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Eric Cancès, Geneviève Dusson, Yvon Maday, Benjamin Stamm, Martin Vohralík. Guaranteed and robust a posteriori bounds for Laplace eigenvalues and eigenvectors: conforming approximations. 2016. hal-01194364v2

**HAL Id: hal-01194364**

**<https://inria.hal.science/hal-01194364v2>**

Preprint submitted on 27 Jun 2016 (v2), last revised 15 Dec 2016 (v3)

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# Guaranteed and robust a posteriori bounds for Laplace eigenvalues and eigenvectors: conforming approximations\*

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June 27, 2016

## Abstract

This paper derives a posteriori error estimates for conforming numerical approximations of the Laplace eigenvalue problem with a homogeneous Dirichlet boundary condition. In particular, upper and lower bounds for the first eigenvalue are given. These bounds are guaranteed, fully computable, and converge with the optimal speed to the exact eigenvalue. They are valid without any restriction on the computational mesh or on the approximate eigenvector; we only need to assume that the approximate eigenvalue is smaller than a computable lower bound on the second smallest eigenvalue, which can be checked in practice. Guaranteed, fully computable, optimally convergent, and polynomial-degree robust bounds on the energy error in the approximation of the first eigenvector are derived as well, under the same hypotheses. Remarkably, there appears no unknown (solution-, regularity-, or polynomial-degree-dependent) constant in our theory, and no convexity/regularity assumption on the computational domain/exact eigenvector(s) is needed. Two improvements of the multiplicative constant appearing in our estimates are presented. First, it is reduced by a fixed factor under an explicit, a posteriori calculable condition on the mesh and on the approximate eigenvector–eigenvalue pair. Second, when an elliptic regularity assumption is satisfied with known constants, it can be brought to the optimal value one. The application of our framework to conforming finite element approximations of arbitrary polynomial degree is provided, along with a numerical illustration on a set of test problems.

**Key words:** Laplace eigenvalue problem, a posteriori estimate, eigenvalue error, eigenvector error, guaranteed bound, conforming finite element method

## 1 Introduction

Precise numerical approximation of eigenvalues and eigenvectors is crucial in countless applications. Thus, there has been a long-standing interest in answering the question: what is the size of the errors in computed eigenvalues and eigenvectors? This question is usually tackled via a posteriori error estimates. For elliptic

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\*This work was supported by the ANR project MANIF “Mathematical and numerical issues in first-principle molecular simulation”. The last author has also received funding from the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation program (grant agreement No 647134 GATIPOR).

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source problems such as the Laplace one, conclusive answers are today given by, in particular, the theory of equilibrated fluxes following Prager and Synge [45], see Destuynder and Métivet [18], Braess *et al.* [8], Ern and Vohralík [20], and the references therein. The structure of the *Laplace eigenvalue* problem with respect to the a posteriori analysis appears rather different from the elliptic source case, and obtaining optimal estimates turns out being far from straightforward.

Recently, though, there has been an important progress in obtaining *guaranteed lower bounds* for the *first eigenvalue*: Armentano and Durán [1], Luo *et al.* [40], Hu *et al.* [30, 31], Carstensen and Gedicke [15], Yang *et al.* [57], or Liu [37] achieve so via the lowest-order nonconforming finite element method, Kuznetsov and Repin [35] and Šebestová and Vejchodský [50] give numerical-method-independent estimates based on flux (functional) estimates, Liu and Oishi [39] elaborate fine a priori approximation estimates for lowest-order conforming finite elements, and, most recently, Xie *et al.* [56] also rely on fluxes. Earlier work comprises Kato [32], Forsythe [22], Weinberger [55], Bazley and Fox [4], Fox and Rheinboldt [23], Moler and Payne [42], Kuttler and Sigillito [33, 34], Still [52], Goerisch and He [25], Plum [44], and Behnke *et al.* [5]. Unfortunately, some improvement typically seems viable: the first eigenvalue  $\lambda_1$  is requested to be the relatively closest eigenvalue to its numerical approximation  $\lambda_h$  in [35, Remark 3.2] and [50, condition (3.6)], a condition on applicability which is difficult to verify; a saturation assumption may be necessary, see the discussion in [30]; the arguments used in [39, 15, 37], of a priori nature, employ the largest mesh element diameter, which potentially entails an overestimation on adaptively generated meshes; the evaluation of the estimate for nonconvex domains requires a solution of an auxiliary eigenvalue problem in [39]; the bounds do not converge with the correct speed in [35, 50]; an auxiliary global flux problem needs to be solved in [56].

The question of precision for both eigenvalues and eigenvectors has also been investigated previously. For conforming finite elements, relying on the a priori error estimates resumed in Babuška and Osborn [2] and Boffi [7], see also the references therein, a posteriori error estimates have been obtained by Verfürth [54], Maday and Patera [41], Larson [36], Heuveline and Rannacher [29], Durán *et al.* [19], Grubišić and Ovall [27], Rannacher *et al.* [46], and Šolín and Giani [51], see also the references therein. Unfortunately, these estimates systematically contain uncomputable terms, claimed higher order on fine enough meshes.

The purpose of the present paper is to derive *guaranteed* and optimally convergent *a posteriori bounds* on both the *first Laplace eigenvalue* and the *associated eigenvector* for conforming variational methods. Nonconforming methods including nonconforming, discontinuous Galerkin, or mixed finite elements are treated in Cancès *et al.* [12]. We describe the setting in details in Section 2. Section 3 and Section 4 then contain a collection of equivalence inequalities between respectively the square root of the first eigenvalue error and the first eigenvector energy error, the first eigenvector energy error and the dual norm of the residual, and between the dual norm of the residual and its computable estimates. These results are valid under three basic assumptions. First,  $\lambda_h$  needs to be smaller than the second smallest eigenvalue  $\lambda_2$ . This can be guaranteed in many cases of practical interest by including the computational domain  $\Omega$  into a larger domain  $\Omega^+$  with known second eigenvalue  $\lambda_2(\Omega^+)$  and by requesting  $\lambda_h < \underline{\lambda}_2 := \lambda_2(\Omega^+)$ , cf. (5.2) below. *Numerical bounds*  $\underline{\lambda}_2 \leq \lambda_2$ , typically on a rather coarse mesh, can also be used, following [39, 15, 37], see Remark 5.2 below. Second, a condition on the Galerkin orthogonality of the residual to the finite element hat functions on patches around the mesh vertices (Assumption 4.1) needs to be satisfied. Finally, the approximation spaces need to consist of appropriate piecewise polynomials (Assumption 4.2). For improved versions of the bounds, we additionally need to check the smallness of the  $L^2(\Omega)$ -norm of the Riesz representation of the residual, see the *a posteriori calculable* conditions (5.5) and (5.8) below. They can be always satisfied by refining the computational mesh/increasing the polynomial degree of the approximate solution.

In Section 5, the results of Section 3–Section 4 are turned into actual a posteriori bounds. First, upper and lower bounds for the first eigenvalue are given in Theorem 5.1 in the form

$$\lambda_h - \eta^2 \leq \lambda_1 \leq \lambda_h - \tilde{\eta}^2, \quad (1.1a)$$

where  $\eta^2$  and  $\tilde{\eta}^2$  are *fully computable quantities* issuing from local residual liftings, by mixed and conforming finite element problems respectively, converging to zero with the speed of convergence of  $\lambda_h$  towards  $\lambda_1$ . First eigenvector energy estimates are given next, see in particular Theorem 5.4 revealing

$$\|\nabla(u_1 - u_h)\| \leq \eta, \quad \eta \leq C\|\nabla(u_1 - u_h)\|, \quad (1.1b)$$

where  $u_1$  is the first exact eigenvector,  $u_h$  its numerical approximation,  $\|\cdot\|$  is the  $L^2(\Omega)$ -norm,  $C$  is a constant that only depends on  $\lambda_1$ ,  $\lambda_h$ ,  $\underline{\lambda}_2$ , on the space dimension  $d$ , on some Poincaré–Friedrichs constants  $C_{\text{cont,PF}}$

on patches of elements (themselves only depending on the shape regularity of the mesh), and on a discrete stability constant  $C_{\text{st}}$ , again only depending on the mesh shape regularity. In particular,  $C$  is independent of the polynomial degree of  $u_h$ , leading to the *polynomial-degree robustness*. Moreover, a computable bound on  $C$  is given. In Cases A and B of Theorems 5.1 and 5.4, these results hold *without any assumption* on the *convexity* of the computational domain  $\Omega$ , on the *regularity* of the eigenfunctions, and on the *space dimension* (we only treat here  $d = 2, 3$  but our arguments are general for  $d \geq 1$ ); case B improves case A under the above-mentioned condition (5.5). The generality of these results is paid by the presence of the factor  $(1 - \frac{\lambda_h}{\lambda_2})^{-1}$  in the estimators  $\eta$  and  $\tilde{\eta}$ . If, additionally, some regularity of the corresponding *source problem* is known and the interpolation and stability constants are computable (typically when  $d = 2$  and  $\Omega$  is convex), we arrive at still improved estimators not featuring this factor, as summarized in Case C of Theorems 5.1 and 5.4.

We show how to apply the above general results to conforming finite elements of arbitrary order in Section 6. Numerical experiments presented in Section 7 fully support the theoretical findings; in particular the necessary conditions hold here from quite coarse meshes. Finally, building on these results, guaranteed error bounds and fully adaptive strategies with dynamic stopping criteria may become possible for nonlinear eigenvalue problems; some of our first results in this direction are summarized in [11].

## 2 Setting

Let  $\Omega \subset \mathbb{R}^d$ ,  $d = 2, 3$ , be a polygonal/polyhedral domain. We denote by  $H^1(\Omega)$  the Sobolev space of  $L^2(\Omega)$  functions with weak gradients in  $[L^2(\Omega)]^d$  and by  $V := H_0^1(\Omega)$  its zero-trace subspace. Similarly,  $\mathbf{H}(\text{div}, \Omega)$  stands for the space of  $[L^2(\Omega)]^d$  functions with weak divergences in  $L^2(\Omega)$ . The notations  $\nabla$  and  $\nabla \cdot$  are used respectively for the weak gradient and divergence.

### 2.1 The Laplace eigenvalue problem

We consider the Laplace eigenvalue problem originating from minimizing the *energy*  $E(\cdot) : V \rightarrow \mathbb{R}$ ,

$$E(v) := \frac{1}{2} \int_{\Omega} |\nabla v|^2 \, \text{d}\mathbf{x},$$

over all functions  $v$  from the energy space  $V$  such that  $\int_{\Omega} v^2 \, \text{d}\mathbf{x} = 1$ , i.e.,

$$u := \arg \inf_{\substack{v \in V, \\ \int_{\Omega} v^2 \, \text{d}\mathbf{x} = 1}} E(v). \quad (2.1)$$

It is well-known that there exists a solution to (2.1), unique up to the sign. The Euler–Lagrange equations of problem (2.1) read: find an eigenvector and eigenvalue pair  $(u, \lambda)$ , with  $u$  satisfying the homogeneous Dirichlet boundary condition over  $\partial\Omega$  and subject to the constraint  $\|u\| = 1$ , such that  $-\Delta u = \lambda u$  in  $\Omega$ .

The weak form of our problem is: find  $(u, \lambda) \in V \times \mathbb{R}^+$  with  $\|u\| = 1$  such that

$$(\nabla u, \nabla v) = \lambda(u, v) \quad \forall v \in V. \quad (2.2)$$

Here, for  $\omega \subset \Omega$ ,  $(\nabla u, \nabla v)_{\omega}$  stands for  $\int_{\omega} \nabla u \cdot \nabla v \, \text{d}\mathbf{x}$  and  $(u, v)_{\omega}$  for  $\int_{\omega} uv \, \text{d}\mathbf{x}$ ; we also denote  $\|\nabla v\|_{\omega}^2 := \int_{\omega} |\nabla v|^2 \, \text{d}\mathbf{x}$  and  $\|v\|_{\omega}^2 := \int_{\omega} v^2 \, \text{d}\mathbf{x}$  and drop the index whenever  $\omega = \Omega$ . The eigenvalue problem (2.2) has infinitely many solution pairs. There exists, cf. Babuška and Osborn [2] or Boffi [7], a countable orthonormal basis  $u_k$ ,  $k \geq 1$ , of  $L^2(\Omega)$  consisting of vectors from  $V$ , and a nondecreasing sequence  $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots$  going to  $+\infty$  such that

$$(\nabla u_k, \nabla v) = \lambda_k(u_k, v) \quad \forall v \in V. \quad (2.3)$$

It follows from (2.3) and the scaling  $\|u_k\| = 1$  that  $\|\nabla u_k\|^2 = \lambda_k$ . In particular, the smallest eigenvalue  $\lambda_1$  is positive, isolated, and the associated eigenvector  $u_1$  is unique under the sign condition  $(u_1, 1) > 0$  ( $u_1$  is actually positive everywhere inside  $\Omega$ ). Remarkably, (2.1) implies  $\frac{\|\nabla v\|^2}{2\|v\|^2} \geq \frac{\|\nabla u_1\|^2}{2} = \frac{\lambda_1}{2}$  for all  $v \in V$ , which is known as

$$\|v\|^2 \leq \frac{1}{\lambda_1} \|\nabla v\|^2 \quad \forall v \in V, \quad (2.4)$$

the Poincaré–Friedrichs inequality. We are only interested here in the numerical approximation of the smallest eigenvalue  $\lambda_1$  and of its associated eigenvector  $u_1$ .

Below, we shall often employ the Parseval identity

$$\|v\|^2 = \sum_{k \geq 1} (v, u_k)^2 \quad (2.5)$$

valid for any  $v \in L^2(\Omega)$ . It is also well known that  $(u_k/\sqrt{\lambda_k})_{k \geq 1}$  forms an orthonormal basis of  $V$ , for which one in particular uses that  $(\nabla u_k, \nabla u_l) = \lambda_k(u_k, u_l) = 0$  for  $k \neq l$ . Then, for  $v \in V$ , we also obtain

$$\|\nabla v\|^2 = \sum_{k \geq 1} \frac{(\nabla v, \nabla u_k)^2}{\lambda_k} = \sum_{k \geq 1} \lambda_k (v, u_k)^2. \quad (2.6)$$

We refer to Gilbarg and Trudinger [24], Babuška and Osborn [2], Boffi [7], and the references therein for more details and background.

## 2.2 Residual and its dual norm

The derivation of a posteriori error estimates usually exploits the concept of the *residual* and of its *dual norm*. We will proceed in this way in the present paper as well. Let  $V'$  stand for the dual of  $V$ .

**Definition 2.1** (Residual and its dual norm). *For any pair  $(u_h, \lambda_h) \in V \times \mathbb{R}$ , define the residual  $\text{Res}(u_h, \lambda_h) \in V'$  by*

$$\langle \text{Res}(u_h, \lambda_h), v \rangle_{V', V} := \lambda_h(u_h, v) - (\nabla u_h, \nabla v) \quad \forall v \in V. \quad (2.7a)$$

*Its dual norm is then*

$$\|\text{Res}(u_h, \lambda_h)\|_{-1} := \sup_{v \in V, \|\nabla v\|=1} \langle \text{Res}(u_h, \lambda_h), v \rangle_{V', V}. \quad (2.7b)$$

We will also often work with the *Riesz representation* of the residual  $\mathfrak{z}_{(h)} \in V$ ,

$$(\nabla \mathfrak{z}_{(h)}, \nabla v) = \langle \text{Res}(u_h, \lambda_h), v \rangle_{V', V} \quad \forall v \in V, \quad (2.8a)$$

$$\|\nabla \mathfrak{z}_{(h)}\| = \|\text{Res}(u_h, \lambda_h)\|_{-1}. \quad (2.8b)$$

## 3 Generic equivalences

We establish in this section *generic equivalence results* between the following three quantities: the square root of the first eigenvalue error  $\sqrt{\lambda_h - \lambda_1}$  (supposing  $\lambda_h \geq \lambda_1$ ), the first eigenvector energy error  $\|\nabla(u_1 - u_h)\|$ , and the dual norm of the residual  $\|\text{Res}(u_h, \lambda_h)\|_{-1}$ . These equivalences may for the moment contain uncomputable terms like the eigenvalues  $\lambda_1, \lambda_2$  or the Riesz representation norm  $\|\mathfrak{z}_{(h)}\|$ , but all such terms will be removed later. A generalization to any self-adjoint operator with compact resolvent can be found in Cancès *et al.* [12].

Our first two lemmas are similar in parts to the developments in [33, 35, 50], giving under very general conditions a computable bound on the  $L^2(\Omega)$  error  $\|u_1 - u_h\|$ :

**Lemma 3.1** ( $L^2(\Omega)$  bound via a quadratic residual inequality). *Let  $(u_h, \lambda_h) \in V \times \mathbb{R}^+$  with  $\|u_h\| = 1$  and  $(u_1, u_h) \geq 0$ . Let also  $\lambda_h < \lambda_2$ . Then*

$$\|u_1 - u_h\| \leq \alpha_h := \sqrt{2} \left(1 - \frac{\lambda_h}{\lambda_2}\right)^{-1} \|\mathfrak{z}_{(h)}\|. \quad (3.1)$$

*Proof.* Characterizations (2.3), (2.7a), and (2.8a) give

$$(\mathfrak{z}_{(h)}, u_k) = \frac{1}{\lambda_k} (\nabla u_k, \nabla \mathfrak{z}_{(h)}) = \frac{1}{\lambda_k} (\lambda_h(u_h, u_k) - (\nabla u_h, \nabla u_k)) = \left(\frac{\lambda_h}{\lambda_k} - 1\right) (u_h, u_k). \quad (3.2)$$

Consequently, the Parseval equality (2.5) with  $v = \mathfrak{z}_{(h)}$  yields

$$\|\mathfrak{z}_{(h)}\|^2 = \sum_{k \geq 1} (\mathfrak{z}_{(h)}, u_k)^2 = \sum_{k \geq 1} \left(1 - \frac{\lambda_h}{\lambda_k}\right)^2 (u_h, u_k)^2. \quad (3.3)$$

Observe that the function  $x \in \mathbb{R}^+ \mapsto \left(1 - \frac{\lambda_h}{x}\right)^2$ , considered for a positive  $x$  and  $\lambda_h > 0$ , reaches its minimum at  $x = \lambda_h$  and is increasing on  $[\lambda_h, \infty)$ . Thus, as we suppose  $\lambda_h < \lambda_2$ , there holds

$$\min_{k \geq 2} \left(1 - \frac{\lambda_h}{\lambda_k}\right)^2 = \left(1 - \frac{\lambda_h}{\lambda_2}\right)^2 =: C_h. \quad (3.4)$$

Further, employing the scalings  $\|u_1\| = 1$  and  $\|u_h\| = 1$ ,

$$(u_h - u_1, u_1) = (u_h, u_1) - \|u_1\|^2 = (u_h, u_1) - \frac{\|u_1\|^2}{2} - \frac{\|u_h\|^2}{2} = -\frac{1}{2}\|u_1 - u_h\|^2. \quad (3.5)$$

As  $(u_1, u_k) = 0$  for  $k \geq 2$  from the orthogonality of  $u_k$  in (2.3), elaborating (3.3) further while adding and subtracting  $C_h(u_h - u_1, u_1)^2$  and using (3.4) and (3.5) gives

$$\begin{aligned} \|\mathfrak{z}_{(h)}\|^2 &= \left(\frac{\lambda_h}{\lambda_1} - 1\right)^2 (u_h, u_1)^2 + \sum_{k \geq 2} \left(1 - \frac{\lambda_h}{\lambda_k}\right)^2 (u_h - u_1, u_k)^2 \\ &\geq \left(\frac{\lambda_h}{\lambda_1} - 1\right)^2 (u_h, u_1)^2 + C_h \sum_{k \geq 1} (u_h - u_1, u_k)^2 - C_h (u_h - u_1, u_1)^2 \\ &= \left(\frac{\lambda_h}{\lambda_1} - 1\right)^2 (u_h, u_1)^2 + C_h \|u_1 - u_h\|^2 - \frac{C_h}{4} \|u_1 - u_h\|^4, \end{aligned} \quad (3.6)$$

where we have also employed (2.5) with  $v = u_h - u_1$ . Dropping the first (nonnegative and presumably small) term on the right-hand side and denoting  $e_h := \|u_1 - u_h\|^2$ , we conclude the validity of the quadratic residual inequality in  $e_h$

$$\frac{C_h}{4} e_h^2 - C_h e_h + \|\mathfrak{z}_{(h)}\|^2 \geq 0. \quad (3.7)$$

From the assumption  $(u_1, u_h) \geq 0$ , employing  $\|u_1\| = \|u_h\| = 1$ ,

$$e_h = \|u_1 - u_h\|^2 = 2 - 2(u_1, u_h) \leq 2, \quad (3.8)$$

so that  $C_h e_h \leq 2\|\mathfrak{z}_{(h)}\|^2$ , i.e., the statement of the theorem. Note that inspecting more closely the quadratic inequality (3.7), the improved bound  $e_h \leq 2 - \sqrt{4 - 2\alpha_h^2}$  (which is  $\sqrt{2}$ -times better for  $e_h$  approaching zero) follows under the additional condition  $\|\mathfrak{z}_{(h)}\|^2 < C_h$  that we prefer to avoid.  $\square$

**Lemma 3.2** ( $L^2(\Omega)$  bound with respect to  $\|\nabla \mathfrak{z}_{(h)}\|$ ). *Under the assumptions of Lemma 3.1, there also holds*

$$\|u_1 - u_h\| \leq \alpha_h := \sqrt{2} \left(1 - \frac{\lambda_h}{\lambda_2}\right)^{-1} \lambda_2^{-\frac{1}{2}} \|\nabla \mathfrak{z}_{(h)}\|. \quad (3.9)$$

*Proof.* Developing (2.6) for  $v = \mathfrak{z}_{(h)}$  via (3.2) gives

$$\|\nabla \mathfrak{z}_{(h)}\|^2 = \sum_{k \geq 1} \lambda_k (\mathfrak{z}_{(h)}, u_k)^2 = \sum_{k \geq 1} \lambda_k \left(1 - \frac{\lambda_h}{\lambda_k}\right)^2 (u_h, u_k)^2. \quad (3.10)$$

Thus, similarly to (3.6)–(3.7), with  $C_h$  given by (3.4) and  $e_h := \|u_1 - u_h\|^2$ ,

$$\frac{C_h \lambda_2}{4} e_h^2 - C_h \lambda_2 e_h + \|\nabla \mathfrak{z}_{(h)}\|^2 \geq 0. \quad (3.11)$$

One then concludes as above in Lemma 3.1.  $\square$

The sign condition  $(u_1, u_h) \geq 0$  necessary in Lemmas 3.1 and 3.2 is typically always satisfied; the following lemma can be used for its rigorous verification:

**Lemma 3.3** (Sign verification). *Recall the assumption  $(u_1, 1) > 0$ . Let  $(u_h, \lambda_h) \in V \times \mathbb{R}$  satisfy  $\|u_h\| = 1$ ,  $(u_h, 1) > 0$ ,  $\lambda_h < \lambda_2$ , and  $\alpha_h \leq |\Omega|^{-\frac{1}{2}}(u_h, 1)$  for  $\alpha_h$  given by (3.1) or (3.9). Then  $(u_1, u_h) \geq 0$ .*

*Proof.* Suppose that  $-(u_1, u_h) > 0$ . Then the bounds of Lemmas 3.1 and 3.2 hold for  $-u_h$  in place of  $u_h$ , i.e.,  $\|u_1 + u_h\| \leq \alpha_h$ . Consequently, a contradiction follows,

$$(u_h, 1) = -(u_1, 1) + (u_1 + u_h, 1) < (u_1 + u_h, 1) \leq \|u_1 + u_h\| |\Omega|^{\frac{1}{2}} \leq (u_h, 1).$$

□

### 3.1 First eigenvalue error equivalences

We first show how to exploit a bound on  $\|u_1 - u_h\|$  for an equivalence between  $\sqrt{\lambda_h - \lambda_1}$  and  $\|\nabla(u_1 - u_h)\|$ , extending some classical results. We need to assume that  $\lambda_h$  is the Rayleigh quotient of  $u_h$ :

**Assumption 3.4** (Conforming variational solution). *Let  $(u_h, \lambda_h) \in V \times \mathbb{R}^+$  with  $\|u_h\| = 1$  be such that  $\|\nabla u_h\|^2 = \lambda_h$ .*

**Theorem 3.5** (Eigenvalue bounds). *Let Assumption 3.4 be satisfied. Suppose also  $\|u_1 - u_h\| \leq \alpha_h$  for some  $\alpha_h \in \mathbb{R}^+$ . Then*

$$\|\nabla(u_1 - u_h)\|^2 - \lambda_1 \alpha_h^2 \leq \lambda_h - \lambda_1 \leq \|\nabla(u_1 - u_h)\|^2. \quad (3.12)$$

*Under the additional assumption  $\alpha_h \leq \sqrt{2}$ , there also holds*

$$\frac{1}{2} \left(1 - \frac{\lambda_1}{\lambda_2}\right) \left(1 - \frac{\alpha_h^2}{4}\right) \|\nabla(u_1 - u_h)\|^2 \leq \lambda_h - \lambda_1. \quad (3.13)$$

*Proof.* Using the weak solution characterization (2.2), Assumption 3.4, and (3.5),

$$\begin{aligned} \lambda_h - \lambda_1 &= \|\nabla u_h\|^2 - \|\nabla u_1\|^2 \\ &= \|\nabla(u_h - u_1)\|^2 + 2(\nabla(u_h - u_1), \nabla u_1) \\ &= \|\nabla(u_h - u_1)\|^2 + 2\lambda_1(u_1, u_h - u_1) \\ &= \|\nabla(u_h - u_1)\|^2 - \lambda_1 \|u_1 - u_h\|^2. \end{aligned} \quad (3.14)$$

Dropping the (nonpositive and presumably small) last term, the upper bound in (3.12) follows; estimating it using  $\|u_1 - u_h\| \leq \alpha_h$ , we arrive at the lower bound in (3.12).

To prove (3.13), we use (2.5)–(2.6) for  $v = u_1 - u_h$ . First,

$$\|\nabla(u_1 - u_h)\|^2 - \lambda_1 \|u_1 - u_h\|^2 = \sum_{k \geq 1} (\lambda_k - \lambda_1) (u_1 - u_h, u_k)^2 = \sum_{k \geq 2} (\lambda_k - \lambda_1) (u_1 - u_h, u_k)^2. \quad (3.15)$$

Using that  $\lambda_k \geq \lambda_2$  for  $k \geq 2$ ,  $\lambda_2 > \lambda_1$ , (3.5), and the Cauchy–Schwarz inequality,

$$\begin{aligned} \|\nabla(u_1 - u_h)\|^2 - \lambda_1 \|u_1 - u_h\|^2 &\geq (\lambda_2 - \lambda_1) \sum_{k \geq 1} (u_1 - u_h, u_k)^2 - (\lambda_2 - \lambda_1) (u_1 - u_h, u_1)^2 \\ &= (\lambda_2 - \lambda_1) \|u_1 - u_h\|^2 - \frac{\lambda_2 - \lambda_1}{4} \|u_1 - u_h\|^4. \end{aligned}$$

Using  $\|u_1 - u_h\| \leq \alpha_h$  and reemploying (2.5) for  $v = u_1 - u_h$ , we arrive at, second,

$$\begin{aligned} \|\nabla(u_1 - u_h)\|^2 - \lambda_1 \|u_1 - u_h\|^2 &\geq (\lambda_2 - \lambda_1) \|u_1 - u_h\|^2 - \alpha_h^2 \frac{\lambda_2 - \lambda_1}{4} \|u_1 - u_h\|^2 \\ &= \sum_{k \geq 1} (\lambda_2 - \lambda_1) \left(1 - \frac{\alpha_h^2}{4}\right) (u_1 - u_h, u_k)^2. \end{aligned}$$

Summing this with (3.15) with weights  $\frac{1}{2}$  yields

$$\|\nabla(u_1 - u_h)\|^2 - \lambda_1 \|u_1 - u_h\|^2 \geq \sum_{k \geq 1} \left\{ \frac{\lambda_k - \lambda_1}{2} + \frac{\lambda_2 - \lambda_1}{2} \left(1 - \frac{\alpha_h^2}{4}\right) \right\} (u_1 - u_h, u_k)^2.$$

Now notice that, using (2.6) for  $v = u_1 - u_h$ ,

$$\frac{1}{2} \left(1 - \frac{\lambda_1}{\lambda_2}\right) \left(1 - \frac{\alpha_h^2}{4}\right) \|\nabla(u_1 - u_h)\|^2 = \sum_{k \geq 1} \frac{\lambda_k}{2} \left(1 - \frac{\lambda_1}{\lambda_2}\right) \left(1 - \frac{\alpha_h^2}{4}\right) (u_1 - u_h, u_k)^2.$$

A simple calculation (note  $\frac{1}{2} \leq \left(1 - \frac{\alpha_h^2}{4}\right) \leq 1$ ) shows that

$$\frac{\lambda_k - \lambda_1}{2} + \frac{\lambda_2 - \lambda_1}{2} \left(1 - \frac{\alpha_h^2}{4}\right) \geq \frac{\lambda_k}{2} \left(1 - \frac{\lambda_1}{\lambda_2}\right) \left(1 - \frac{\alpha_h^2}{4}\right) \quad k \geq 1.$$

Thus

$$\|\nabla(u_1 - u_h)\|^2 - \lambda_1 \|u_1 - u_h\|^2 \geq \frac{1}{2} \left(1 - \frac{\lambda_1}{\lambda_2}\right) \left(1 - \frac{\alpha_h^2}{4}\right) \|\nabla(u_1 - u_h)\|^2,$$

and (3.13) follows using (3.14).  $\square$

### 3.2 First eigenvector error equivalences

We next investigate the equivalence between  $\|\nabla(u_1 - u_h)\|$  and the dual residual norm  $\|\text{Res}(u_h, \lambda_h)\|_{-1}$ :

**Theorem 3.6** (Eigenvector bounds). *Let Assumption 3.4 be satisfied. Suppose also  $\|u_1 - u_h\| \leq \alpha_h$  for some  $\alpha_h \in \mathbb{R}^+$  and  $\lambda_h < \lambda_2$ . Then*

$$\begin{aligned} & \left( \frac{\|\nabla(u_1 - u_h)\|^2}{\lambda_1} + 1 \right)^{-1} \|\text{Res}(u_h, \lambda_h)\|_{-1}^2 \\ & \leq \|\nabla(u_1 - u_h)\|^2 \leq \|\text{Res}(u_h, \lambda_h)\|_{-1}^2 + 2\lambda_h \alpha_h^2. \end{aligned} \quad (3.16)$$

Under the additional assumption  $\alpha_h \leq \sqrt{2}$ , there also holds

$$\|\nabla(u_1 - u_h)\|^2 \leq \left(1 - \frac{\lambda_h}{\lambda_2}\right)^{-2} \left(1 - \frac{\alpha_h^2}{4}\right)^{-1} \|\text{Res}(u_h, \lambda_h)\|_{-1}^2. \quad (3.17)$$

*Proof.* Starting from (3.10), adding and subtracting  $C_h \lambda_1 (u_h - u_1, u_1)^2$ , using  $(u_1, u_k) = 0$  for  $k \geq 2$ , (3.4), (3.5), and the Cauchy–Schwarz inequality, we observe

$$\begin{aligned} \|\nabla \mathfrak{z}_{(h)}\|^2 & \geq \lambda_1 \left( \frac{\lambda_h}{\lambda_1} - 1 \right)^2 (u_h, u_1)^2 + C_h \sum_{k \geq 1} \lambda_k (u_h - u_1, u_k)^2 - C_h \lambda_1 (u_h - u_1, u_1)^2 \\ & = \lambda_1 \left( \frac{\lambda_h}{\lambda_1} - 1 \right)^2 (u_h, u_1)^2 + C_h \|\nabla(u_1 - u_h)\|^2 - \frac{C_h}{4} \lambda_1 \|u_1 - u_h\|^4 \\ & \geq C_h \|\nabla(u_1 - u_h)\|^2 - \frac{C_h}{4} \lambda_1 \|u_1 - u_h\|^4. \end{aligned}$$

Using  $\|u_1 - u_h\| \leq \alpha_h$  and the Poincaré–Friedrichs inequality (2.4) for  $v = u_1 - u_h$ ,

$$\|\nabla \mathfrak{z}_{(h)}\|^2 \geq C_h \|\nabla(u_1 - u_h)\|^2 - \frac{C_h}{4} \|\nabla(u_1 - u_h)\|^2 \alpha_h^2,$$

so that (3.17) follows employing (2.8b).

As  $\lambda_h < \lambda_2$ ,  $\max_{k \geq 2} \left(1 - \frac{\lambda_h}{\lambda_k}\right)^2 \leq 1$  (recall that  $\lambda_h \geq 0$ ). Thus the Cauchy–Schwarz inequality and  $\|u_1\| = \|u_h\| = 1$  give

$$\|\nabla \mathfrak{z}_{(h)}\|^2 \leq \lambda_1 \left( \frac{\lambda_h}{\lambda_1} - 1 \right)^2 + \sum_{k \geq 2} \lambda_k (u_h - u_1, u_k)^2 \leq \lambda_1 \left( \frac{\lambda_h}{\lambda_1} - 1 \right)^2 + \|\nabla(u_1 - u_h)\|^2.$$

As  $\lambda_h \geq \lambda_1$ , the right inequality in (3.12) implies the left inequality in (3.16).

Finally, the right inequality in (3.16) can be seen as in, e.g., Carstensen and Gedicke [14, Lemma 3.1] combined with  $\|u_1 - u_h\| \leq \alpha_h$  and  $\lambda_1 \leq \lambda_h$ .  $\square$

## 4 Dual norm of the residual equivalences

In addition to the results of Section 3, we now need to estimate the dual residual norm  $\|\text{Res}(u_h, \lambda_h)\|_{-1}$ . For the upper bound, following [45, 18, 8, 20] and the references therein, we introduce an *equilibrated flux reconstruction*. This is a vector field  $\boldsymbol{\sigma}_h$  constructed from the *local residual* of  $(u_h, \lambda_h)$  by solving patchwise mixed finite element problems in such a way that

$$\boldsymbol{\sigma}_h \in \mathbf{V}_h \subset \mathbf{H}(\text{div}, \Omega), \quad (4.1a)$$

$$\nabla \cdot \boldsymbol{\sigma}_h = \lambda_h u_h. \quad (4.1b)$$

Inversely, local conforming residual liftings following [3, Section 5.1], [47, Section 4.1.1], [20, Section 3.3] will allow us to construct  $r_h \in X_h \subset V$  leading to a lower bound on  $\|\text{Res}(u_h, \lambda_h)\|_{-1}$ .

### 4.1 Meshes and discrete spaces

We first introduce some more notation. Let henceforth  $\{\mathcal{T}_h\}_h$  be a family of matching simplicial partitions of the domain  $\Omega$ , shape regular in the sense that the ratio of each element diameter to the diameter of its largest inscribed ball is uniformly bounded by a constant  $\kappa_{\mathcal{T}} > 0$ . We denote by  $K$  a generic element of  $\mathcal{T}_h$ . The set of vertices is denoted by  $\mathcal{V}_h$ , with interior vertices  $\mathcal{V}_h^{\text{int}}$ , vertices located on the boundary  $\mathcal{V}_h^{\text{ext}}$ , and a generic vertex  $\mathbf{a}$ . We call  $\mathcal{T}_{\mathbf{a}}$  the patch of elements of  $\mathcal{T}_h$  which share the vertex  $\mathbf{a} \in \mathcal{V}_h$ ,  $\omega_{\mathbf{a}}$  the corresponding subdomain, and  $\mathbf{n}_{\omega_{\mathbf{a}}}$  its outward unit normal. We often tacitly extend functions defined on  $\omega_{\mathbf{a}}$  by zero outside of  $\omega_{\mathbf{a}}$ , whereas  $V_h(\omega_{\mathbf{a}})$  stands for the restriction of the space  $V_h$  to  $\omega_{\mathbf{a}}$ . Next,  $\psi_{\mathbf{a}}$  for  $\mathbf{a} \in \mathcal{V}_h$  stands for the piecewise affine ‘‘hat’’ function taking value 1 at the vertex  $\mathbf{a}$  and zero at the other vertices. Remarkably,  $(\psi_{\mathbf{a}})_{\mathbf{a} \in \mathcal{V}_h}$  form a partition of unity via  $\sum_{\mathbf{a} \in \mathcal{V}_h} \psi_{\mathbf{a}} = 1|_{\Omega}$ .

Let  $\mathbb{P}_s(K)$ ,  $s \geq 0$ , stand for polynomials of total degree at most  $s$  on  $K \in \mathcal{T}_h$ , and  $\mathbb{P}_s(\mathcal{T}_h)$  for piecewise polynomials on  $\mathcal{T}_h$ , without any continuity requirement. Let also  $\mathbf{V}_h \times Q_h \subset \mathbf{H}(\text{div}, \Omega) \times L^2(\Omega)$  stand for the Raviart–Thomas–Nédélec (RTN) mixed finite element spaces: for degree  $s \geq 0$ ,  $\mathbf{V}_h := \{\mathbf{v}_h \in \mathbf{H}(\text{div}, \Omega); \mathbf{v}_h|_K \in [\mathbb{P}_s(K)]^d + \mathbb{P}_s(K)\mathbf{x}\}$  and  $Q_h := \mathbb{P}_s(\mathcal{T}_h)$ , see Brezzi and Fortin [10] or Roberts and Thomas [48]. We also denote by  $\Pi_{Q_h}$  the  $L^2(\Omega)$ -orthogonal projection onto  $Q_h$ .

### 4.2 Equilibrated flux reconstruction

To proceed further, we assume:

**Assumption 4.1** (Galerkin orthogonality of the residual to  $\psi_{\mathbf{a}}$ ). *There holds*

$$\lambda_h(u_h, \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} - (\nabla u_h, \nabla \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} = \langle \text{Res}(u_h, \lambda_h), \psi_{\mathbf{a}} \rangle_{V', V} = 0 \quad \forall \mathbf{a} \in \mathcal{V}_h^{\text{int}}.$$

**Assumption 4.2** (Piecewise polynomials). *There holds*

$$u_h \in \mathbb{P}_p(\mathcal{T}_h), p \geq 1, \text{ and the spaces } \mathbf{V}_h \times Q_h \text{ are of order } p + 1.$$

We construct concretely  $\boldsymbol{\sigma}_h$  by solving the following *homogeneous local Neumann* (Neumann–Dirichlet close to the boundary) discrete *problems* on the patches  $\omega_{\mathbf{a}}$ :

**Definition 4.3** (Equilibrated flux reconstruction). *Let  $u_h \in V$  satisfy Assumption 4.1. For  $\mathbf{a} \in \mathcal{V}_h$ , set*

$$\begin{aligned} \mathbf{V}_h^{\mathbf{a}} &:= \{\mathbf{v}_h \in \mathbf{V}_h(\omega_{\mathbf{a}}); \mathbf{v}_h \cdot \mathbf{n}_{\omega_{\mathbf{a}}} = 0 \text{ on } \partial\omega_{\mathbf{a}}\}, & \mathbf{a} \in \mathcal{V}_h^{\text{int}}, \\ Q_h^{\mathbf{a}} &:= \{q_h \in Q_h(\omega_{\mathbf{a}}); (q_h, 1)_{\omega_{\mathbf{a}}} = 0\}, \\ \mathbf{V}_h^{\mathbf{a}} &:= \{\mathbf{v}_h \in \mathbf{V}_h(\omega_{\mathbf{a}}); \mathbf{v}_h \cdot \mathbf{n}_{\omega_{\mathbf{a}}} = 0 \text{ on } \partial\omega_{\mathbf{a}} \setminus \partial\Omega\}, & \mathbf{a} \in \mathcal{V}_h^{\text{ext}}, \\ Q_h^{\mathbf{a}} &:= Q_h(\omega_{\mathbf{a}}), \end{aligned}$$

Then define  $\sigma_h := \sum_{\mathbf{a} \in \mathcal{V}_h} \sigma_h^{\mathbf{a}} \in \mathbf{V}_h$ , where  $\sigma_h^{\mathbf{a}} \in \mathbf{V}_h^{\mathbf{a}}$  solve

$$\sigma_h^{\mathbf{a}} := \arg \min_{\substack{\mathbf{v}_h \in \mathbf{V}_h^{\mathbf{a}}, \\ \nabla \cdot \mathbf{v}_h = \Pi_{Q_h}(\psi_{\mathbf{a}} \lambda_h u_h - \nabla u_h \cdot \nabla \psi_{\mathbf{a}})}} \|\psi_{\mathbf{a}} \nabla u_h + \mathbf{v}_h\|_{\omega_{\mathbf{a}}} \quad \forall \mathbf{a} \in \mathcal{V}_h. \quad (4.2)$$

Note that the Euler–Lagrange equations for (4.2) give the standard *mixed finite element formulation*, cf. [20, Remark 3.7]: find  $\sigma_h^{\mathbf{a}} \in \mathbf{V}_h^{\mathbf{a}}$  and  $p_h^{\mathbf{a}} \in Q_h^{\mathbf{a}}$  such that

$$(\sigma_h^{\mathbf{a}}, \mathbf{v}_h)_{\omega_{\mathbf{a}}} - (p_h^{\mathbf{a}}, \nabla \cdot \mathbf{v}_h)_{\omega_{\mathbf{a}}} = -(\psi_{\mathbf{a}} \nabla u_h, \mathbf{v}_h)_{\omega_{\mathbf{a}}} \quad \forall \mathbf{v}_h \in \mathbf{V}_h^{\mathbf{a}}, \quad (4.3a)$$

$$(\nabla \cdot \sigma_h^{\mathbf{a}}, q_h)_{\omega_{\mathbf{a}}} = (\psi_{\mathbf{a}} \lambda_h u_h - \nabla u_h \cdot \nabla \psi_{\mathbf{a}}, q_h)_{\omega_{\mathbf{a}}} \quad \forall q_h \in Q_h^{\mathbf{a}}. \quad (4.3b)$$

Consequently, under Assumption 4.2,  $\nabla \cdot \sigma_h = \lambda_h u_h$ , cf., e.g., [20, Lemma 3.5].

### 4.3 Conforming local residual liftings

To estimate  $\|\text{Res}(u_h, \lambda_h)\|_{-1}$  from below, we solve conforming primal counterparts of problems (4.3). Note here that

$$-(\psi_{\mathbf{a}} \nabla u_h, \nabla v)_{\omega_{\mathbf{a}}} + (\psi_{\mathbf{a}} \lambda_h u_h - \nabla u_h \cdot \nabla \psi_{\mathbf{a}}, v)_{\omega_{\mathbf{a}}} = \langle \text{Res}(u_h, \lambda_h), \psi_{\mathbf{a}} v \rangle_{V', V} \quad \forall v \in V,$$

for all  $\mathbf{a} \in \mathcal{V}_h$ . On each patch  $\omega_{\mathbf{a}}$  around the vertex  $\mathbf{a} \in \mathcal{V}_h$ , define

$$H_*^1(\omega_{\mathbf{a}}) := \{v \in H^1(\omega_{\mathbf{a}}); (v, 1)_{\omega_{\mathbf{a}}} = 0\}, \quad \mathbf{a} \in \mathcal{V}_h^{\text{int}}, \quad (4.4a)$$

$$H_*^1(\omega_{\mathbf{a}}) := \{v \in H^1(\omega_{\mathbf{a}}); v = 0 \text{ on } \partial\omega_{\mathbf{a}} \cap \partial\Omega\}, \quad \mathbf{a} \in \mathcal{V}_h^{\text{ext}}, \quad (4.4b)$$

and let  $X_h^{\mathbf{a}}$  be an arbitrary discrete subspace of  $H_*^1(\omega_{\mathbf{a}})$ , typically  $\mathbb{P}_{p+1}(\mathcal{T}_{\mathbf{a}}) \cap H_*^1(\omega_{\mathbf{a}})$ .

**Definition 4.4** (Conforming local Neumann problems). Define  $r_h^{\mathbf{a}} \in X_h^{\mathbf{a}}$  by

$$(\nabla r_h^{\mathbf{a}}, \nabla v_h)_{\omega_{\mathbf{a}}} = \langle \text{Res}(u_h, \lambda_h), \psi_{\mathbf{a}} v_h \rangle_{V', V} \quad \forall v_h \in X_h^{\mathbf{a}}$$

for each  $\mathbf{a} \in \mathcal{V}_h$ . Then set  $r_h := \sum_{\mathbf{a} \in \mathcal{V}_h} \psi_{\mathbf{a}} r_h^{\mathbf{a}}$ .

The functions  $r_h^{\mathbf{a}}$  are discrete Riesz projections of the local residual with hat-weighted test functions. As all  $\psi_{\mathbf{a}} r_h^{\mathbf{a}} \in H_0^1(\omega_{\mathbf{a}})$ ,  $r_h \in V$ , though  $r_h^{\mathbf{a}} \notin V$ .

### 4.4 Dual norm of the residual equivalences

Following Carstensen and Funken [13, Theorem 3.1], Braess *et al.* [8, Section 3], or [20, Lemma 3.12], there exists a constant  $C_{\text{cont,PF}}$  only depending on the mesh regularity parameter  $\kappa_{\mathcal{T}}$  such that

$$\|\nabla(\psi_{\mathbf{a}} v)\|_{\omega_{\mathbf{a}}} \leq C_{\text{cont,PF}} \|\nabla v\|_{\omega_{\mathbf{a}}} \quad \forall v \in H_*^1(\omega_{\mathbf{a}}), \forall \mathbf{a} \in \mathcal{V}_h. \quad (4.5)$$

Moreover, the key result of Braess *et al.* [8, Theorem 7], see also Ern and Vohralík [20, Corollary 3.16] and [21], states that the reconstructions of Definition 4.3 under Assumptions 4.1 and 4.2 satisfy the following *stability* property,

$$\|\psi_{\mathbf{a}} \nabla u_h + \sigma_h^{\mathbf{a}}\|_{\omega_{\mathbf{a}}} \leq C_{\text{st}} \sup_{v \in H_*^1(\omega_{\mathbf{a}}); \|\nabla v\|_{\omega_{\mathbf{a}}}=1} \langle \text{Res}(u_h, \lambda_h), \psi_{\mathbf{a}} v \rangle_{V', V}. \quad (4.6)$$

The constant  $C_{\text{st}} > 0$  again only depends on  $\kappa_{\mathcal{T}}$ , and a computable upper bound on  $C_{\text{st}}$  is given in [20, Lemma 3.23]. With these results, we can summarize the main result of this section:

**Theorem 4.5** (Residual equivalences). Let  $(u_h, \lambda_h) \in V \times \mathbb{R}$  verifying Assumptions 4.1 and 4.2 be arbitrary. Then, for  $\sigma_h$  of Definition 4.3 and  $r_h$  of Definition 4.4,

$$\frac{\langle \text{Res}(u_h, \lambda_h), r_h \rangle_{V', V}}{\|\nabla r_h\|} \leq \|\text{Res}(u_h, \lambda_h)\|_{-1} \leq \|\nabla u_h + \sigma_h\|, \quad (4.7a)$$

$$\|\nabla u_h + \sigma_h\| \leq (d+1) C_{\text{st}} C_{\text{cont,PF}} \|\text{Res}(u_h, \lambda_h)\|_{-1}. \quad (4.7b)$$

*Proof.* Fix  $v \in V$  with  $\|\nabla v\| = 1$ . Using definition (2.7a), adding and subtracting  $(\boldsymbol{\sigma}_h, \nabla v)$ , and employing the Green theorem and the equilibrium property (4.1b) yields

$$(\text{Res}(u_h, \lambda_h), v)_{V',V} = \lambda_h(u_h, v) - (\nabla u_h, \nabla v) = -(\nabla u_h + \boldsymbol{\sigma}_h, \nabla v).$$

Thus, definition (2.7b) of the dual norm of the residual and the Cauchy–Schwarz inequality yield the upper bound in (4.7a). This actually also holds for  $\mathbf{V}_h$  being the cheaper RTN space of order  $p$  and not  $p+1$ , as the key property (4.1b) still holds. The lower bound in (4.7a) is trivial from (2.7b) by taking  $v = r_h \in V$ . Importantly, this can further be bounded by a Hilbertian sum of  $\|\nabla r_h^{\mathbf{a}}\|_{\omega_{\mathbf{a}}}$ , which can be seen as in [43, proof of Theorem 2]. Thus, this bound is meaningful. Finally, to prove (4.7b), we proceed as in Ern and Vohralík [20, Lemmas 3.22 and 3.23], while treating the weak norm  $\|\text{Res}(u_h, \lambda_h)\|_{-1}$  as in Ciarlet and Vohralík [17, Theorems 4.7 and 5.1]. One builds here crucially on inequalities (4.5) and (4.6) and relies on the choice  $p+1$ .  $\square$

## 5 Guaranteed and fully computable upper and lower bounds

We combine here the different results of the previous sections to derive the actual guaranteed and fully computable bounds for eigenvalues (in Section 5.1) and eigenvectors (in Section 5.2). A discussion of the results is provided in Section 5.3. We will sometimes use  $\zeta_{(h)} \in V$ , the solution of the Laplace *source problem*  $-\Delta \zeta_{(h)} = \boldsymbol{\varkappa}_{(h)}$  in  $\Omega$ ,  $\zeta_{(h)} = 0$  on  $\partial\Omega$ , i.e.,

$$(\nabla \zeta_{(h)}, \nabla v) = (\boldsymbol{\varkappa}_{(h)}, v) \quad \forall v \in V. \quad (5.1)$$

We also denote by  $V_h := \mathbb{P}_1(\mathcal{T}_h) \cap V$  the lowest-order conforming finite element space, i.e., the span of  $\psi_{\mathbf{a}}$  over all  $\mathbf{a} \in \mathcal{V}_h^{\text{int}}$ , and by  $h$  the maximal diameter of all  $K \in \mathcal{T}_h$ .

### 5.1 Eigenvalues

We first tackle the upper and lower bounds for the first eigenvalue  $\lambda_1$ . We discuss the necessary auxiliary eigenvalue bounds below in Remark 5.2.

**Theorem 5.1** (Guaranteed bounds for the first eigenvalue). *Let  $\underline{\lambda}_2 > 0$  and  $\underline{\lambda}_1 > 0$  be such that  $\underline{\lambda}_2 \leq \lambda_2$  and  $\underline{\lambda}_1 \leq \lambda_1$ . Let  $(u_h, \lambda_h)$  be any element of  $V \times \mathbb{R}^+$  verifying Assumption 3.4 and the inequality*

$$\lambda_h < \underline{\lambda}_2. \quad (5.2)$$

*Let next Assumptions 4.1 and 4.2 hold and let  $\boldsymbol{\sigma}_h$  be constructed following Definition 4.3 and  $r_h$  following Definition 4.4. Then*

$$\lambda_h - \eta^2 \leq \lambda_1 \leq \lambda_h - \tilde{\eta}^2, \quad (5.3)$$

*where we distinguish the following three cases:*

**Case A** (No smallness assumption) *If  $(u_1, u_h) \geq 0$  is known to hold, set  $\bar{\gamma}_h := (1 - \frac{\lambda_h}{\lambda_2})^{-1} \underline{\lambda}_2^{-\frac{1}{2}}$ ; if only  $(u_h, 1) > 0$  holds, take  $\bar{\gamma}_h := (1 - \frac{\lambda_h}{\lambda_2})^{-1} \underline{\lambda}_2^{-\frac{1}{2}} (1 - \|u_h - \Pi_0 u_h\|)^{-\frac{1}{2}}$ . Then (5.3) holds with*

$$\eta^2 := (1 + 4\lambda_h \bar{\gamma}_h^2) \|\nabla u_h + \boldsymbol{\sigma}_h\|^2, \quad (5.4a)$$

$$\tilde{\eta}^2 := -2\lambda_h \bar{\gamma}_h^2 \|\nabla u_h + \boldsymbol{\sigma}_h\|^2 + \frac{\lambda_1}{2} \left( \sqrt{1 + \frac{4}{\underline{\lambda}_1} \frac{\langle \text{Res}(u_h, \lambda_h), r_h \rangle_{V',V}^2}{\|\nabla r_h\|^2}} - 1 \right). \quad (5.4b)$$

**Case B** (Improved estimates under a smallness assumption) *Let  $(u_h, 1) > 0$  and*

$$\bar{\alpha}_h := \sqrt{2} \left( 1 - \frac{\lambda_h}{\lambda_2} \right)^{-1} \underline{\lambda}_2^{-\frac{1}{2}} \|\nabla u_h + \boldsymbol{\sigma}_h\| \leq \min \left\{ \sqrt{2}, |\Omega|^{-\frac{1}{2}} (u_h, 1) \right\}. \quad (5.5)$$

Then (5.3) holds with

$$\eta^2 := \left(1 - \frac{\lambda_h}{\lambda_2}\right)^{-2} \left(1 - \frac{\bar{\alpha}_h^2}{4}\right)^{-1} \|\nabla u_h + \sigma_h\|^2, \quad (5.6a)$$

$$\tilde{\eta}^2 := \frac{1}{2} \left(1 - \frac{\lambda_h}{\lambda_2}\right) \left(1 - \frac{\bar{\alpha}_h^2}{4}\right) \frac{\lambda_1}{2} \left( \sqrt{1 + \frac{4}{\lambda_1} \frac{\langle \text{Res}(u_h, \lambda_h), r_h \rangle_{V', V}^2}{\|\nabla r_h\|^2}} - 1 \right). \quad (5.6b)$$

**Case C** (Optimal estimates under elliptic regularity assumption) *Let  $(u_h, 1) > 0$  and assume that the solution  $\zeta_{(h)}$  of problem (5.1) belongs to the space  $H^{1+\delta}(\Omega)$ ,  $0 < \delta \leq 1$ , so that the approximation and stability estimates*

$$\inf_{v_h \in V_h} \|\nabla(\zeta_{(h)} - v_h)\| \leq C_I h^\delta |\zeta_{(h)}|_{H^{1+\delta}(\Omega)}, \quad (5.7a)$$

$$|\zeta_{(h)}|_{H^{1+\delta}(\Omega)} \leq C_S \|\mathbf{z}_{(h)}\| \quad (5.7b)$$

are satisfied. Set  $\bar{\gamma}_h := \left(1 - \frac{\lambda_h}{\lambda_2}\right)^{-1} C_I C_S h^\delta$  and let

$$\sqrt{2\bar{\gamma}_h} \|\nabla u_h + \sigma_h\| \leq |\Omega|^{-\frac{1}{2}} (u_h, 1). \quad (5.8)$$

Then (5.3) holds with  $\eta^2$  and  $\tilde{\eta}^2$  given by (5.4).

**Remark 5.2** (Auxiliary bounds  $\lambda_1$  and  $\lambda_2$ ). *A straightforward consequence of the min-max principle for self-adjoint operators, see, e.g., Gilbarg and Trudinger [24], is that  $\Omega \subset \Omega^+ \Rightarrow \lambda_k(\Omega^+) \leq \lambda_k$  for all  $k \geq 1$ , where  $\lambda_k(\Omega^+)$  is the  $k$ -th eigenvalue on  $\Omega^+$ . We can often obtain  $\lambda_2$  and  $\lambda_1$  necessary in Theorem 5.1 by this domain inclusion for  $\Omega^+$  with known exact eigenvalues (typically rectangular  $d$ -parallelepipeds or  $d$ -spheres, cf. [53]). When such analytic bounds are too rough to be useful, namely for  $\lambda_2$  in (5.2), guaranteed and easily computable numerical bounds can be used from Liu and Oishi [39] (on convex domains for  $d = 2$ ), Carstensen and Gedicke [15], or Liu [37]. In what concerns  $\lambda_1$ , a very precise choice is to use  $\lambda_1 := \lambda_h - \eta^2$ , where  $\eta^2$  was first computed with a rather rough bound  $\lambda_1$ . Finally, as a ‘‘practical gratis’’ strategy for  $\lambda_2$ , one may simply use  $\lambda_{h,2}$  computed by the linear algebra toolbox when solving for  $(\lambda_h, u_h)$ , see, e.g., Saad [49] and the references therein. Then Theorem 5.1 (and 5.4 below) may no longer hold stricto sensu, but sharp upper and lower bounds are still expected/observed in practice.*

**Remark 5.3** (Constants  $C_I$  and  $C_S$ ). *Let  $\Omega$  be a convex polygon in  $\mathbb{R}^2$ . Then it is classical that the solution  $\zeta_{(h)}$  of (5.1) belongs to  $H^2(\Omega)$  and  $|\zeta_{(h)}|_{H^2(\Omega)} = \|\Delta \zeta_{(h)}\| = \|\mathbf{z}_{(h)}\|$ , so that  $\delta = 1$  and  $C_S = 1$ , see Grisvard [26, Theorem 4.3.1.4]. In this situation, calculable bounds on  $C_I$  can be found in Liu and Kikuchi [38] and Carstensen et al. [16], see also Liu and Oishi [39, Section 2] and the references therein; in particular, for a mesh formed by isosceles right-angled triangles,  $C_I \leq \frac{0.493}{\sqrt{2}}$ .*

We now prove Theorem 5.1, separately for each case:

*Proof (Case A).* If  $(u_1, u_h) \geq 0$  is known to hold, we can start from the  $L^2(\Omega)$  bound (3.9). If this is not the case but  $(u_h, 1) > 0$  holds, we first inspect the proof of Lemma 3.2 to obtain an alternative  $L^2(\Omega)$  estimate. We have  $-2(u_1, u_h) = -2(u_1, u_h - \Pi_0 u_h) - 2(u_1, \Pi_0 u_h)$ , where  $\Pi_0 u_h$  stands for the mean value of  $u_h$  on  $\Omega$ . Note that the second term is negative by the sign assumption  $(u_1, 1) > 0$  on  $u_1$ . Thus, instead of (3.8), as  $\|u_1\| = 1$  and  $\|u_h - \Pi_0 u_h\| < 1$ ,

$$\|u_1 - u_h\|^2 \leq 2 + 2\|u_h - \Pi_0 u_h\| =: \delta_h < 4.$$

Consequently, (3.11) implies  $\|u_1 - u_h\|^2 \leq \|\nabla \mathbf{z}_{(h)}\|^2 C_h^{-1} \lambda_2^{-1} (1 - \delta_h/4)^{-1}$ . Thus, the upper bound in (4.7a) and the assumption (5.2) enable us to give a computable upper bound on the  $L^2(\Omega)$  error. Consequently, the lower bound in (5.3) follows by combining the upper bounds in (3.12), (3.16), and once again (4.7a). The converse inequality starts from the lower bound in (3.12). We then need to bound  $\|\nabla(u_1 - u_h)\|^2$  from below. From the lower bound in (3.16), we have  $\|\text{Res}(u_h, \lambda_h)\|_{-1}^2 \leq \|\nabla(u_1 - u_h)\|^4 / \lambda_1 + \|\nabla(u_1 - u_h)\|^2$ . Developing this quadratic inequality gives

$$\|\nabla(u_1 - u_h)\|^2 \geq \frac{\lambda_1}{2} \left( \sqrt{1 + \frac{4}{\lambda_1} \|\text{Res}(u_h, \lambda_h)\|_{-1}^2} - 1 \right). \quad (5.9)$$

Then the lower bound in (4.7a) and the inequality  $1 - \frac{\lambda_h}{\lambda_2} \leq 1 - \frac{\lambda_1}{\lambda_2}$  (obtained from  $0 < \lambda_1 \leq \lambda_h < \lambda_2 \leq \lambda_2$ ) finish the proof.  $\square$

*Proof (Case B).* The proof proceeds as above. Note that conditions in (5.5) imply that  $\alpha_h \leq \sqrt{2}$  and  $\alpha_h \leq |\Omega|^{-\frac{1}{2}}(u_h, 1)$  for  $\alpha_h$  of (3.9). We can thus use Lemma 3.3 to find that  $(u_1, u_h)$  is indeed non-negative, Lemma 3.2 for the  $L^2(\Omega)$  bound, and the improved estimates (3.13) of Theorem 3.5 and (3.17) of Theorem 3.6.  $\square$

*Proof (Case C).* The proof is as in Case A (with  $(u_1, u_h) \geq 0$ ), but it relies on Lemma 3.1. It additionally uses the Aubin–Nitsche trick, cf. [9, Theorem 5.4.8], [26, Theorem 4.3.1.4], or [6]. By (5.1), (2.8a), and the orthogonality of Assumption 4.1

$$\|\boldsymbol{z}_{(h)}\|^2 = (\nabla \zeta_{(h)}, \nabla \boldsymbol{z}_{(h)}) = (\nabla(\zeta_{(h)} - \zeta_h), \nabla \boldsymbol{z}_{(h)}),$$

where  $\zeta_h \in V_h$  is arbitrary. Thus the Cauchy–Schwarz inequality and estimates (5.7) give  $\|\boldsymbol{z}_{(h)}\|^2 \leq C_I C_S h^\delta \|\boldsymbol{z}_{(h)}\| \|\nabla \boldsymbol{z}_{(h)}\|$ . Consequently, the characterization (2.8b) proves

$$\|\boldsymbol{z}_{(h)}\| \leq C_I C_S h^\delta \|\text{Res}(u_h, \lambda_h)\|_{-1}.$$

$\square$

## 5.2 Eigenvectors

We now summarize our estimate on the energy error in the approximation of the first eigenvector, as well as its efficiency and robustness:

**Theorem 5.4** (Guaranteed and robust bound for the first eigenvector error). *Let the assumptions of Theorem 5.1 be verified. Then the energy error can be bounded via*

$$\|\nabla(u_1 - u_h)\| \leq \eta, \tag{5.10}$$

where  $\eta$  is defined in the three considered cases by (5.4a) or (5.6a), with appropriate  $\bar{\gamma}_h$ . All these estimators  $\eta$  are efficient as

$$\|\nabla u_h + \boldsymbol{\sigma}_h\|^2 \leq (d+1)^2 C_{\text{st}}^2 C_{\text{cont,PF}}^2 \left( \frac{\|\nabla(u_1 - u_h)\|^2}{\lambda_1} + 1 \right) \|\nabla(u_1 - u_h)\|^2. \tag{5.11}$$

*Proof.* The guaranteed error bound (5.10) follows upon combining the upper bounds in estimates (3.17) or (3.16) together with (4.7a). The efficiency (5.11) is a consequence of (4.7b) and of the lower bound in (3.16).  $\square$

## 5.3 Comments

We collect here a few comments about Theorems 5.1 and 5.4.

**Remark 5.5** (Sharpness and comparison of the different bounds of Theorems 5.1 and 5.4). *The advantage of Case A is that it holds on an arbitrarily coarse mesh, provided that only the structural assumption (5.2) holds. It may, however, happen that the overestimation is rather crude. On very rough meshes, a minimum between  $\eta^2$  of (5.4a) and the estimators of [35, 50], with  $\boldsymbol{\sigma}_h$  of Definition 4.3, can then be used, and  $\lambda_h$  can be used simply in place of  $\lambda_h - \tilde{\eta}^2$ . Case B, under the “fine enough mesh” condition (5.5), then significantly improves the multiplicative factor in front of the central term  $\|\nabla u_h + \boldsymbol{\sigma}_h\|$ . The bound of Case B still holds without any regularity/convexity/dimension assumption and all the quantities appearing are known. Finally, also the factor  $(1 - \frac{\lambda_h}{\lambda_2})^{-1}$  is asymptotically removed in Case C, when  $\delta > 0$  and  $h \rightarrow 0$ . Here, however, elliptic regularity is needed, see Remark 5.3.*

**Remark 5.6** (Dependence on the maximal element diameter  $h$ ). *The maximal element diameter  $h$  is not present at all in Cases A and B of Theorem 5.1 and it does not necessarily need to tend to zero in Case C: it only appears as a multiplicative factor of the principal estimator  $\|\nabla u_h + \boldsymbol{\sigma}_h\|$ . (Note that  $h \rightarrow 0$  may not be needed in a posteriori convergence proofs.) This stands in contrast to previous guaranteed results like [39, Theorem 4.3], [15, Theorem 3.2], or [37, Theorem 2.1].*

**Remark 5.7** (Polynomial-degree robustness). *Noting that  $\frac{\|\nabla(u_1 - u_h)\|^2}{\lambda_1} \leq \frac{2(\lambda_1 + \lambda_h)}{\lambda_1}$  (in fact this term becomes negligible with mesh refinement/increasing the polynomial degree), we conclude that the result of Theorem 5.4 is fully robust with respect to the polynomial degree  $p$  of  $u_h$ : all the constants in the comparison between the error  $\|\nabla(u_1 - u_h)\|$  and the estimate featuring  $\|\nabla u_h + \sigma_h\|$  are independent of  $p$ .*

**Remark 5.8** (Error localization and mesh adaptivity). *Since  $\|\nabla u_h + \sigma_h\|^2 = \sum_{K \in \mathcal{T}_h} \|\nabla u_h + \sigma_h\|_K^2$ , a crucial advantage of the estimators of Theorems 5.1 and 5.4 is that these local contributions can be used in adaptive mesh refinement based on marking strategies. This is tightly linked to Remark 5.6.*

## 6 Application to conforming finite elements

We verify in this section the conditions of the application of our results to the conforming finite element method.

Let  $V_h := \mathbb{P}_p(\mathcal{T}_h) \cap V$  for a given polynomial degree  $p \geq 1$ . In the finite element method, one looks for  $(u_h, \lambda_h) \in V_h \times \mathbb{R}^+$  with  $\|u_h\| = 1$  such that

$$(\nabla u_h, \nabla v_h) = \lambda_h(u_h, v_h) \quad \forall v_h \in V_h, \quad (6.1)$$

with the sign ideally fixed by  $(u_1, u_h) \geq 0$ , practically by  $(u_h, 1) > 0$ . Assumption 3.4 follows by taking  $v_h = u_h$  in (6.1), whereas the choice  $v_h = \psi_{\mathbf{a}}$  implies Assumption 4.1. Taking the mixed finite element reconstruction spaces  $\mathbf{V}_h \times Q_h$  of order  $p + 1$  validates Assumption 4.2. Thus, upon verifying (5.2) and possibly checking (5.5) or (5.8), all the results of Theorems 5.1 and 5.4 hold for any  $p \geq 1$ .

## 7 Numerical experiments

We finally numerically illustrate our estimates on three test cases in  $\mathbb{R}^2$ , for conforming finite elements (6.1) of order  $p = 1$ . We actually only use the cheaper Raviart–Thomas–Nédélec space of degree  $p = 1$  for the flux equilibration instead of  $p + 1$ . This still gives guaranteed bounds, see the proof of Theorem 4.5, and we do not observe any asymptotic loss of efficiency. The implementation was done in the FreeFem++ code [28].

### 7.1 Unit square

We start by testing the framework on a geometry where everything is known explicitly: the unit square  $\Omega = (0, 1)^2$ . The eigenvalues on a square of size  $H$  being  $\pi^2(k^2 + l^2)/H^2$ ,  $k, l = 1, \dots, \infty$ , the first and second eigenvalues are  $\lambda_1 = 2\pi^2$  and  $\lambda_2 = 5\pi^2$ , respectively. In consequence, we can easily choose different  $\lambda_1 \leq \lambda_1$  and  $\lambda_2 \leq \lambda_2$  and analyze the results with respect to these choices. The first eigenfunction is given by  $u_1(x, y) = \sin(\pi x) \sin(\pi y)$ . We can here apply the refined elliptic regularity of Case C, since  $d = 2$  and the domain is convex. We use constants  $C_S = 1$  and  $\delta = 1$  given in Remark 5.3.

#### 7.1.1 Structured mesh

We first illustrate in Table 1 how quickly the computable conditions (5.2) and (5.8) are satisfied under a uniform refinement of a structured mesh. We take  $C_I = \frac{0.493}{\sqrt{2}}$  following Remark 5.3 and consider  $N = 3, 4, 5$  subdivisions of each boundary of  $\Omega$  for the two choices  $\lambda_1 = 1.5\pi^2$ ,  $\lambda_2 = 4.5\pi^2$  and  $\lambda_1 = 0.5\pi^2$ ,  $\lambda_2 = 3\pi^2$ , respectively. Note that the finite element space on the coarsest mesh such that all conditions are satisfied contains 25, resp. 36, degrees of freedom only. Indeed, it turns out that our conditions are rather mild.

Next, Figure 1 (left) illustrates the convergence of the error  $\lambda_h - \lambda_1$  as well as of its lower and upper bounds  $\tilde{\eta}^2, \eta^2$  given by Case C of Theorem 5.1. We also plot the energy error in the eigenfunction  $\|\nabla(u_1 - u_h)\|$  and its upper bound  $\eta$  of Theorem 5.4, Case C. The convergence rates are optimal as expected from the theory.

We present in Table 2 precise numbers of the lower and upper bounds  $\lambda_h - \eta^2 \leq \lambda_1 \leq \lambda_h - \tilde{\eta}^2$  on the exact eigenvalue  $\lambda_1$ , the effectivity indices of the lower and upper bounds  $\tilde{\eta}^2 \leq \lambda_h - \lambda_1 \leq \eta^2$  of the error  $\lambda_h - \lambda_1$ , and the effectivity index of the upper bound  $\|\nabla(u_1 - u_h)\| \leq \eta$ , given respectively by

$$I_{\lambda, \text{eff}}^{\text{lb}} := \frac{\lambda_h - \lambda_1}{\tilde{\eta}^2}, \quad I_{\lambda, \text{eff}}^{\text{ub}} := \frac{\eta^2}{\lambda_h - \lambda_1}, \quad I_{u, \text{eff}}^{\text{ub}} := \frac{\eta}{\|\nabla(u_1 - u_h)\|}. \quad (7.1)$$

	$N$	$h$	ndof	$\underline{\lambda}_2 - \lambda_h$ (5.2)	$ \Omega ^{-\frac{1}{2}}(u_h, 1) - \sqrt{2}\gamma_h \ \nabla u_h + \sigma_h\ $ (5.8)
$\underline{\lambda}_1 = 1.5\pi^2$ $\underline{\lambda}_2 = 4.5\pi^2$	3	0.4714	16	19.04 (✓)	-0.64 (×)
	4	0.3536	25	21.55 (✓)	0.12 (✓)
	5	0.2828	36	22.69 (✓)	0.40 (✓)
$\underline{\lambda}_1 = 0.5\pi^2$ $\underline{\lambda}_2 = 3\pi^2$	3	0.4714	16	4.233 (✓)	-3.49 (×)
	4	0.3536	25	6.743 (✓)	-0.66 (×)
	5	0.2828	36	7.887 (✓)	0.02 (✓)

Table 1: [Unit square, structured mesh] Validation of assumptions (5.2), (5.8)

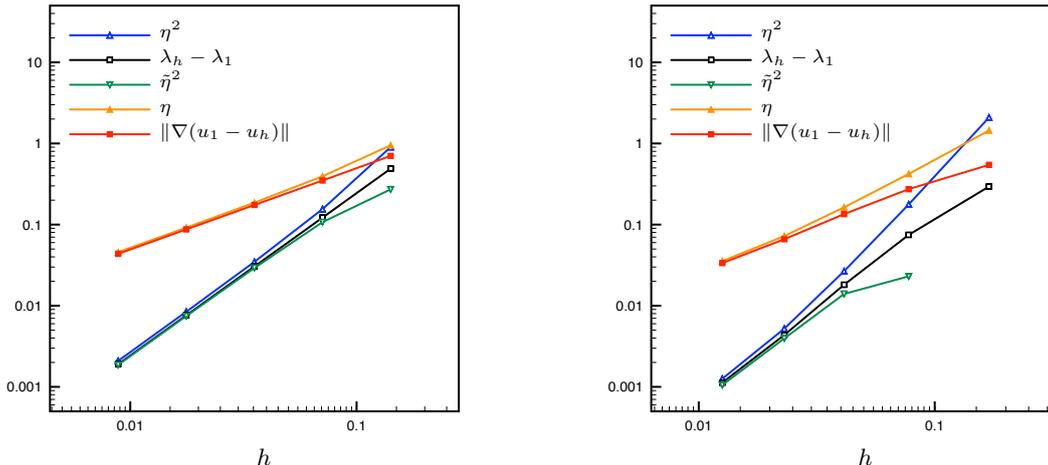


Figure 1: [Unit square] Error in the eigenvalue and eigenvector approximation, its lower bound (eigenvalue only) and its upper bound for the choice  $\underline{\lambda}_1 = 1.5\pi^2$ ,  $\underline{\lambda}_2 = 4.5\pi^2$ ; sequence of structured (left) and unstructured but quasi-uniform (right) meshes; Case C

We observe rather sharp results, and this also for the relative error

$$E_{\lambda, \text{rel}} := 2 \frac{(\lambda_h - \tilde{\eta}^2) - (\lambda_h - \eta^2)}{(\lambda_h - \tilde{\eta}^2) + (\lambda_h - \eta^2)}. \quad (7.2)$$

### 7.1.2 Unstructured mesh

Consider now a sequence of unstructured quasi-uniform meshes, obtained by an initial partition of each boundary edge into  $N$  intervals. Conditions (5.2), (5.8) turn here to be satisfied similarly as in Table 1.

The convergence plots for this case are presented in Figure 1 (right), showing a similar behavior as for the structured meshes. This time, we use the upper bound on  $C_1$  according to [38, Eqn. (46)]:  $C_1 = 0.493 \max_{K \in \mathcal{T}_h} \frac{1 + |\cos(\theta_K)|}{\sin(\theta_K)} \sqrt{\frac{\nu_+(\alpha_K, \theta_K)}{2}} \frac{h_K^{[38]}}{h_K}$ . We refer to [38] for the definition of  $h_K^{[38]}$  and other notation. We observe in Table 3 that the results are similar to structured meshes; in particular the case of  $\underline{\lambda}_1 = 0.5\pi^2$ ,  $\underline{\lambda}_2 = 3\pi^2$  is less sensitive to the unstructured mesh (not presented).

## 7.2 L-shaped domain

We next consider the L-shaped domain  $\Omega := (-1, 1)^2 \setminus [0, 1] \times [-1, 0]$ , where  $\lambda_1 \approx 9.6397238440$  is known to high accuracy [53]. Including  $\Omega$  into the square  $\Omega^+ = (-1, 1)^2$ , cf. Remark 5.2, we take  $\underline{\lambda}_1 = \lambda_1(\Omega^+) = \pi^2/2$ , whereas  $\underline{\lambda}_2 = 15.1753$  from Table 1 of [37] is employed. We test here the Cases A and B.

$N$	$h$	ndof	$\lambda_1$	$\lambda_h$	$\lambda_h - \eta^2$	$\lambda_h - \tilde{\eta}^2$	$I_{\lambda, \text{eff}}^{\text{lb}}$	$I_{\lambda, \text{eff}}^{\text{ub}}$	$E_{\lambda, \text{rel}}$	$I_{u, \text{eff}}^{\text{ub}}$
10	0.1414	121	19.7392	20.2284	19.3256	19.9566	1.80	1.85	3.21e-02	1.35
20	0.0707	441	19.7392	19.8611	19.7058	19.7539	1.14	1.27	2.44e-03	1.13
40	0.0354	1681	19.7392	19.7697	19.7349	19.7404	1.04	1.14	2.79e-04	1.07
80	0.0177	6561	19.7392	19.7468	19.7384	19.7394	1.02	1.11	4.91e-05	1.05
160	0.0088	25921	19.7392	19.7411	19.7390	19.7392	1.02	1.10	1.14e-05	1.05

Table 2: [Unit square, structured mesh] Lower and upper bounds on the exact eigenvalue  $\lambda_1$ , the effectivity indices, and the relative eigenvalue error;  $\underline{\lambda}_1 = 1.5\pi^2$ ,  $\underline{\lambda}_2 = 4.5\pi^2$ ; Case C

$N$	$h$	ndof	$\lambda_1$	$\lambda_h$	$\lambda_h - \eta^2$	$\lambda_h - \tilde{\eta}^2$	$I_{\lambda, \text{eff}}^{\text{lb}}$	$I_{\lambda, \text{eff}}^{\text{ub}}$	$E_{\lambda, \text{rel}}$	$I_{u, \text{eff}}^{\text{ub}}$
10	0.1698	143	19.7392	20.0336	17.9458	20.6491	-0.48	7.09	1.40e-01	2.65
20	0.0776	523	19.7392	19.8139	19.6366	19.7909	3.24	2.37	7.83e-03	1.54
40	0.0413	1975	19.7392	19.7573	19.7307	19.7434	1.30	1.47	6.42e-04	1.21
80	0.0230	7704	19.7392	19.7436	19.7383	19.7396	1.10	1.20	6.41e-05	1.09
160	0.0126	30666	19.7392	19.7403	19.7391	19.7393	1.07	1.12	1.04e-05	1.06

Table 3: [Unit square, unstructured mesh] Lower and upper bounds on the exact eigenvalue  $\lambda_1$ , the effectivity indices, and the relative eigenvalue error;  $\underline{\lambda}_1 = 1.5\pi^2$ ,  $\underline{\lambda}_2 = 4.5\pi^2$ ; Case C

### 7.2.1 Unstructured mesh with quasi-uniform refinement

We first consider a sequence of unstructured quasi-uniform meshes, with  $N$  elements partitioning the edges of  $\Omega$  of length 2 and  $N/2$  elements the edges of length 1. Table 4 (top) illustrates whether the conditions (5.2), (5.5) are satisfied or not depending on  $N$ . Compared to the square domain, a finer mesh is needed until all conditions are satisfied. On the other hand, as the solution has low regularity, a finer mesh is naturally needed to get some minimal accuracy with a uniform mesh refinement.

Figure 2 (left) illustrates the convergence of  $\lambda_h - \lambda_1$  and of its lower and upper bounds  $\tilde{\eta}^2$ ,  $\eta^2$  given by (5.6) of Theorem 5.1, Case B. The energy error  $\|\nabla(u_1 - u_h)\|$  and its upper bound  $\eta$  of Theorem 5.4, Case B, are also given, with details in Table 5. All the theoretical results are nicely confirmed. The somewhat worse effectivity indices appear because of the multiplicative factor  $(1 - \frac{\lambda_h}{\underline{\lambda}_2})^{-1}$  as discussed in Remark 5.5.

### 7.2.2 Adaptive refinement

We finally test adaptive refinement using the local character of our estimators, see Remark 5.8. We employ the Dörfler marking with  $\theta = 0.6$  and the newest vertex bisection mesh refinement. The same lower bounds  $\underline{\lambda}_1 = \pi^2/2$  and  $\underline{\lambda}_2 = 15.1753$  as for the uniform refinement have been used.

Table 4 (bottom) illustrates whether the conditions (5.2), (5.5) are satisfied under this adaptive refinement. Figure 2 (right) illustrates the error in the eigenvalue and the eigenvector and their bounds using (5.6) in (5.3) and (5.10). The optimal convergence rates are indicated by dashed lines. The initial mesh is structured with 22 degrees of freedom and the conditions (5.2) and (5.5) are all satisfied starting from 140 degrees of freedom. The transition from Case A to Case B in Theorem 5.1 is marked by a dotted line. Table 6 then presents more details.

## 7.3 Domain with a hole

We next consider a domain with a polygonal hole, see Figure 3 (left) illustrating the mesh used at iteration 20 of our adaptive mesh refinement strategy. The lower bounds  $\underline{\lambda}_1$  and  $\underline{\lambda}_2$  on the first and second eigenvalue have been obtained once and for all before starting the adaptive algorithm following the estimates derived in [37], on a uniform mesh with 15231 nodes. Figure 3 (right) shows the interval between our lower  $(\lambda_h - \eta^2)$  and upper  $(\lambda_h - \tilde{\eta}^2)$  bounds on the first eigenvalue, relying on Case B of Theorem 5.1, whose assumptions hold starting from 1886 degrees of freedom; Table 7 states the numbers. Note that the interval size  $(\lambda_h - \tilde{\eta}^2) - (\lambda_h - \eta^2) = \eta^2 - \tilde{\eta}^2$  behaves like  $1/\text{ndof}$ .

	$N$	$h$	ndof	$\underline{\lambda}_2 - \lambda_h$ (5.2)	$\bar{\alpha}_h/\sqrt{2}$ (5.5)	$ \Omega ^{-\frac{1}{2}}(u_h, 1) - \bar{\alpha}_h$ (5.5)
uniform refinement	4	0.7500	22	1.822 (✓)	8.57 (×)	-11.38 (×)
	12	0.2652	148	5.098 (✓)	0.99 (✓)	-0.61 (×)
	22	0.1435	453	5.385 (✓)	0.55 (✓)	0.01 (✓)
adaptive refinement	4	0.7071	34	3.88 (✓)	1.80 (×)	-1.79 (×)
	8	0.5000	85	4.935 (✓)	0.83 (✓)	-0.39 (×)
	11	0.4507	185	5.277 (✓)	0.50 (✓)	0.07 (✓)

Table 4: [L-shaped domain, unstructured mesh and adaptive mesh refinement] Validation of the assumptions (5.2), (5.5) for  $\underline{\lambda}_1 = \pi^2/2$  and  $\underline{\lambda}_2 = 15.1753$

$N$	$h$	ndof	$\lambda_1$	$\lambda_h$	$\lambda_h - \eta^2$	$\lambda_h - \tilde{\eta}^2$	$I_{\lambda, \text{eff}}^{\text{lb}}$	$I_{\lambda, \text{eff}}^{\text{ub}}$	$E_{\lambda, \text{rel}}$	$I_{u, \text{eff}}^{\text{ub}}$
25	0.1263	556	9.6397	9.7637	8.3825	9.7473	7.57	11.14	1.51e-01	3.35
50	0.0634	2286	9.6397	9.6783	9.2904	9.6726	6.77	10.06	4.03e-02	3.19
100	0.0397	8691	9.6397	9.6536	9.5173	9.6515	6.61	9.84	1.40e-02	3.17
200	0.0185	34206	9.6397	9.6448	9.5946	9.6440	6.59	9.85	5.14e-03	3.20
400	0.0094	136062	9.6397	9.6416	9.6226	9.6413	6.68	9.96	1.94e-03	3.33

Table 5: [L-shaped domain, unstructured mesh] Lower and upper bounds on the exact eigenvalue  $\lambda_1$ , the effectivity indices, and the relative eigenvalue error;  $\underline{\lambda}_1 = \pi^2/2$  and  $\underline{\lambda}_2 = 15.1753$ ; Case B

## 7.4 Comparison with existing results

We finally compare our results with some existing ones from [15, 39]. For the unit square, Table 2 reveals that our estimates are sharper compared to the estimates presented in [15], see Figure 6.2 therein. For the L-shaped domain and uniformly refined meshes, we obtain better results than those presented in [15, Figure 6.4], not efficient in this case. Compared to the results presented in [39, Table 5.5], we observe that our lower bound  $\lambda_h - \eta^2$  of the exact eigenvalue is a little less sharp, whereas the upper bound  $\lambda_h - \tilde{\eta}^2$  is not present in [39]. Recall also from Section 1 that our estimates are much cheaper here than those of [39] (there is no auxiliary eigenvalue problem to solve). For adaptive meshes, we observe that our efficiency as measured in [15] by  $\frac{1}{2}(\eta^2 - \tilde{\eta}^2)/|\lambda_1 - \lambda_h + \frac{1}{2}(\tilde{\eta}^2 + \eta^2)|$  is approaching 1.086 which is much better than in [15, Figure 6.5]. Recall finally that an important advantage of our estimates is that they also give a guaranteed control of the eigenvector error and are theoretically sound for adaptive mesh refinement.

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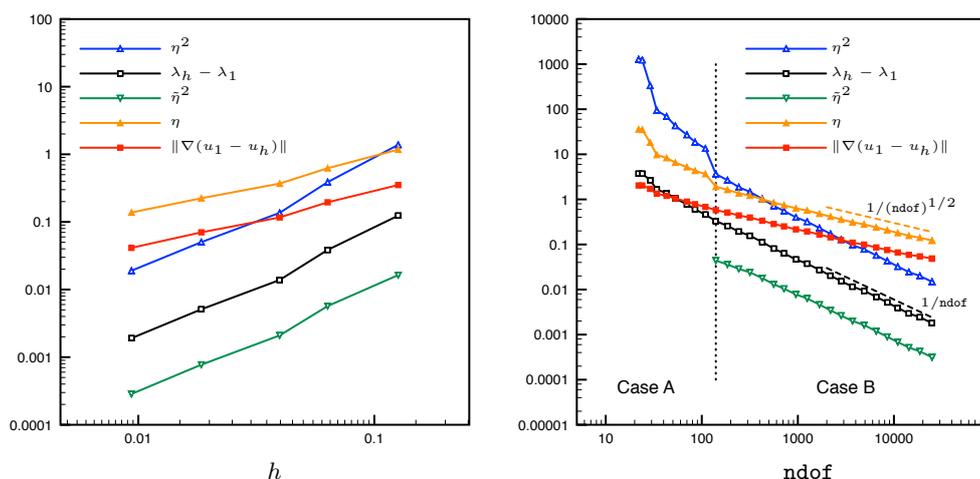


Figure 2: [L-shaped domain, unstructured and adaptive mesh refinement] Error in the eigenvalue and eigenvector approximation, its lower bound (eigenvalue only) and its upper bound for a quasi-uniform refinement (left, Case B) and adaptive refinement (right, Cases A and B).

Level	ndof	$\lambda_1$	$\lambda_h$	$\lambda_h - \eta^2$	$\lambda_h - \tilde{\eta}^2$	$I_{\lambda, \text{eff}}^{\text{lb}}$	$I_{\lambda, \text{eff}}^{\text{ub}}$	$E_{\lambda, \text{rel}}$	$I_{u, \text{eff}}^{\text{ub}}$
10	140	9.6397	9.9700	6.3175	9.9260	7.50	11.06	4.44e-01	3.31
15	561	9.6397	9.7207	9.0035	9.7075	6.17	8.86	7.53e-02	2.98
20	2188	9.6397	9.6601	9.4887	9.6566	5.88	8.43	1.75e-02	2.88
25	8513	9.6397	9.6449	9.6019	9.6440	5.77	8.31	4.37e-03	2.75
30	24925	9.6397	9.6415	9.6266	9.6412	5.73	8.26	1.51e-03	2.51

Table 6: [L-shaped domain, adaptive mesh refinement] Lower and upper bounds on the exact eigenvalue  $\lambda_1$ , the effectivity indices, and the relative eigenvalue error;  $\underline{\lambda}_1 = \pi^2/2$  and  $\underline{\lambda}_2 = 15.1753$ , Case B

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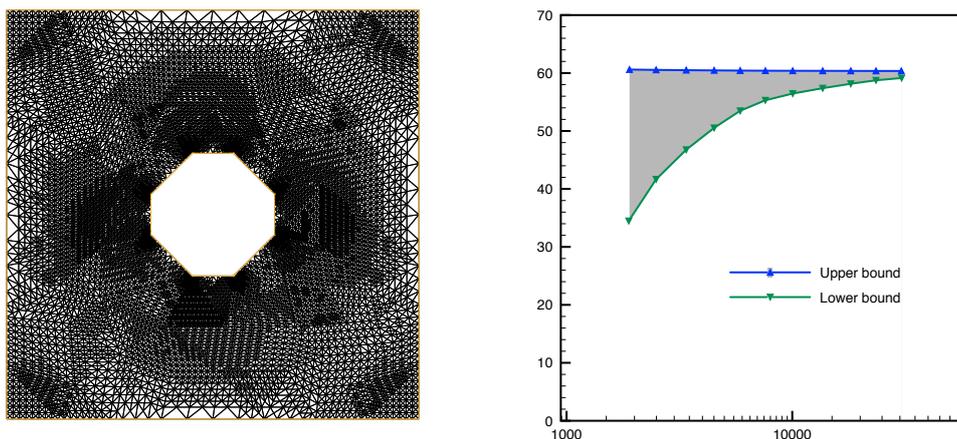


Figure 3: [Domain with a hole, adaptive mesh refinement] Mesh of the adaptive algorithm at iteration 20 (left) and the lower and upper bounds for the exact eigenvalue  $\lambda_1$  (right); Case B

ndof	1886	2494	3390	4508	5879	7602	10047	13640	18163	23494	30533
$\lambda_h - \tilde{\eta}^2$	60.600	60.541	60.493	60.454	60.422	60.401	60.387	60.376	60.367	60.359	60.356
$\lambda_h - \eta^2$	34.481	41.670	46.765	50.539	53.485	55.301	56.459	57.385	58.149	58.746	59.142

Table 7: [Domain with a hole, adaptive mesh refinement] Lower and upper bounds on the exact eigenvalue  $\lambda_1$  as a function of the degrees of freedom; Case B

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