}02^{(m-p-d_1-d_2-\dots-d_p)}, \quad p + \sum_{i=1}^p d_i \leq m,$$

where $p \geq 1$ is the number of 0s and $d_i > 0$ for $1 \leq i \leq p$.

[B2] The support of the matrix B , that is, the matrix $[a_{i,j}]_{m \times n}$, satisfies the adjacency condition [A2].

[B3] For $1 \leq k \leq n$, the k th column of the matrix B satisfies these conditions:

- (a) If $b_{i,k} > 0$, where $1 \leq i \leq m$, then $b_{i-1,k}, b_{i+1,k} \in \{b_{i,k}, 0\}$.
- (b) If $b_{p_1,k}, b_{p_2,k}, \dots, b_{p_l,k}$, where $l \leq \lfloor m/2 \rfloor$, and $p_1 < p_2 < \dots < p_l$, are the first appearance of the elements from C^+ in the k th column, then $b_{p_i,k} = i + 1$.
- (c) If $k \geq 2$, $1 \leq i, j \leq m$, $i \neq j$, $b_{i,k-1} = b_{j,k-1}$, and $a_{i,k} = a_{j,k} = a_{i,k-1} = a_{j,k-1} = 1$, then $b_{i,k} = b_{j,k}$.
- (d) If $k \geq 2$, $1 \leq i, j \leq m$, $i \neq j$, $b_{i,k-1} = b_{j,k-1}$, $b_{i,k} = b_{j,k} = b$, and $a_{i,k-1} = a_{i,k} = 1$, then the k th column does not contain any b -factor that contains both $b_{i,k}$ and $b_{j,k}$.
- (e) If $k \geq 2$ and if v and u are two different maximal nonzero b -factors in the k th column, then there is exactly one sequence $v = v_1, v_2, \dots, v_p = u$ of $p > 1$ different maximal b -factors in the k th column with the property that for every i with $1 \leq i \leq p-1$, in the $(k-1)$ th column, there exists exactly one letter $b_{j_i,k-1}$ with $a_{j_i,k-1} = a_{j_i,k}$ for which $b_{j_i,k} \in v_i$, and there exists exactly one letter $b_{s_{i+1},k-1}$ with $a_{s_{i+1},k-1} = a_{s_{i+1},k}$ for which $b_{s_{i+1},k} \in v_{i+1}$ and $b_{j_i,k-1} = b_{s_{i+1},k-1}$; and $j_i \neq s_i$ for $1 < i < p$ (see Figure 6).
- (f) For $k \geq 2$ and for each number $b \in C^+$ that appears in the $(k-1)$ th column, there must exist an integer i , where $1 \leq i \leq m$, for which $b_{i,k-1} = b$ and $b_{i,k} > 0$.
- (g) Every column has both positive and zero entries.

[B4] The last column $b_{1,n}b_{2,n} \dots b_{m,n}$ is

- For HC^{nc} ,

$$0^{d_1}20^{d_2}30^{d_3} \dots p0^{d_p}(p+1)^{m-p-d_1-d_2-\dots-d_p}, \quad p + \sum_{i=1}^p d_i = m,$$

or

$$20^{d_1}30^{d_2}40^{d_3} \dots (p+1)0^{d_p}, \quad p + \sum_{i=1}^p d_i \leq m,$$

where $p \geq 1$ is the number of positive integers and $d_i > 0$ for $1 \leq i \leq p$.

- For HC^c ,

$$2^{d_1}02^{d_2}02^{d_3} \dots 02^{d_p}02^{m-p-d_1-d_2-\dots-d_p}, \quad p + \sum_{i=1}^p d_i = m,$$

or

$$02^{d_1}02^{d_2}02^{d_3} \dots 02^{d_p}, \quad p + \sum_{i=1}^p d_i \leq m,$$

where $p \geq 1$ is the number of 0s and $d_i > 0$ for $1 \leq i \leq p$.

Proof: First, a few remarks.

- [B1] and [B2] follow from the definition of the matrix B .
- [B3a]: Two windows belonging to the same equivalence class must be associated with the same number.
- [B3b]: This follows from the definition of the matrix B .
- [B3c]: If $w_{i,k-1}$ and $w_{j,k-1}$ are $(k-1)$ -SIST, and if the windows $w_{i,k}$, $w_{j,k}$, $w_{i,k-1}$ and $w_{j,k-1}$ are from the positive region, then the windows $w_{i,k}$ and $w_{j,k}$ must be k -SIST.
- [B3d]: If the opposite is true, we would obtain a cycle in a positive tree, which is impossible.
- [B3e]: If we can conclude by knowing the first k columns that v and u are in the same tree, then there is exactly one path from v to u in their positive tree via some windows from the previous column, that is, the $(k-1)$ th column.
- [B3f]: Every positive tree must “reach” the last column.
- [B3g]: For a HC^{nc} , the unique path in $W_{m,n}$ starting in a positive window from the first column and finishing in the last column must cross every column. For a HC^c , the unique split tree must cross every column as well. Furthermore, the occurrence of a column with no zero window would imply that the corresponding subgraph in $C_m \times P_{n+1}$ is not connected, which is impossible.
- [B4]: This follows from the definition of the matrix B .

Based on these remarks, it is not difficult to verify the properties listed in the theorem. \square

Theorem 4.2 Every integer matrix $B = [b_{i,j}]_{m \times n}$ with entries from $C^+ \cup \{0\}$ satisfying properties [B1]–[B4] determines a unique HC on $C_m \times P_{n+1}$.

Proof: It suffices to show that the support of B (which could be either B^{nc} or B^c) satisfies conditions [A1]–[A4] in Theorem 3.1. It is clear that properties [B1], [B2] and [B4] imply conditions [A1], [A2] and [A4], respectively. Properties [B3d] and [B3e] yield the forest structure for the subgraph of $W_{m,n}$ induced by positive windows (since no cycle can occur). The properties [B3c], [B3f] and [B4] for B^{nc} assert that every positive tree in $W_{m,n}$ has exactly one right root. For B^c , the property [B3f] implies that for every positive window there exists a path starting from this window and finishing in the last column of $W_{m,n}$, and the property [B4] guarantees that the subgraph of $W_{m,n}$ induced by the positive windows is connected. \square

The computation of the generating functions

$$\mathcal{H}_m^{nc}(x) = \sum_{n \geq 0} h_m^{nc}(n+1)x^n \quad \text{and} \quad \mathcal{H}_m^c(x) = \sum_{n \geq 0} h_m^c(n+1)x^n$$

is rather routine (see Theorem 4.7.2 in [18]). It is obvious that

$$\mathcal{H}_m(x) = \sum_{n \geq 0} h_m(n+1)x^n = \mathcal{H}_m^{nc}(x) + \mathcal{H}_m^c(x). \quad (1)$$

These generating functions are rational functions. Their denominators are determined by the characteristic polynomials of the adjacency matrices. Table 1 displays, for $3 \leq m \leq 10$, the numbers of vertices in \mathcal{F}_m , \mathcal{D}_m and \mathcal{D}_m^* , as well as the degrees of the denominators in these generating functions, which determine the orders of the recurrence relations for h_m^{nc} and h_m^c .

We find an interesting upper bound of $|V(\mathcal{D}_m)|$. A column in the matrix $[b_{i,j}]_{m \times n}$ can be viewed as a word. Let its maximal nonzero b -factors, in the order of their appearance, be p_1 -factor, p_2 -factor, \dots , p_k -factor. Call $p_1 p_2 \dots p_k$ a **positive truncated word**. For example, the positive truncated words correspond to the 2nd and 6th columns of B^c in Figure 5 are 22233 and 2322, respectively. Every truncated word v has two properties:

- If a letter $s \geq 3$ appears in v , then, in accordance with the property [B3b], each number from $\{2, \dots, s-1\}$ must have appeared at least once before it. In other words, if we remove the duplicated letters, the remaining letters will form the word 234...
- If $abab$ is a subsequence of the word v , then $a = b$ (because of the properties [B3e] and [B1]).

A word over the alphabet $\{2, \dots, k+1\}$ that possesses the above-mentioned properties is called a **color word**. The number of color words of length k is the Catalan number $C_k = \frac{1}{k+1} \binom{2k}{k}$, see [3, 16]. Using a relation between Catalan and Motzkin numbers described in [6], we obtain the following corollary.

Corollary 5.1 *An upper bound on the number of vertices of digraph \mathcal{D}_m is*

$$|V(\mathcal{D}_m)| \leq 2 \sum_{k=1}^{\lfloor m/2 \rfloor} \binom{m}{2k} C_k = 2(M_m - 1),$$

where C_m is the m th Catalan number and M_m is m th Motzkin number.

In light of Corollary 5.1, we would like to remark that we could use Motzkin words to encode the columns. See, for example, [19].

6 Computational Results

Based on the discussion in the previous section, we use Pascal programs to compute the adjacency matrices of the multidigraphs \mathcal{D}_m^* , from which we obtain $\mathcal{H}_m^{nc}(x)$ and $\mathcal{H}_m^c(x)$. The results are summarized in Table 1. Notice that the numbers $|V(\mathcal{D}_m)|$ and $2(M_m - 1)$ are equal when m is odd.

m	3	4	5	6	7	8	9	10
$ \mathcal{F}_m = L_m - 1$	3	6	10	17	28	46	75	122
$2(M_m - 1)$	6	16	40	100	252	644	1668	4374
$ V(\mathcal{D}_m) $	6	12	40	64	252	364	1668	2234
$ V(\mathcal{D}_m^*) $	2	4	8	14	30	44	128	172
deg den. $\mathcal{H}_m(x)$	1	2	3	7	12	20	51	74
deg den. $\mathcal{K}_m(x)$	1	2	3	6	12	20	51	67
deg den. $\mathcal{H}_m^{nc}(x), \mathcal{H}_m^c(x)$	2	4	6	13	24	40	102	141

Tab. 1: The computational results from Pascal programs.

Since $\mathcal{H}_m^{nc}(x)$ and $\mathcal{H}_m^c(x)$ are derived from the same transfer matrix, their denominators are identical. After adding the two rational functions to form $\mathcal{H}_m(x)$, the new denominator may have a lesser degree. In fact, numerical data reveal that the degree is reduced by roughly one-half, see Table 1.

Upon further examination of the factorization of the denominator, we conclude that a better way to study them is to introduce the function

$$\mathcal{K}_m(x) = \mathcal{H}_m^c(x) - \mathcal{H}_m^{nc}(x), \quad (2)$$

such that, together with (1),

$$\mathcal{H}_m^{nc}(x) = \frac{1}{2}(\mathcal{H}_m(x) - \mathcal{K}_m(x)), \quad (3)$$

$$\mathcal{H}_m^c(x) = \frac{1}{2}(\mathcal{H}_m(x) + \mathcal{K}_m(x)). \quad (4)$$

Since both $\mathcal{H}_m(x)$ and $\mathcal{K}_m(x)$ are rational functions, we can express them as

$$\mathcal{H}_m(x) = \overline{\mathcal{H}}_m(x) + \frac{p_m(x)}{q_m(x)} \quad \text{and} \quad \mathcal{K}_m(x) = \overline{\mathcal{K}}_m(x) + \frac{r_m(x)}{s_m(x)},$$

for some polynomials $\overline{\mathcal{H}}_m(x)$, $\overline{\mathcal{K}}_m(x)$, $p_m(x)$, $q_m(x)$, $r_m(x)$ and $s_m(x)$, such that $\deg(p_m) < \deg(q_m)$ and $\deg(r_m) < \deg(s_m)$.

The denominator $q_m(x)$ of the generating function $\mathcal{H}_m(x)$ provides important information about the numbers $h_m(n)$. Let its degree be d_m . Then $\chi_m(t) = t^{d_m} q_m(1/t)$ is the characteristic polynomial which determines the recurrence relation that $h_m(n)$ satisfies. It has d_m nonzero roots (the characteristic roots) over \mathbb{C} , name them $\lambda_{m,i}$ so that $|\lambda_{m,1}| \geq |\lambda_{m,2}| \geq \dots \geq |\lambda_{m,d_m}|$. We can write

$$q_m(x) = \prod_{i=1}^{d_m} (1 - \lambda_{m,i}x).$$

Note that the zeros of $q_m(x)$ are $\lambda_{m,i}^{-1}$. For the sake of brevity, we shall still call $\lambda_{m,i}$ s the **characteristic roots** of $q_m(x)$. It is a routine exercise to show that, if $\lambda_{m,i}$ s are simple (hence distinct) roots, then

$$\frac{p_m(x)}{q_m(x)} = \sum_{i=1}^{d_m} \frac{\alpha_i}{1 - \lambda_{m,i}x},$$

so that for sufficiently large n

$$h_m(n+1) = \sum_{i=1}^{d_m} \alpha_i \lambda_{m,i}^n,$$

where $\alpha_i = -\lambda_{m,i} p_m(\lambda_{m,i}^{-1}) / q'_m(\lambda_{m,i}^{-1})$. The solution is more complicated if some of the $\lambda_{m,i}$ s are repeated roots. Nonetheless, if $\lambda_{m,1}$ is a simple positive root such that $\lambda_{m,1} > |\lambda_{m,2}|$, then

$$h_m(n+1) \sim \alpha_1 \lambda_{m,1}^n,$$

in which the formula for α_1 given above still holds. See the following sections for illustrations of our discussion.

6.1 The Thin Grid Cylinder $C_2 \times P_{n+1}$

We find $h_2^{nc}(n) = 2$ and $h_2^c(n) = 2$, hence $h_2(n) = 4$, for all $n \geq 1$.

6.2 The Thin Grid Cylinder $C_3 \times P_{n+1}$

Let $V(\mathcal{D}_3) = \{v_1, v_2, \dots, v_6\}$. We obtain the following:

$$\begin{array}{l} v_1 = (2, 2, 0) \\ v_2 = (2, 0, 2) \\ v_3 = (0, 2, 2) \\ v_4 = (0, 0, 2) \\ v_5 = (0, 2, 0) \\ v_6 = (2, 0, 0) \end{array} \quad T_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} \mathcal{F}_3 = \{v_1, v_2, v_3\} \\ \mathcal{L}_3^{nc} = \{v_4, v_5, v_6\} \\ \mathcal{L}_3^c = \{v_1, v_2, v_3\} \end{array}$$

$$T_3^* = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$$

$$\begin{array}{ll} h_3^{nc}(2k-1) = 0, & k \geq 1 \\ h_3^{nc}(2) = 6 \\ h_3^{nc}(4) = 24 \\ h_3^{nc}(6) = 96 \\ h_3^{nc}(8) = 384 \end{array} \quad \begin{array}{ll} h_3^c(2k) = 0, & k \geq 1 \\ h_3^c(1) = 3 \\ h_3^c(3) = 12 \\ h_3^c(5) = 48 \\ h_3^c(7) = 192 \end{array}$$

The characteristic polynomial of T_3^* is $x^2 - 4$. Because of Cayley-Hamilton theorem, we obtain the recurrence relations $h_3^{nc}(n) = 4h_3^{nc}(n-2)$ and $h_3^c(n) = 4h_3^c(n-2)$. The generating functions are

$$\begin{aligned} \mathcal{H}_3^{nc}(x) &= \frac{6x}{1-4x^2} = \frac{3}{2(1-2x)} - \frac{3}{2(1+2x)}, \\ \mathcal{H}_3^c(x) &= \frac{3}{1-4x^2} = \frac{3}{2(1-2x)} + \frac{3}{2(1+2x)}. \end{aligned}$$

Therefore,

$$\mathcal{H}_3(x) = \frac{3}{1-2x} \quad \text{and} \quad \mathcal{K}_3(x) = \frac{3}{1+2x}.$$

The denominator of $\mathcal{H}_3(x)$ yields the recurrence relation

$$h_3(n) = 2h_3(n-1), \quad n \geq 2.$$

Since $\frac{3}{1-2x} = 3 \sum_{k=0}^{\infty} 2^k x^k$, we obtain the following simple formula for $h_3(n)$.

Theorem 6.1 For $n \geq 1$, the number of Hamiltonian cycles in $C_3 \times P_{n+1}$ is

$$h_3(n) = 3 \cdot 2^{n-1}.$$

6.3 The Thin Grid Cylinder $C_4 \times P_{n+1}$

Let $V(\mathcal{D}_4) = \{v_1, v_2, \dots, v_{12}\}$. We obtain the following:

$$\begin{array}{l}
 v_1 = (2, 2, 2, 0) \\
 v_2 = (2, 2, 0, 2) \\
 v_3 = (2, 0, 2, 2) \\
 v_4 = (2, 0, 3, 0) \\
 v_5 = (0, 2, 2, 2) \\
 v_6 = (0, 2, 0, 3) \\
 v_7 = (0, 0, 0, 2) \\
 v_8 = (0, 0, 2, 0) \\
 v_9 = (0, 2, 0, 0) \\
 v_{10} = (2, 0, 0, 0) \\
 v_{11} = (2, 0, 2, 0) \\
 v_{12} = (0, 2, 0, 2)
 \end{array}
 \quad
 T_4 =
 \begin{bmatrix}
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\
 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\
 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\
 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1
 \end{bmatrix}$$

$$\begin{array}{l}
 \mathcal{F}_4 = \{v_1, v_2, \dots, v_6\} \\
 \mathcal{L}_4^{nc} = \{v_4, v_6, v_7, v_8, v_9, v_{10}\} \\
 \mathcal{L}_4^c = \{v_1, v_2, v_3, v_5, v_{11}, v_{12}\}
 \end{array}
 \quad
 \begin{array}{l}
 h_4^{nc}(1) = 2 \\
 h_4^{nc}(2) = 14 \\
 h_4^{nc}(3) = 34 \\
 h_4^{nc}(4) = 170 \\
 h_4^{nc}(5) = 530 \\
 h_4^{nc}(6) = 2230 \\
 h_4^{nc}(7) = 7714 \\
 h_4^{nc}(8) = 30258 \\
 h_4^{nc}(9) = 109378 \\
 h_4^{nc}(10) = 416766 \\
 h_4^{nc}(11) = 1534722 \\
 h_4^{nc}(12) = 5777562 \\
 h_4^{nc}(13) = 21441682
 \end{array}
 \quad
 \begin{array}{l}
 h_4^c(1) = 4 \\
 h_4^c(2) = 8 \\
 h_4^c(3) = 48 \\
 h_4^c(4) = 136 \\
 h_4^c(5) = 612 \\
 h_4^c(6) = 2032 \\
 h_4^c(7) = 8192 \\
 h_4^c(8) = 29104 \\
 h_4^c(9) = 112164 \\
 h_4^c(10) = 410040 \\
 h_4^c(11) = 1550960 \\
 h_4^c(12) = 5738360 \\
 h_4^c(13) = 21536324
 \end{array}$$

$$T_4^* =
 \begin{bmatrix}
 0 & 0 & 3 & 1 \\
 2 & 1 & 0 & 0 \\
 3 & 1 & 0 & 0 \\
 0 & 0 & 2 & 1
 \end{bmatrix}$$

The generating functions are:

$$\begin{aligned}
 \mathcal{H}_4^{nc}(x) &= \frac{2(1 + 5x - 5x^2 + x^3)}{(1 - 4x + x^2)(1 + 2x - x^2)} = \frac{3 - x}{1 - 4x + x^2} - \frac{1 - x}{1 + 2x - x^2}, \\
 \mathcal{H}_4^c(x) &= \frac{4}{(1 - 4x + x^2)(1 + 2x - x^2)} = \frac{3 - x}{1 - 4x + x^2} + \frac{1 - x}{1 + 2x - x^2},
 \end{aligned}$$

from which we obtain

$$\mathcal{H}_4(x) = \frac{2(3 - x)}{1 - 4x + x^2} \quad \text{and} \quad \mathcal{K}_4(x) = \frac{2(1 - x)}{1 + 2x - x^2},$$

and the recurrence relation

$$h_4(n) = 4h_4(n-1) - h_4(n-2), \quad n \geq 3.$$

After decomposing into partial fractions, we find

$$\frac{2(3-x)}{1-4x+x^2} = \frac{9+5\sqrt{3}}{3} \cdot \frac{1}{1-(2+\sqrt{3})x} + \frac{9-5\sqrt{3}}{3} \cdot \frac{1}{1-(2-\sqrt{3})x}.$$

This leads to the next result.

Theorem 6.2 For $n \geq 1$, the number of Hamiltonian cycles in $C_4 \times P_{n+1}$ is

$$h_4(n) = \frac{1}{3} [(9+5\sqrt{3})(2+\sqrt{3})^{n-1} + (9-5\sqrt{3})(2-\sqrt{3})^{n-1}],$$

and $h_4(n) \sim \frac{1}{3} (9+5\sqrt{3})(2+\sqrt{3})^{n-1}$.

6.4 The Thin Grid Cylinder $C_5 \times P_{n+1}$

We find $|V(\mathcal{D}_5)| = 40$, $V(\mathcal{D}_5^*) = \{v_1, \dots, v_8\}$, and

$$\begin{array}{l} v_1 = (2, 2, 2, 2, 0) \\ v_2 = (2, 2, 0, 3, 0) \\ v_3 = (0, 0, 0, 0, 2) \\ v_4 = (0, 0, 2, 0, 3) \\ v_5 = (2, 0, 2, 0, 0) \\ v_6 = (2, 0, 0, 2, 2) \\ v_7 = (2, 0, 2, 2, 0) \\ v_8 = (2, 0, 0, 0, 2) \end{array} \quad T_5^* = \begin{bmatrix} 0 & 0 & 4 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 2 & 0 & 0 \\ 4 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 3 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 2 & 0 & 0 \end{bmatrix}$$

$$\begin{array}{ll} h_5^{nc}(2k-1) = 0, & k \geq 1 \\ h_5^{nc}(2) = 30 & h_5^c(2k) = 0, \quad k \geq 1 \\ h_5^{nc}(4) = 850 & h_5^c(1) = 5 \\ h_5^{nc}(6) = 24040 & h_5^c(3) = 160 \\ h_5^{nc}(8) = 680040 & h_5^c(5) = 4520 \\ h_5^{nc}(10) = 19236840 & h_5^c(7) = 127860 \\ & h_5^c(9) = 3616880 \end{array}$$

We obtain

$$\begin{aligned} \mathcal{H}_5^{nc}(x) &= \frac{10x(x^2+3)}{1-28x^2-8x^4-4x^6} = \frac{5}{1-6x+4x^2-2x^3} - \frac{5}{1+6x+4x^2+2x^3}, \\ \mathcal{H}_5^c(x) &= \frac{5(4x^2+1)}{1-28x^2-8x^4-4x^6} = \frac{5}{1-6x+4x^2-2x^3} + \frac{5}{1+6x+4x^2+2x^3}. \end{aligned}$$

Hence,

$$\mathcal{H}_5(x) = \frac{10}{1-6x+4x^2-2x^3} \quad \text{and} \quad \mathcal{K}_5(x) = \frac{10}{1+6x+4x^2+2x^3}.$$

Due to its complexity, we will not display the explicit formula for $h_5(n)$. Numerically, $\lambda_{5,1} \approx 5.31863$, and $\lambda_{5,2}, \lambda_{5,3} \approx 0.34069 \pm 0.50987i$.

6.5 The Thin Grid Cylinder $C_6 \times P_{n+1}$

$$\begin{aligned}
\mathcal{H}_6^{nc}(x) &= 2 + 62x + 278x^2 + 4178x^3 + 27710x^4 + 314354x^5 + 2468810x^6 + 24770708x^7 \\
&\quad + 210413420x^8 + 1998760352x^9 + 17601771968x^{10} + 163119159176x^{11} \\
&\quad + 1460403914672x^{12} + 13382718140000x^{13} + 120722781112208x^{14} \\
&\quad + 1100628776882000x^{15} + 9962793339446672x^{16} \\
&\quad + 90619491133658576x^{17} + 821568683907144752x^{18} \\
&\quad + 7464893093725073072x^{19} + 67726216376743239056x^{20} + \dots, \\
\mathcal{H}_6^c(x) &= 6 + 24x + 498x^2 + 2832x^3 + 35964x^4 + 263736x^5 + 2779014x^6 + 22869384x^7 \\
&\quad + 222067212x^8 + 1927331160x^9 + 18039580560x^{10} + 160435712688x^{11} \\
&\quad + 1476851478768x^{12} + 13281906604320x^{13} + 121340682078768x^{14} \\
&\quad + 1096841495972016x^{15} + 9986006600900208x^{16} + 90477210822238320x^{17} \\
&\quad + 822440758133272176x^{18} + 7459547916670820976x^{19} \\
&\quad + 67758978401907276048x^{20} + \dots, \\
\mathcal{H}_6(x) &= \frac{2(4 + 7x + x^2 - 27x^3 - 26x^4 - 20x^5 - 3x^6)}{1 - 9x - 10x^3 + 28x^4 + 36x^5 + 32x^6 + 12x^7}, \\
\mathcal{K}_6(x) &= \frac{2(2 - 11x + 14x^2 - 11x^3 - x^4 + x^5)}{1 + 4x - 10x^2 + 16x^3 - 16x^4 + 4x^5 + 4x^6}.
\end{aligned}$$

The denominator $q_6(x)$ has seven simple roots, three real and four complex, and $\lambda_{6,1} \approx 9.07807$.

6.6 The Thin Grid Cylinder $C_7 \times P_{n+1}$

We find $\mathcal{H}_7(x) = p_7(x)/q_7(x)$, and $\mathcal{K}_7(x) = r_7(x)/s_7(x)$, where

$$\begin{aligned}
\mathcal{H}_7^{nc}(x) &= 126x + 18452x^3 + 2861964x^5 + 444486280x^7 + 69048910000x^9 \\
&\quad + 10726732430288x^{11} + 1666401898058352x^{13} + 258876295158900832x^{15} \\
&\quad + 40216553455854426560x^{17} + 6247660438430706481984x^{19} + \dots, \\
\mathcal{H}_7^c(x) &= 7 + 1484x^2 + 229698x^4 + 35663964x^6 + 5539931796x^8 + 860620499760x^{10} \\
&\quad + 133697577587000x^{12} + 20769976722986288x^{14} + 3226625529605854320x^{16} \\
&\quad + 501257787787122948736x^{18} + 77870632467402116097056x^{20} + \dots, \\
p_7(x) &= 7(1 + 6x - 22x^2 - 120x^3 - 178x^4 + 72x^5 + 580x^6 + 616x^7 + 264x^8 + 72x^9 + 16x^{10}), \\
q_7(x) &= 1 - 12x - 18x^2 + 112x^3 + 440x^4 + 772x^5 + 196x^6 \\
&\quad - 2064x^7 - 3724x^8 - 2040x^9 - 496x^{10} - 128x^{11} + 16x^{12},
\end{aligned}$$

and $r_7(x) = p_7(-x)$, and $s_7(x) = q_7(-x)$.

6.7 The Thin Grid Cylinder $C_8 \times P_{n+1}$

Again, we have $\overline{\mathcal{H}}_8(x) = \overline{\mathcal{K}}_8(x) = 0$,

$$\begin{aligned}
\mathcal{H}_8^{nc}(x) &= 2 + 254x + 1794x^2 + 82138x^3 + 1012930x^4 + 30717374x^5 + 481369234x^6 \\
&\quad + 12070287370x^7 + 214585144402x^8 + 4886085696654x^9 \\
&\quad + 92880601782338x^{10} + 2011688161424970x^{11} + 39622707294281746x^{12} \\
&\quad + 836009740378418718x^{13} + 16778455639135020178x^{14} + \dots, \\
\mathcal{H}_8^c(x) &= 8 + 64x + 4320x^2 + 44288x^3 + 1575288x^4 + 22337664x^5 + 605992784x^6 \\
&\quad + 10215798448x^7 + 242178636928x^8 + 4475508186384x^9 \\
&\quad + 98989761676840x^{10} + 1920787160180224x^{11} + 40975264449253872x^{12} \\
&\quad + 815884428197037360x^{13} + 17077909293201385648x^{14} + \dots, \\
p_8(x) &= 2(5 + 44x - 430x^2 + 33x^3 + 93x^4 + 1471x^5 + 4596x^6 + 6807x^7 \\
&\quad + 8263x^8 + 2751x^9 - 2482x^{10} - 5126x^{11} - 4711x^{12} - 2094x^{13} \\
&\quad - 1406x^{14} + 450x^{15} + 580x^{16} - 132x^{17} + 32x^{18} + 40x^{19}), \\
q_8(x) &= 1 - 23x + 34x^2 + 345x^3 + 218x^4 - 22x^5 - 2919x^6 - 5041x^7 \\
&\quad - 8806x^8 - 11998x^9 - 5873x^{10} + 1318x^{11} + 4467x^{12} + 11373x^{13} \\
&\quad + 3848x^{14} - 584x^{15} + 1018x^{16} - 928x^{17} + 84x^{18} + 72x^{19} - 40x^{20}, \\
r_8(x) &= 2(3 - 80x + 476x^2 - 1143x^3 + 303x^4 + 4917x^5 - 8670x^6 - 2291x^7 \\
&\quad + 19477x^8 - 13315x^9 - 16780x^{10} + 19224x^{11} + 6103x^{12} - 9974x^{13} \\
&\quad - 1352x^{14} + 3926x^{15} - 1796x^{16} + 644x^{17} - 168x^{18} + 16x^{19}), \\
s_8(x) &= 1 + 5x - 104x^2 + 529x^3 - 1548x^4 + 1830x^5 + 3915x^6 - 13527x^7 \\
&\quad + 7182x^8 + 20914x^9 - 31027x^{10} - 9214x^{11} + 35037x^{12} + 1205x^{13} \\
&\quad - 19590x^{14} + 890x^{15} + 5770x^{16} - 2048x^{17} + 588x^{18} - 184x^{19} + 16x^{20}.
\end{aligned}$$

6.8 The Thin Grid Cylinder $C_9 \times P_{n+1}$

$$\begin{aligned}
\mathcal{H}_9^{nc}(x) &= 510x + 351258x^3 + 276018090x^5 + 218915964618x^7 + 173923080282474x^9 \\
&\quad + 138226113213225360x^{11} + 109864493967924549384x^{13} \\
&\quad + 87323767337933601800838x^{15} + 69407973132514050824027916x^{17} \\
&\quad + 55167927811346067821770238916x^{19} \\
&\quad + 43849442381504976630009404305836x^{21} + \dots, \\
\mathcal{H}_9^c(x) &= 9 + 12348x^2 + 9806292x^4 + 7769376972x^6 + 6169925169414x^8 \\
&\quad + 4903042542453720x^{10} + 3896923927019062734x^{12} \\
&\quad + 3097380080814655131414x^{14} + 2461902328199084994926838x^{16} \\
&\quad + 1956807009306757665486727506x^{18} \\
&\quad + 1555340096869096304430909957438x^{20} + \dots.
\end{aligned}$$

We find $\overline{\mathcal{H}}_9(x) = \overline{\mathcal{K}}_9(x) = 0$. Like the cases of $m = 3, 5, 7$, we also have $r_9(x) = p_9(-x)$ and $s_9(x) = q_9(-x)$. However, since $\deg(p_9) + 1 = \deg(q_9) = 51$, we will not attempt to list these polynomials in their entirety.

6.9 The Thin Grid Cylinder $C_{10} \times P_{n+1}$

$$\begin{aligned}\mathcal{H}_{10}^{nc}(x) &= 2 + 1022x + 10652x^2 + 1505612x^3 + 32718482x^4 + 2701992092x^5 + 79977736982x^6 \\ &\quad + 5099841986502x^7 + 179765502917052x^8 + 9933064485778002x^9 \\ &\quad + 387981888303174142x^{10} + 19745599426500473672x^{11} \\ &\quad + 819563054782862759352x^{12} + 39759941758256449144532x^{13} \\ &\quad + 1710706207634346787583712x^{14} + 80696804239003472593910602x^{15} + \dots \\ \mathcal{H}_{10}^c(x) &= 10 + 160x + 34850x^2 + 621720x^3 + 62999960x^4 + 1641664580x^5 + 116791523380x^6 \\ &\quad + 3817933082020x^7 + 224360971248960x^8 + 8381173203185000x^9 \\ &\quad + 441980748032029010x^{10} + 17866610320162579120x^{11} \\ &\quad + 884945074721799980580x^{12} + 37484874131377414126080x^{13} \\ &\quad + 178987055278304706976120x^{14} + 77942162101044243981212480x^{15} + \dots\end{aligned}$$

We close by mentioning that $\overline{\mathcal{H}}_{10}(x) = \overline{\mathcal{K}}_{10}(x) = 0$, $\deg(p_{10}) + 1 = \deg(q_{10}) = 74$, and $\deg(r_{10}) + 1 = \deg(s_{10}) = 67$.

7 Asymptotic Values

Let ρ_m be the radius of convergence for $\mathcal{H}_m(x)$. The coefficients of $\mathcal{H}_m(x)$ are non-negative, Pringsheim's Theorem (see, for example, [8]) states that it has a singularity at $x = \rho_m$. Since we assume that $q_m(x) = \prod_{i=1}^{d_m} (1 - \lambda_{m,i}x)$, where $|\lambda_{m,1}| \geq |\lambda_{m,2}| \geq \dots \geq |\lambda_{m,d_m}| \neq 0$, one of the characteristic roots with the largest moduli must be real, positive, and equal to $1/\rho_m$. We may assume it is $\lambda_{m,1}$. For brevity, we denote it θ_m . If $\theta_m = \lambda_{m,1} > |\lambda_{m,2}|$, then θ_m is the dominant root, and

$$h_m(n+1) \sim a_m \theta_m^n,$$

where $a_m = -\theta_m p_m(\theta_m^{-1})/q'_m(\theta_m^{-1})$. Do we always have $|\lambda_{m,1}| > |\lambda_{m,2}|$? The fact that the transfer matrix T_m^* is nonnegative points to the Perron-Frobenius theorem for an answer.

Let M be a nonnegative square matrix. We say that M is irreducible if, for every i and j , there exists a positive integer $k = k(i, j)$ such that $(M^k)_{ij} > 0$. This is equivalent to saying that the multidigraph G_M with adjacency matrix M is strongly connected. The matrix M is said to be primitive if $A^\gamma > 0$ for some positive integer γ . For example, the matrix

$$T_3^* = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$$

is irreducible but not primitive, and T_4^* is primitive because $T_4^{*2} > 0$. The period or order of cyclicity of M , labeled by g , can be defined as the greatest common divisor of the lengths of the directed cycles in G_M [12]. From the Perron-Frobenius theory (see, for example, [12]), g is the number of eigenvalues of

M having the largest modulus. In particular, a primitive matrix is an irreducible nonnegative matrix with $g = 1$, it has exactly one dominant characteristic root. We find that, for $3 \leq m \leq 10$, the transfer matrix T_m^* is irreducible, but it is primitive only when m is even. Accordingly, we shall study the cases of odd and even m separately.

When m is odd, Theorem 2.1 implies that

$$h_m(n) = \begin{cases} h_m^{nc}(n) & \text{if } n \text{ is even,} \\ h_m^c(n) & \text{if } n \text{ is odd.} \end{cases} \quad (5)$$

Hence $\mathcal{H}_m^{nc}(x)$ comes from the odd terms of $\mathcal{H}_m(x)$, and $\mathcal{H}_m^c(x)$ from the even terms. This means

$$\begin{aligned} \mathcal{H}_m^{nc}(x) &= \frac{1}{2} (\mathcal{H}_m(x) - \mathcal{H}_m(-x)), \\ \mathcal{H}_m^c(x) &= \frac{1}{2} (\mathcal{H}_m(x) + \mathcal{H}_m(-x)). \end{aligned}$$

Hence, $\mathcal{K}_m(x) = \mathcal{H}_m(-x)$ when m is odd. Since $\mathcal{H}_m^{nc}(x)$ and $\mathcal{H}_m^c(x)$ share the same denominator $q_m(x)q_m(-x)$, both sequences h_m^{nc} and h_m^c satisfy a linear recurrence relation of order $2d_m$. However, $q_m(x)q_m(-x)$ is an even function, so it is a polynomial of degree d_m in x^2 . Thus, the subsequences of nonzero terms $\{h_m^{nc}(2n)\}_{n \geq 1}$ and $\{h_m^c(2n-1)\}_{n \geq 1}$ satisfy a linear recurrence relation of order d_m . Because of (5), it is clear that, for the nonzero terms, the asymptotic behavior of $h_m^{nc}(n)$ and $h_m^c(n)$ is same as that of $h_m(n)$. More precisely, $h_m^{nc}(2n) \sim h_m(2n)$, and $h_m^c(2n+1) \sim h_m(2n+1)$.

If the transfer matrix \mathcal{D}_m^* is irreducible, then it would have g_m dominant characteristic roots, where g_m denotes the period of \mathcal{D}_m^* . The fact that $q_m(x)q_m(-x)$, the denominator that $\mathcal{H}_m^{nc}(x)$ and $\mathcal{H}_m^c(x)$ share, is a polynomial in x^2 suggests that \mathcal{D}_m^* is a bipartite graph. If this can be confirmed, then g_m must be even. In fact, it contains the following directed cycle of length 2: $u_1 u_2 u_1$, where u_1 and u_2 are vertices (written as words) in \mathcal{D}_m^*

$$u_1 = 222 \cdots 20, \quad u_2 = 000 \cdots 02.$$

See Figure 4 and note that, because of the rotational symmetry, the vertex $22 \cdots 202$ is identified to $22 \cdots 220$, as well $20 \cdots 00$ to $00 \cdots 02$. We conclude that $g_m = 2$, so the two dominant characteristic roots of \mathcal{D}_m^* must be $\pm \theta_m$. This in turn implies that θ_m is the sole dominant characteristic root of $q_m(x)$. Consequently, we deduce that, for odd m ,

$$h_m^{nc}(2n) \sim a_m \theta_m^{2n-1} \quad \text{and} \quad h_m^c(2n+1) \sim a_m \theta_m^{2n},$$

provided that T_m^* is irreducible, and \mathcal{D}_m^* is a bipartite graph. Our computational data reveal that \mathcal{D}_3^* , \mathcal{D}_5^* , \mathcal{D}_7^* , \mathcal{D}_9^* are bipartite multidigraphs.

For even m , we note that \mathcal{D}_m^* contains loops. For example, there is a loop around the vertex representing the word $2030 \cdots ((m+2)/2)0$, see Figure 4. We conclude that, if T_m^* is irreducible (recall that our computational data confirm that the matrix T_m^* is indeed irreducible for $m \leq 10$), then $g_m = 1$. But the dominant characteristic root can come from either $\mathcal{H}_m(x)$ or $\mathcal{K}_m(x)$. Our computational data reveal that, for $m \leq 10$, the radius of convergence for $\mathcal{K}_m(x)$ is greater than that of $\mathcal{H}_m(x)$. This, together with (3) and (4), imply that the dominant characteristic root of both $\mathcal{H}_m^{nc}(x)$ and $\mathcal{H}_m^c(x)$ comes from $\mathcal{H}_m(x)$. Hence,

$$h_m^{nc}(n+1) \sim \frac{a_m}{2} \theta_m^n \quad \text{and} \quad h_m^c(n+1) \sim \frac{a_m}{2} \theta_m^n.$$

This immediately proves that $h_m^{nc}(n) \sim h_m^c(n)$ when $m = 2, 4, 6, 8, 10$. Is it always true when m is even?

8 Concluding Remarks and Open Problems

Our computational data affirm that for $3 \leq m \leq 10$, the denominator $q_m(x)$ has only one real positive dominant characteristic root θ_m , see Table 2. Our main conjecture is:

m	θ_m	a_m
3	2	3
4	3.73205080756887729352744634151	5.8867513459481288225457439025
5	5.31862821775018565910968015332	5.6485507137110988135657454508
6	9.07807499686426137037316693063	9.3759765980423268475201653010
7	12.46396683154921167484924057847	9.5114780466647699643291510197
8	20.49548062885849319891140410573	14.6698889618659187804647562240
9	28.1928327984540292722773603077	15.4543604331204162432381530254
10	45.31795107579019470088202555080	22.7172562899371282508816262267

Tab. 2: The approximate values of θ_m and a_m .

Conjecture 1 For each even $m \geq 4$,

$$h_m^{nc}(n+1) \sim h_m^c(n+1) \sim \frac{a_m}{2} \theta_m^n,$$

where $a_m = -\theta_m p_m(\theta_m^{-1})/q'_m(\theta_m^{-1})$.

As we have discussed in the previous section, the validity this conjecture, and other related asymptotic relations, can be completely resolved if we can settle the following open problems:

1. Is T_m^* irreducible for all $m \geq 3$? Note that this has been confirmed for $m \leq 10$.
2. Is \mathcal{D}_m^* bipartite when m is odd? Again, this has been confirmed up to $m = 9$.
3. Does the dominant characteristic root remain in \mathcal{H}_m when m is even? This is equivalent to showing that the radius of convergence for $\mathcal{K}_m(x)$ is greater than that of $\mathcal{H}_m(x)$.

Our computational data suggest further problems for investigation:

4. Is the sequence generated by $\mathcal{K}_m(x)$ always alternating? In other words, do we always have $h_m^c(2k) < h_m^{nc}(2k)$ and $h_m^c(2k+1) > h_m^{nc}(2k+1)$?

We close our discussion with three more interesting questions:

6. What is an appropriate combinatorial interpretation for $\mathcal{K}_m(x)$?
7. Can we define a labeling of the windows such that a single transfer matrix can be used to obtain the generating function $\mathcal{H}_m(x)$ directly? If such a transfer matrix does exist, its characteristic polynomial should be $\chi_m(t)$ mentioned in Section 6. (Recall that the matrices obtained in [2] for graph $C_m \times P_n$ are transfer matrices for sequences $h_m(n)$, but obtained by a labeling of the vertices of $C_m \times P_n$. We additionally verified that they are indeed primitive for $m \leq 12$.)
8. Can we find some similar properties of sequences $h_m(n)$, $h_m^{nc}(n)$ and $h_m^c(n)$ for the case of thick cylinder $P_m \times C_n$ (m is kept constant, whereas n grows)?

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