



HAL
open science

Guaranteed and robust a posteriori bounds for Laplace eigenvalues and eigenvectors: conforming approximations

Eric Cancès, Geneviève Dusson, Yvon Maday, Benjamin Stamm, Martin Vohralík

► To cite this version:

Eric Cancès, Geneviève Dusson, Yvon Maday, Benjamin Stamm, Martin Vohralík. Guaranteed and robust a posteriori bounds for Laplace eigenvalues and eigenvectors: conforming approximations. 2015. hal-01194364v1

HAL Id: hal-01194364

<https://inria.hal.science/hal-01194364v1>

Preprint submitted on 6 Sep 2015 (v1), last revised 15 Dec 2016 (v3)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Guaranteed and robust a posteriori bounds for Laplace eigenvalues and eigenvectors: conforming approximations*

Eric Cancès[†] Geneviève Dusson^{‡§} Yvon Maday^{‡¶} Benjamin Stamm[‡]
Martin Vohralík^{**}

September 6, 2015

Abstract

This paper derives a posteriori error estimates for conforming numerical approximations of the Laplace eigenvalue problem with a homogeneous Dirichlet boundary condition. In particular, upper and lower bounds for the first eigenvalue are given. These bounds are guaranteed, fully computable, and converge with the optimal speed to the exact eigenvalue. They are valid under explicit, a posteriori conditions on the computational mesh and on the approximate solution; we also need to assume that the approximate eigenvalue is smaller than a computable lower bound on the second smallest eigenvalue, which can be verified in many cases of practical interest by including the computational domain into a rectangular parallelepiped or a d -sphere. Guaranteed, fully computable, and polynomial-degree robust bounds for the energy error in the approximation of the first eigenvector are derived as well, under the same conditions. Remarkably, there appears no unknown (solution-, regularity-, or polynomial-degree-dependent) constant in our theory, and no convexity/regularity assumption on the computational domain/exact eigenvector(s) is needed. An improvement of the multiplicative constant appearing in our estimates is possible when an elliptic regularity assumption is satisfied with known constants, typically for convex two-dimensional domains. The application of our framework to conforming finite element approximations of arbitrary polynomial degree is provided, along with a numerical illustration on a set of test problems.

Key words: Laplace eigenvalue problem, a posteriori estimate, eigenvalue error, eigenvector error, guaranteed bound, conforming finite element method

1 Introduction

Precise numerical approximation of eigenvalues and eigenvectors is crucial in countless applications. Thus, there has been a long-standing interest in answering the question: what is the size of the errors in computed eigenvalues and eigenvectors? This question is usually tackled via a posteriori error estimates, and in particular the theory based on equilibrated fluxes following Prager and Synge [43], Destuynder and Métivet [18], Braess *et al.* [7], Ern and Vohralík [22], and the references therein enables to give conclusive answers for elliptic source problems such as the Laplace one. The structure of the *Laplace eigenvalue* problem with respect to the a posteriori analysis appears rather different from the elliptic source case, and obtaining optimal estimates turns out being far from straightforward.

*This work was supported by the ANR project MANIF “Mathematical and numerical issues in first-principle molecular simulation”.

[†]Université Paris Est, CERMICS, Ecole des Ponts and INRIA, 6 & 8 Av. Pascal, 77455 Marne-la-Vallée, France (cances@cermics.enpc.fr).

[‡]Sorbonne Universités, UPMC Univ Paris 06, CNRS, UMR 7598, Laboratoire Jacques-Louis Lions, 4 place Jussieu 75005, Paris, France (dusson@ann.jussieu.fr, maday@ann.jussieu.fr, stamm@ann.jussieu.fr).

[§]Sorbonne Universités, UPMC Univ. Paris 06, Institut du Calcul et de la Simulation, 75005, Paris, France.

[¶]Institut Universitaire de France, 75005, Paris, France.

^{||}Division of Applied Mathematics, Brown University, 182 George St, Providence, RI 02912, USA.

**INRIA Paris-Rocquencourt, B.P. 105, 78153 Le Chesnay, France (martin.vohralik@inria.fr).

Recently, though, there has been an important progress in particular in obtaining *guaranteed lower bounds* for the *first eigenvalue*: Armentano and Durán [1], Hu *et al.* [31, 32], Carstensen and Gedicke [14], and Yang *et al.* [53] achieve so via the lowest-order nonconforming finite element method, Kuznetsov and Repin [36] and Šebestová and Vejchodský [48] give numerical-method-independent estimates based on flux (functional) estimates, and Liu and Oishi [39] elaborate fine a priori approximation estimates for lowest-order conforming finite elements. Earlier work comprises Kato [33], Forsythe [23], Weinberger [52], Bazley and Fox [4], Fox and Rheinboldt [24], Moler and Payne [41], Kuttler and Sigillito [34, 35], Still [49], Goerisch and He [26], and Plum [42]. Unfortunately, there typically appears a condition on applicability which is difficult to verify, like in [36, Remark 3.2] and [48, condition (3.6)], a saturation assumption may be necessary, see the discussion in [31], the arguments used are of a priori nature and employ the diameter of the largest mesh element, which potentially entails an overestimation, as in [14, 39], the evaluation of the estimate (for nonconvex domains) requires a solution of an auxiliary eigenvalue problem as in [39], or the bounds do not converge with the correct speed as in [36, 48].

The question of precision for both eigenvalues and eigenvectors has also been investigated previously. For conforming finite elements, relying on the a priori error estimates resumed in Babuška and Osborn [2] and Boffi [6], see also the references therein, a posteriori error estimates have been obtained by Verfürth [51], Maday and Patera [40], Larson [37], Heuveline and Rannacher [30], Durán *et al.* [20], Grubišić and Ovall [28], and Rannacher *et al.* [44], see also the references therein. Unfortunately, these estimates systematically contain uncomputable terms which are claimed higher order on fine enough meshes via a priori arguments.

The purpose of the present paper is to derive *guaranteed* and *optimal a posteriori bounds* on both the *first Laplace eigenvalue* and the *associated eigenvector* for conforming variational methods. Nonconforming methods are treated in Cancès *et al.* [11]. We describe the setting in details in Section 2. Section 3 and Section 4 then contain a collection of equivalence inequalities between respectively the square root of the first eigenvalue error and the first eigenvector energy error, the first eigenvector energy error and the dual norm of the residual, and between the dual norm of the residual and its computable estimates. These results are valid under four hypotheses. First, Assumption 3.1 below requests λ_h , the numerical approximation of the smallest eigenvalue λ_1 , to be smaller than the second smallest eigenvalue λ_2 . Second, there are two conditions in Assumption 3.2 requesting essentially the smallness of the $L^2(\Omega)$ -norm of the Riesz representation of the residual. Finally, a condition on the Galerkin orthogonality of the residual to the finite element hat functions on patches around the mesh vertices (Assumption 4.1) needs to be satisfied, and the approximation spaces need to consist of appropriate piecewise polynomials (Assumption 4.2).

The first condition above can be guaranteed in many cases of practical interest by including the computational domain Ω into a larger domain Ω^+ (see Section 5.2), where the second smallest eigenvalue $\lambda_2(\Omega^+)$ is known (typically a rectangular parallelepiped or a d -sphere), and by requesting $\lambda_h < \underline{\lambda}_2 := \lambda_2(\Omega^+)$, as $\lambda_2(\Omega^+)$ is a lower bound on λ_2 , cf. (6.1) below. Computable estimates on the Riesz representation of the residual (Section 5.1) make the second condition explicit and *a posteriori calculable*, cf. (6.2) below. It can then be always satisfied by refining the computational mesh/increasing the polynomial degree of the approximate solution. More details can be found in Remarks 6.6 and 6.7.

In Section 6, the results of Section 3–Section 5 are turned into guaranteed a posteriori bounds. First, upper and lower bounds for the first eigenvalue are given in Theorem 6.1 in the form

$$\lambda_h - \eta^2 \leq \lambda_1 \leq \lambda_h - \tilde{\eta}^2, \quad (1.1a)$$

where η^2 and $\tilde{\eta}^2$ are *fully computable quantities* issuing from flux equilibration, converging to zero with the speed of convergence of λ_h towards λ_1 . First eigenvector energy estimates are given next, see in particular Theorem 6.2 revealing

$$\|\nabla(u_1 - u_h)\| \leq \eta, \quad \eta \leq C \|\nabla(u_1 - u_h)\|, \quad (1.1b)$$

where u_1 is the first exact eigenvector, u_h its numerical approximation, $\|\cdot\|$ is the $L^2(\Omega)$ -norm, C is a constant that only depends on λ_1 , λ_h , $\underline{\lambda}_2$, on some Poincaré–Friedrichs constants $C_{\text{cont,PF}}$ on patches of elements (themselves only depending on the shape regularity of the mesh), and on a certain stability constant C_{st} , again only depending on the shape regularity of the mesh. In particular, C is independent of the polynomial degree of u_h , leading to the *polynomial-degree robustness*. Moreover, a computable bound on C is given. These results hold *without any assumption* on the *convexity* of the computational domain Ω , on the *regularity* of the eigenfunctions, or on the *space dimension* (we only treat here $d = 2, 3$ but our arguments are general for $d \geq 1$). The price we pay for this generality is the presence of the (roughly constant) factor $1 - \frac{\lambda_h}{\underline{\lambda}_2}$ in our

estimators η and $\tilde{\eta}$. If some regularity of the corresponding *source problem* is known and the interpolation and stability constants are computable (typically when $d = 2$ and Ω is convex), we arrive at improved bounds of the structure (1.1a)–(1.1b) not featuring this factor, as summarized in Theorems 6.3 and 6.4 and Corollary 6.5.

The application of the above results only requests the verification of a few precise assumptions, in generalization of Ern and Vohralík [22], which is undertaken in Section 7 for conforming finite elements of arbitrary order. Numerical experiments presented in Section 8 fully support the theoretical findings for a collection of model problems; in particular the conditions for the validity of our results are here satisfied already on very coarse meshes. Finally, remarks and an outlook are given in Section 9.

2 Setting

Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a polygonal/polyhedral domain. We denote by $H^1(\Omega)$ the Sobolev space of $L^2(\Omega)$ functions with weak gradients in $[L^2(\Omega)]^d$ and by $V := H_0^1(\Omega)$ its zero-trace subspace. Similarly, $\mathbf{H}(\text{div}, \Omega)$ stands for the space of $[L^2(\Omega)]^d$ functions with weak divergences in $L^2(\Omega)$. The notations ∇ and $\nabla \cdot$ are used respectively for the weak gradient and divergence.

2.1 The Laplace eigenvalue problem

We consider the Laplace eigenvalue problem originating from minimizing the *energy* $E(\cdot) : V \rightarrow \mathbb{R}$,

$$E(v) := \frac{1}{2} \int_{\Omega} |\nabla v|^2 \, d\mathbf{x},$$

over all functions v from the energy space V such that $\int_{\Omega} v^2 \, d\mathbf{x} = 1$, i.e.,

$$u := \arg \inf_{\substack{v \in V, \\ \int_{\Omega} v^2 \, d\mathbf{x} = 1}} E(v). \quad (2.1)$$

It is well-known that there exists a solution to (2.1), unique up to the sign. The Euler–Lagrange equations of problem (2.1) read: find an eigenvector and eigenvalue pair (u, λ) , with u satisfying the homogeneous Dirichlet boundary condition over $\partial\Omega$ and subject to the constraint $\|u\| = 1$, such that $-\Delta u = \lambda u$ in Ω .

The weak form of our problem is: find $(u, \lambda) \in V \times \mathbb{R}^+$ with $\|u\| = 1$ such that

$$(\nabla u, \nabla v) = \lambda(u, v) \quad \forall v \in V. \quad (2.2)$$

Here, for $\omega \subset \Omega$, $(\nabla u, \nabla v)_{\omega}$ stands for $\int_{\omega} \nabla u \cdot \nabla v \, d\mathbf{x}$ and $(u, v)_{\omega}$ for $\int_{\omega} uv \, d\mathbf{x}$; we also denote $\|\nabla v\|_{\omega}^2 := \int_{\omega} |\nabla v|^2 \, d\mathbf{x}$ and $\|v\|_{\omega}^2 := \int_{\omega} v^2 \, d\mathbf{x}$ and drop the index whenever $\omega = \Omega$. Finally, $\langle u, v \rangle := \int_{\partial\Omega} uv \, d\gamma(\mathbf{x})$. The eigenvalue problem (2.2) has infinitely many solution pairs. There exists, cf. Babuška and Osborn [2] or Boffi [6], a countable orthonormal basis u_k , $k \geq 1$, of $L^2(\Omega)$ consisting of vectors from V , and a nondecreasing sequence $0 < \lambda_1 < \lambda_2 \leq \dots \leq \lambda_k \rightarrow \infty$ such that

$$(\nabla u_k, \nabla v) = \lambda_k(u_k, v) \quad \forall v \in V. \quad (2.3)$$

It follows from (2.3) and the scaling $\|u_k\| = 1$ that $\|\nabla u_k\|^2 = \lambda_k$. In particular, the smallest eigenvalue λ_1 is positive, isolated, and the associated eigenvector u_1 is unique under the sign condition $(u_1, 1) > 0$ (u_1 is actually positive everywhere inside Ω). Remarkably, (2.1) implies $\frac{\|\nabla v\|^2}{2\|v\|^2} \geq \frac{\|\nabla u_1\|^2}{2} = \frac{\lambda_1}{2}$ for all $v \in V$, which is known as

$$\|v\|^2 \leq \frac{1}{\lambda_1} \|\nabla v\|^2 \quad \forall v \in V, \quad (2.4)$$

the Poincaré–Friedrichs inequality. We are only interested here in the numerical approximation of the smallest eigenvalue λ_1 and of its associated eigenvector u_1 .

Below, we shall often employ the Parseval identity

$$\|v\|^2 = \sum_{k \geq 1} (v, u_k)^2 \quad (2.5)$$

valid for any $v \in L^2(\Omega)$. It is also well known that $(u_k/\sqrt{\lambda_k})_{k \geq 1}$ forms an orthonormal basis of V , for which one in particular uses that $(\nabla u_k, \nabla u_l) = \lambda_k(u_k, u_l) = 0$ for $k \neq l$. Then, for $v \in V$, we also obtain

$$\|\nabla v\|^2 = \sum_{k \geq 1} \frac{(\nabla v, \nabla u_k)^2}{\lambda_k} = \sum_{k \geq 1} \lambda_k (v, u_k)^2. \quad (2.6)$$

We refer to Gilbarg and Trudinger [25], Babuška and Osborn [2], Boffi [6], and the references therein for more details and background.

2.2 Numerical approximation

The results below are valid for any function $u_h \in V$ and positive number λ_h verifying the given hypotheses, so that they are not restricted to the solutions of a specific numerical method. It is shown that these hypotheses in particular hold when dealing with conforming finite elements in Section 7; the reader can for simplicity think of (u_h, λ_h) as given by (7.1) throughout the paper.

2.3 Residual and its dual norm

The derivation of a posteriori error estimates usually exploits the concept of the *residual* and of its *dual norm*. We will proceed in this way in the present paper as well. Let V' stand for the dual of V .

Definition 2.1 (Residual and its dual norm). *For any pair $(u_h, \lambda_h) \in V \times \mathbb{R}$, define the residual $\text{Res}(u_h, \lambda_h) \in V'$ by*

$$\langle \text{Res}(u_h, \lambda_h), v \rangle_{V', V} := \lambda_h(u_h, v) - (\nabla u_h, \nabla v) \quad \forall v \in V. \quad (2.7)$$

Its dual norm is then

$$\|\text{Res}(u_h, \lambda_h)\|_{-1} := \sup_{v \in V, \|\nabla v\|=1} \langle \text{Res}(u_h, \lambda_h), v \rangle_{V', V}. \quad (2.8)$$

We make the following classical remark on the *Riesz representation* of the residual:

Remark 2.2 (Riesz representation of the residual). *Let $\mathfrak{z}_{(h)} \in V$ be given by*

$$(\nabla \mathfrak{z}_{(h)}, \nabla v) = \langle \text{Res}(u_h, \lambda_h), v \rangle_{V', V} \quad \forall v \in V. \quad (2.9)$$

Then

$$\|\nabla \mathfrak{z}_{(h)}\| = \|\text{Res}(u_h, \lambda_h)\|_{-1}. \quad (2.10)$$

Note that $\mathfrak{z}_{(h)}$ depends on u_h and λ_h but cannot be easily calculated in practice.

3 Generic equivalences

We establish in this section *generic equivalence results* between the following three quantities: the square root of the first eigenvalue error $\sqrt{\lambda_h - \lambda_1}$, the first eigenvector energy error $\|\nabla(u_1 - u_h)\|$, and the dual norm of the residual $\|\text{Res}(u_h, \lambda_h)\|_{-1}$. These equivalences, forming our building blocks, may for the moment contain uncomputable terms like the continuous eigenvalues λ_1, λ_2 , or the $L^2(\Omega)$ norm of the Riesz residual representation $\|\mathfrak{z}_{(h)}\|$, but all such terms will be removed later. The results presented in this section can be generalized to any self-adjoint operator with compact resolvent, see Cancès *et al.* [11].

We start by defining some assumptions for the forthcoming results. We first request the approximate eigenvalue λ_h to be smaller than the exact second eigenvalue:

Assumption 3.1 (Approximate eigenvalue bound). *There holds*

$$\lambda_h < \lambda_2.$$

The second assumption is concerned with $\mathfrak{z}_{(h)} \in V$ of Remark 2.2:

Assumption 3.2 (Bound on the Riesz residual representation). *There holds*

$$\begin{aligned}\beta_h &:= \left(1 - \frac{\lambda_h}{\lambda_2}\right)^{-1} \|\mathbf{z}_{(h)}\| < 1, \\ \alpha_h &:= \sqrt{2 \left(1 - \sqrt{1 - \beta_h^2}\right)} \leq |\Omega|^{-\frac{1}{2}} |(u_h, 1)|.\end{aligned}$$

Note that β_h is well-defined under Assumption 3.1 and that α_h satisfies $0 \leq \alpha_h \leq \sqrt{2}\beta_h$, so that in particular $\alpha_h < \sqrt{2}$ and $\alpha_h \rightarrow 0$ as $\beta_h \rightarrow 0$.

Our first lemma is similar in parts to the developments in [34, 36, 48], giving (under the above assumptions) a computable bound on the $L^2(\Omega)$ error $\|u_1 - u_h\|$:

Lemma 3.3 ($L^2(\Omega)$ bound via a quadratic residual inequality). *Let $(u_h, \lambda_h) \in V \times \mathbb{R}^+$ with $\|u_h\| = 1$ and $(u_h, 1) > 0$. Let Assumptions 3.1 and 3.2 hold. Then*

$$\|u_1 - u_h\| \leq \alpha_h. \quad (3.1)$$

Proof. Characterizations (2.3), (2.7), and (2.9) give

$$(z_{(h)}, u_k) = \frac{1}{\lambda_k} (\nabla u_k, \nabla z_{(h)}) = \frac{1}{\lambda_k} (\lambda_h (u_h, u_k) - (\nabla u_h, \nabla u_k)) = \left(\frac{\lambda_h}{\lambda_k} - 1\right) (u_h, u_k). \quad (3.2)$$

Consequently, the Parseval equality (2.5) with $v = z_{(h)}$ yields

$$\|z_{(h)}\|^2 = \sum_{k \geq 1} (z_{(h)}, u_k)^2 = \sum_{k \geq 1} \left(1 - \frac{\lambda_h}{\lambda_k}\right)^2 (u_h, u_k)^2. \quad (3.3)$$

Observe that the function $x \in \mathbb{R}^+ \mapsto \left(1 - \frac{\lambda_h}{x}\right)^2$, considered for a positive x and $\lambda_h > 0$, reaches its minimum at $x = \lambda_h$ and is increasing on $[\lambda_h, \infty)$. Thus, as we suppose $\lambda_h < \lambda_2$ in Assumption 3.1, there holds

$$\min_{k \geq 2} \left(1 - \frac{\lambda_h}{\lambda_k}\right)^2 = \left(1 - \frac{\lambda_h}{\lambda_2}\right)^2 =: C_h. \quad (3.4)$$

Further, employing the scalings $\|u_1\| = 1$ and $\|u_h\| = 1$,

$$(u_h - u_1, u_1) = (u_h, u_1) - \|u_1\|^2 = (u_h, u_1) - \frac{\|u_1\|^2}{2} - \frac{\|u_h\|^2}{2} = -\frac{1}{2} \|u_1 - u_h\|^2. \quad (3.5)$$

As $(u_1, u_k) = 0$ for $k \geq 2$ from the orthogonality of u_k in (2.3), elaborating (3.3) further while adding and subtracting $C_h (u_h - u_1, u_1)^2$ and using (3.4) and (3.5) gives

$$\begin{aligned}\|z_{(h)}\|^2 &= \left(\frac{\lambda_h}{\lambda_1} - 1\right)^2 (u_h, u_1)^2 + \sum_{k \geq 2} \left(1 - \frac{\lambda_h}{\lambda_k}\right)^2 (u_h - u_1, u_k)^2 \\ &\geq \left(\frac{\lambda_h}{\lambda_1} - 1\right)^2 (u_h, u_1)^2 + C_h \sum_{k \geq 1} (u_h - u_1, u_k)^2 - C_h (u_h - u_1, u_1)^2 \\ &= \left(\frac{\lambda_h}{\lambda_1} - 1\right)^2 (u_h, u_1)^2 + C_h \|u_1 - u_h\|^2 - \frac{C_h}{4} \|u_1 - u_h\|^4,\end{aligned}$$

where we have also employed (2.5) with $v = u_h - u_1$. Dropping the first (nonnegative and presumably small) term on the right-hand side and denoting $e_h := \|u_1 - u_h\|^2$, we conclude the validity of the quadratic residual inequality in e_h

$$\frac{C_h}{4} e_h^2 - C_h e_h + \|z_{(h)}\|^2 \geq 0.$$

Its discriminant is given by $C_h^2 - C_h \|z_{(h)}\|^2 = C_h (C_h - \|z_{(h)}\|^2)$ and it is positive under the first condition of Assumption 3.2. Thus the error e_h has to be either confined between 0 and $(C_h - \sqrt{C_h (C_h - \|z_{(h)}\|^2)}) / \frac{C_h}{2}$,

or be greater or equal to $(C_h + \sqrt{C_h(C_h - \|z_h\|^2)})/\frac{C_h}{2}$. Using again the first condition of Assumption 3.2, a little rougher bounds for e_h are

$$e_h \leq 2 \left(1 - \sqrt{1 - \beta_h^2}\right) \quad \text{or} \quad e_h \geq 2(1 + \sqrt{1 - \beta_h^2}).$$

Suppose now the sign condition $(u_1, u_h) \geq 0$. Then, employing $\|u_1\| = \|u_h\| = 1$,

$$\|u_1 - u_h\|^2 = 2 - 2(u_1, u_h) \leq 2.$$

Thus $\|u_1 - u_h\|^2$ cannot be greater than or equal to $2 + 2\sqrt{1 - \beta_h^2}$, and (3.1) follows.

We are left to show that the sign condition $(u_1, u_h) \geq 0$ holds true, and we employ for this purpose the premise $(u_1, 1) > 0$ on the exact solution u_1 , the discrete sign condition $(u_h, 1) > 0$, and the second condition of Assumption 3.2. By contradiction, suppose that $-(u_1, u_h) > 0$. Then all the results above hold for $-u_h$ in place of u_h , with $\|u_1 + u_h\| \leq \alpha_h$ in particular. We would therefore have

$$\begin{aligned} (u_h, 1) &= -(u_1, 1) + (u_1 + u_h, 1) \leq -(u_1, 1) + \|u_1 + u_h\| |\Omega|^{\frac{1}{2}} \\ &\leq -(u_1, 1) + \alpha_h |\Omega|^{\frac{1}{2}} \leq -(u_1, 1) + (u_h, 1) < (u_h, 1), \end{aligned}$$

which is obviously not true. This completes the proof. \square

3.1 First eigenvalue error equivalences

We will now show how to exploit Lemma 3.3 for an equivalence between $\sqrt{\lambda_h - \lambda_1}$ and $\|\nabla(u_1 - u_h)\|$, in extension of some classical results, cf. [2, 6]. We need to assume that the approximate solution is conforming and that the approximate eigenvalue is its Rayleigh quotient:

Assumption 3.4 (Conforming solution). *Let $(u_h, \lambda_h) \in V \times \mathbb{R}^+$ with $\|u_h\| = 1$ and $(u_h, 1) > 0$ be such that $\|\nabla u_h\|^2 = \lambda_h$.*

Theorem 3.5 (Eigenvalue bounds). *Under Assumptions 3.1, 3.2, and 3.4,*

$$\frac{1}{2} \left(1 - \frac{\lambda_1}{\lambda_2}\right) \left(1 - \frac{\alpha_h^2}{4}\right) \|\nabla(u_1 - u_h)\|^2 \leq \lambda_h - \lambda_1 \leq \|\nabla(u_1 - u_h)\|^2. \quad (3.6)$$

There also holds

$$\|\nabla(u_1 - u_h)\|^2 - \lambda_1 \alpha_h^2 \leq \lambda_h - \lambda_1. \quad (3.7)$$

Proof. Using the weak solution characterization (2.2), Assumption 3.4, and (3.5),

$$\begin{aligned} \lambda_h - \lambda_1 &= \|\nabla u_h\|^2 - \|\nabla u_1\|^2 \\ &= \|\nabla(u_h - u_1)\|^2 + 2(\nabla(u_h - u_1), \nabla u_1) \\ &= \|\nabla(u_h - u_1)\|^2 + 2\lambda_1(u_1, u_h - u_1) \\ &= \|\nabla(u_h - u_1)\|^2 - \lambda_1 \|u_1 - u_h\|^2. \end{aligned} \quad (3.8)$$

Dropping the (nonpositive and presumably small) last term, the upper bound in (3.6) follows; estimating this term via (3.1), we arrive at (3.7).

To prove the lower bound in (3.6), we use (2.5)–(2.6) for $v = u_1 - u_h$. First,

$$\|\nabla(u_1 - u_h)\|^2 - \lambda_1 \|u_1 - u_h\|^2 = \sum_{k \geq 1} (\lambda_k - \lambda_1) (u_1 - u_h, u_k)^2 = \sum_{k \geq 2} (\lambda_k - \lambda_1) (u_1 - u_h, u_k)^2. \quad (3.9)$$

Using that $\lambda_k \geq \lambda_2$ for $k \geq 2$, $\lambda_2 > \lambda_1$, (3.5), and the Cauchy–Schwarz inequality,

$$\begin{aligned} \|\nabla(u_1 - u_h)\|^2 - \lambda_1 \|u_1 - u_h\|^2 &\geq (\lambda_2 - \lambda_1) \sum_{k \geq 1} (u_1 - u_h, u_k)^2 - (\lambda_2 - \lambda_1) (u_1 - u_h, u_1)^2 \\ &= (\lambda_2 - \lambda_1) \|u_1 - u_h\|^2 - \frac{\lambda_2 - \lambda_1}{4} \|u_1 - u_h\|^4. \end{aligned}$$

Using the $L^2(\Omega)$ bound (3.1) of Lemma 3.3 and reemploying (2.5) for $v = u_1 - u_h$, we arrive at, second,

$$\begin{aligned} \|\nabla(u_1 - u_h)\|^2 - \lambda_1 \|u_1 - u_h\|^2 &\geq (\lambda_2 - \lambda_1) \|u_1 - u_h\|^2 - \alpha_h^2 \frac{\lambda_2 - \lambda_1}{4} \|u_1 - u_h\|^2 \\ &= \sum_{k \geq 1} (\lambda_2 - \lambda_1) \left(1 - \frac{\alpha_h^2}{4}\right) (u_1 - u_h, u_k)^2. \end{aligned}$$

Summing this with (3.9) with weights $\frac{1}{2}$ yields

$$\|\nabla(u_1 - u_h)\|^2 - \lambda_1 \|u_1 - u_h\|^2 \geq \sum_{k \geq 1} \left\{ \frac{\lambda_k - \lambda_1}{2} + \frac{\lambda_2 - \lambda_1}{2} \left(1 - \frac{\alpha_h^2}{4}\right) \right\} (u_1 - u_h, u_k)^2.$$

Now notice that, using (2.6) for $v = u_1 - u_h$,

$$\frac{1}{2} \left(1 - \frac{\lambda_1}{\lambda_2}\right) \left(1 - \frac{\alpha_h^2}{4}\right) \|\nabla(u_1 - u_h)\|^2 = \sum_{k \geq 1} \frac{\lambda_k}{2} \left(1 - \frac{\lambda_1}{\lambda_2}\right) \left(1 - \frac{\alpha_h^2}{4}\right) (u_1 - u_h, u_k)^2.$$

A simple calculation (note $\frac{1}{2} < \left(1 - \frac{\alpha_h^2}{4}\right) \leq 1$) shows that

$$\frac{\lambda_k - \lambda_1}{2} + \frac{\lambda_2 - \lambda_1}{2} \left(1 - \frac{\alpha_h^2}{4}\right) \geq \frac{\lambda_k}{2} \left(1 - \frac{\lambda_1}{\lambda_2}\right) \left(1 - \frac{\alpha_h^2}{4}\right) \quad k \geq 1.$$

Thus

$$\|\nabla(u_1 - u_h)\|^2 - \lambda_1 \|u_1 - u_h\|^2 \geq \frac{1}{2} \left(1 - \frac{\lambda_1}{\lambda_2}\right) \left(1 - \frac{\alpha_h^2}{4}\right) \|\nabla(u_1 - u_h)\|^2,$$

and the lower bound in (3.6) follows using (3.8). \square

3.2 First eigenvector error equivalences

We next investigate the equivalence between $\|\nabla(u_1 - u_h)\|$ and the dual residual norm $\|\text{Res}(u_h, \lambda_h)\|_{-1}$:

Theorem 3.6 (Eigenvector bounds). *Under Assumptions 3.1, 3.2, and 3.4,*

$$\begin{aligned} &\left(\frac{\|\nabla(u_1 - u_h)\|^2}{\lambda_1} + 1 \right)^{-1} \|\text{Res}(u_h, \lambda_h)\|_{-1}^2 \\ &\leq \|\nabla(u_1 - u_h)\|^2 \leq \left(1 - \frac{\lambda_h}{\lambda_2}\right)^{-2} \left(1 - \frac{\alpha_h^2}{4}\right)^{-1} \|\text{Res}(u_h, \lambda_h)\|_{-1}^2. \end{aligned} \tag{3.10}$$

There also holds

$$\|\nabla(u_1 - u_h)\|^2 \leq \|\text{Res}(u_h, \lambda_h)\|_{-1}^2 + 2\lambda_h \alpha_h^2. \tag{3.11}$$

Proof. Developing (2.6) for $v = z_h$ via (3.2) gives

$$\|\nabla z_h\|^2 = \sum_{k \geq 1} \lambda_k (z_h, u_k)^2 = \sum_{k \geq 1} \lambda_k \left(1 - \frac{\lambda_h}{\lambda_k}\right)^2 (u_h, u_k)^2.$$

As in the proof of Lemma 3.3, adding and subtracting $C_h \lambda_1 (u_h - u_1, u_1)^2$, using $(u_1, u_k) = 0$ for $k \geq 2$, (3.4), (3.5), and the Cauchy-Schwarz inequality, we observe

$$\begin{aligned} \|\nabla z_h\|^2 &= \lambda_1 \left(\frac{\lambda_h}{\lambda_1} - 1\right)^2 (u_h, u_1)^2 + \sum_{k \geq 2} \lambda_k \left(1 - \frac{\lambda_h}{\lambda_k}\right)^2 (u_h - u_1, u_k)^2 \\ &\geq \lambda_1 \left(\frac{\lambda_h}{\lambda_1} - 1\right)^2 (u_h, u_1)^2 + C_h \sum_{k \geq 1} \lambda_k (u_h - u_1, u_k)^2 - C_h \lambda_1 (u_h - u_1, u_1)^2 \\ &= \lambda_1 \left(\frac{\lambda_h}{\lambda_1} - 1\right)^2 (u_h, u_1)^2 + C_h \|\nabla(u_1 - u_h)\|^2 - \frac{C_h}{4} \lambda_1 \|u_1 - u_h\|^4 \\ &\geq C_h \|\nabla(u_1 - u_h)\|^2 - \frac{C_h}{4} \lambda_1 \|u_1 - u_h\|^4. \end{aligned}$$

Employing now (3.1) and the Poincaré–Friedrichs inequality (2.4) for $v = u_1 - u_h$,

$$\|\nabla \boldsymbol{\varepsilon}_{(h)}\|^2 \geq C_h \|\nabla(u_1 - u_h)\|^2 - \frac{C_h}{4} \|\nabla(u_1 - u_h)\|^2 \alpha_h^2,$$

so that the right inequality in (3.10) follows using (2.10).

It is a consequence of Assumption 3.1 that $\max_{k \geq 2} \left(1 - \frac{\lambda_h}{\lambda_k}\right)^2 \leq 1$ (recall that $\lambda_h \geq 0$). Thus the Cauchy–Schwarz inequality and $\|u_1\| = \|u_h\| = 1$ give from the above development for $\|\nabla \boldsymbol{\varepsilon}_{(h)}\|^2$

$$\begin{aligned} \|\nabla \boldsymbol{\varepsilon}_{(h)}\|^2 &\leq \lambda_1 \left(\frac{\lambda_h}{\lambda_1} - 1\right)^2 (u_h, u_1)^2 + \sum_{k \geq 2} \lambda_k (u_h - u_1, u_k)^2 \\ &\leq \lambda_1 \left(\frac{\lambda_h}{\lambda_1} - 1\right)^2 + \|\nabla(u_1 - u_h)\|^2. \end{aligned}$$

Noting that $\lambda_h \geq \lambda_1$ by $\|\nabla u_h\|^2 = \lambda_h$ from Assumption 3.4 and by (2.1), the right inequality in (3.6) implies the left inequality in (3.10).

Finally, the bound (3.11) is a consequence of

$$\begin{aligned} \|\nabla(u_1 - u_h)\|^2 &= (\nabla(u_1 - u_h), \nabla(u_1 - u_h)) \\ &= \lambda_1 (u_1, u_1 - u_h) + \langle \text{Res}(u_h, \lambda_h), u_1 - u_h \rangle_{V', V} - \lambda_h (u_h, u_1 - u_h) \\ &= \langle \text{Res}(u_h, \lambda_h), u_1 - u_h \rangle_{V', V} + \frac{\lambda_1 + \lambda_h}{2} \|u_1 - u_h\|^2 \end{aligned}$$

that one obtains simply using (2.2) and (2.7) for $v = u_1 - u_h$ and (3.5), of the Young inequality and of definition (2.8) leading to

$$\|\nabla(u_1 - u_h)\|^2 \leq \|\text{Res}(u_h, \lambda_h)\|_{-1}^2 + (\lambda_1 + \lambda_h) \|u_1 - u_h\|^2,$$

and of (3.1) with the bound $\lambda_1 \leq \lambda_h$. □

Remark 3.7 (Use in a priori analysis). *The upper bounds in (3.6) and (3.10) enable the control of both $\lambda_h - \lambda_1$ and $\|\nabla(u_1 - u_h)\|$ in terms of the dual norm of the residual $\|\text{Res}(u_h, \lambda_h)\|_{-1}$. This can be further bounded following Larson [37] to obtain precise a priori estimates in the conforming finite element setting.*

4 Dual norm of the residual equivalences

In addition to the results of Section 3, we now need to estimate from above and from below the dual norm of the residual $\|\text{Res}(u_h, \lambda_h)\|_{-1}$. This section introduces a comprehensive methodology enabling computable and accurate estimates, following [43, 18, 7, 22] and the references therein.

To motivate, note that from (2.2), it is straightforward that $-\nabla u_1 \in \mathbf{H}(\text{div}, \Omega)$, with the weak divergence equal to $\lambda_1 u_1$. On the discrete level, however, we have, in general, $-\nabla u_h \notin \mathbf{H}(\text{div}, \Omega)$, and a fortiori $\nabla \cdot (-\nabla u_h) \neq \lambda_h u_h$. We thus introduce an *equilibrated flux reconstruction*, a vector field $\boldsymbol{\sigma}_h$ constructed from (u_h, λ_h) , satisfying

$$\boldsymbol{\sigma}_h \in \mathbf{H}(\text{div}, \Omega), \tag{4.1a}$$

$$\nabla \cdot \boldsymbol{\sigma}_h = \lambda_h u_h. \tag{4.1b}$$

Naturally, only specific reconstructions $\boldsymbol{\sigma}_h$ will give tight estimates, and we present in the following a *local construction* relying on the solution of homogeneous Neumann (Neumann–Dirichlet close to the boundary) problems by mixed finite elements.

4.1 Meshes and discrete spaces

Let henceforth $\{\mathcal{T}_h\}_h$ be a family of meshes, i.e., matching simplicial partitions of the domain Ω . We suppose that it is shape regular in the sense that there exists a constant $\kappa_{\mathcal{T}} > 0$ such that the ratio of the element diameter and of the diameter of its largest inscribed ball is uniformly bounded by $\kappa_{\mathcal{T}}$, cf. Ciarlet [17]. We denote by K a generic element of \mathcal{T}_h . The set of vertices is denoted by \mathcal{V}_h , with interior vertices $\mathcal{V}_h^{\text{int}}$, vertices located on the boundary $\mathcal{V}_h^{\text{ext}}$, and a generic vertex \mathbf{a} . We call $\mathcal{T}_{\mathbf{a}}$ the patch of elements of \mathcal{T}_h which share the vertex $\mathbf{a} \in \mathcal{V}_h$, $\omega_{\mathbf{a}}$ the corresponding subdomain, and $\mathbf{n}_{\omega_{\mathbf{a}}}$ its outward unit normal. We will often tacitly extend functions defined on $\omega_{\mathbf{a}}$ by zero outside of $\omega_{\mathbf{a}}$, whereas $V_h(\omega_{\mathbf{a}})$ stands for the restriction of the space V_h to $\omega_{\mathbf{a}}$. Next, $\psi_{\mathbf{a}}$ for $\mathbf{a} \in \mathcal{V}_h$ stands for the piecewise affine “hat” function taking value 1 at the vertex \mathbf{a} and zero at the other vertices. Remarkably, $(\psi_{\mathbf{a}})_{\mathbf{a} \in \mathcal{V}_h}$ form a partition of unity via $\sum_{\mathbf{a} \in \mathcal{V}_h} \psi_{\mathbf{a}} = 1|_{\Omega}$.

Let $\mathbb{P}_s(K)$, $s \geq 0$, stand for polynomials of total degree at most s on $K \in \mathcal{T}_h$, and $\mathbb{P}_s(\mathcal{T}_h)$ for piecewise polynomials on \mathcal{T}_h , without any continuity requirement. Let also $\mathbf{V}_h \times Q_h \subset \mathbf{H}(\text{div}, \Omega) \times L^2(\Omega)$ stand for Raviart–Thomas–Nédélec (RTN) mixed finite element spaces. We thus have, for degree $s \geq 0$, $\mathbf{V}_h := \{\mathbf{v}_h \in \mathbf{H}(\text{div}, \Omega); \mathbf{v}_h|_K \in [\mathbb{P}_s(K)]^d + \mathbb{P}_s(K)\mathbf{x}\}$ and $Q_h := \mathbb{P}_s(\mathcal{T}_h)$, see Brezzi and Fortin [9] or Roberts and Thomas [46]. We also denote by Π_{Q_h} the $L^2(\Omega)$ -orthogonal projection onto Q_h .

4.2 Equilibrated flux reconstruction

To proceed further, we will need to make some more assumptions. In particular, we will suppose that: i) the pair (u_h, λ_h) satisfies a Galerkin orthogonality with respect to the hat functions $\psi_{\mathbf{a}}$; ii) u_h is a piecewise polynomial from $\mathbb{P}_p(\mathcal{T}_h)$, $p \geq 1$; and iii) the construction of σ_h is carried out in a sufficiently rich (order $p+1$) RTN space:

Assumption 4.1 (Galerkin orthogonality of the residual to $\psi_{\mathbf{a}}$). *There holds*

$$\lambda_h(u_h, \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} - (\nabla u_h, \nabla \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} = \langle \text{Res}(u_h, \lambda_h), \psi_{\mathbf{a}} \rangle_{V', V} = 0 \quad \forall \mathbf{a} \in \mathcal{V}_h^{\text{int}}.$$

Assumption 4.2 (Piecewise polynomials). *There holds*

$$u_h \in \mathbb{P}_p(\mathcal{T}_h), p \geq 1, \text{ and the spaces } \mathbf{V}_h \times Q_h \text{ are of order } p+1.$$

We construct concretely σ_h by the following *local constrained minimizations*:

Definition 4.3 (Equilibrated flux reconstruction). *Let $u_h \in V$ satisfy Assumption 4.1. For $\mathbf{a} \in \mathcal{V}_h$, set*

$$\begin{aligned} \mathbf{V}_h^{\mathbf{a}} &:= \{\mathbf{v}_h \in \mathbf{V}_h(\omega_{\mathbf{a}}); \mathbf{v}_h \cdot \mathbf{n}_{\omega_{\mathbf{a}}} = 0 \text{ on } \partial\omega_{\mathbf{a}}\}, & \mathbf{a} \in \mathcal{V}_h^{\text{int}}, \\ Q_h^{\mathbf{a}} &:= \{q_h \in Q_h(\omega_{\mathbf{a}}); (q_h, 1)_{\omega_{\mathbf{a}}} = 0\}, \\ \mathbf{V}_h^{\mathbf{a}} &:= \{\mathbf{v}_h \in \mathbf{V}_h(\omega_{\mathbf{a}}); \mathbf{v}_h \cdot \mathbf{n}_{\omega_{\mathbf{a}}} = 0 \text{ on } \partial\omega_{\mathbf{a}} \setminus \partial\Omega\}, & \mathbf{a} \in \mathcal{V}_h^{\text{ext}}, \\ Q_h^{\mathbf{a}} &:= Q_h(\omega_{\mathbf{a}}), \end{aligned}$$

Then define $\sigma_h := \sum_{\mathbf{a} \in \mathcal{V}_h} \sigma_h^{\mathbf{a}} \in \mathbf{V}_h$, where $\sigma_h^{\mathbf{a}} \in \mathbf{V}_h^{\mathbf{a}}$ solve

$$\sigma_h^{\mathbf{a}} := \arg \min_{\substack{\mathbf{v}_h \in \mathbf{V}_h^{\mathbf{a}}, \\ \nabla \cdot \mathbf{v}_h = \Pi_{Q_h}(\psi_{\mathbf{a}} \lambda_h u_h - \nabla u_h \cdot \nabla \psi_{\mathbf{a}})}} \|\psi_{\mathbf{a}} \nabla u_h + \mathbf{v}_h\|_{\omega_{\mathbf{a}}} \quad \forall \mathbf{a} \in \mathcal{V}_h. \quad (4.2)$$

Note that the Euler–Lagrange equations for (4.2) give the standard *mixed finite element formulation*, cf. [22, Remark 3.7]: find $\sigma_h^{\mathbf{a}} \in \mathbf{V}_h^{\mathbf{a}}$ and $q_h^{\mathbf{a}} \in Q_h^{\mathbf{a}}$ such that

$$(\sigma_h^{\mathbf{a}}, \mathbf{v}_h)_{\omega_{\mathbf{a}}} - (p_h^{\mathbf{a}}, \nabla \cdot \mathbf{v}_h)_{\omega_{\mathbf{a}}} = -(\psi_{\mathbf{a}} \nabla u_h, \mathbf{v}_h)_{\omega_{\mathbf{a}}} \quad \forall \mathbf{v}_h \in \mathbf{V}_h^{\mathbf{a}}, \quad (4.3a)$$

$$(\nabla \cdot \sigma_h^{\mathbf{a}}, q_h)_{\omega_{\mathbf{a}}} = (\psi_{\mathbf{a}} \lambda_h u_h - \nabla u_h \cdot \nabla \psi_{\mathbf{a}}, q_h)_{\omega_{\mathbf{a}}} \quad \forall q_h \in Q_h^{\mathbf{a}}. \quad (4.3b)$$

Consequently, under Assumption 4.2, $\nabla \cdot \sigma_h = \lambda_h u_h$, cf., e.g., [22, Lemma 3.5]. This will be vital for giving an upper bound to $\|\text{Res}(u_h, \lambda_h)\|_{-1}$.

4.3 Conforming local residual liftings

In order to estimate as precisely as possible $\|\text{Res}(u_h, \lambda_h)\|_{-1}$ from below, it will be useful, following Babuška and Strouboulis [3, Section 5.1], Repin [45, Section 4.1.1], or Ern and Vohralík [22, Section 3.3], see also the references therein, to introduce primal conforming local residual Riesz representations.

We propose to solve *homogeneous local Neumann* (Neumann–Dirichlet close to the boundary) *problems* on the patches $\omega_{\mathbf{a}}$ via conforming primal counterparts of problems (4.3). Note that in particular in this respect, for all $\mathbf{a} \in \mathcal{V}_h$,

$$-(\psi_{\mathbf{a}} \nabla u_h, \nabla v)_{\omega_{\mathbf{a}}} + (\psi_{\mathbf{a}} \lambda_h u_h - \nabla u_h \cdot \nabla \psi_{\mathbf{a}}, v)_{\omega_{\mathbf{a}}} = \langle \text{Res}(u_h, \lambda_h), \psi_{\mathbf{a}} v \rangle_{V', V} \quad \forall v \in V.$$

Let $\mathbf{a} \in \mathcal{V}_h$ and consider a patch $\omega_{\mathbf{a}}$ around the vertex \mathbf{a} . Define

$$H_*^1(\omega_{\mathbf{a}}) := \{v \in H^1(\omega_{\mathbf{a}}); (v, 1)_{\omega_{\mathbf{a}}} = 0\}, \quad \mathbf{a} \in \mathcal{V}_h^{\text{int}}, \quad (4.4a)$$

$$H_*^1(\omega_{\mathbf{a}}) := \{v \in H^1(\omega_{\mathbf{a}}); v = 0 \text{ on } \partial\omega_{\mathbf{a}} \cap \partial\Omega\}, \quad \mathbf{a} \in \mathcal{V}_h^{\text{ext}}, \quad (4.4b)$$

and let $X_h^{\mathbf{a}}$ stand for an arbitrary finite-dimensional subspace of $H_*^1(\omega_{\mathbf{a}})$. Typically, for $u_h \in \mathbb{P}_p(\mathcal{T}_h)$, we choose $X_h^{\mathbf{a}} := \mathbb{P}_{p+1}(\mathcal{T}_{\mathbf{a}}) \cap H_*^1(\omega_{\mathbf{a}})$.

Definition 4.4 (Conforming local Neumann problems). *Define $r_h^{\mathbf{a}} \in X_h^{\mathbf{a}}$ by*

$$(\nabla r_h^{\mathbf{a}}, \nabla v_h)_{\omega_{\mathbf{a}}} = \langle \text{Res}(u_h, \lambda_h), \psi_{\mathbf{a}} v_h \rangle_{V', V} \quad \forall v_h \in X_h^{\mathbf{a}}$$

for each $\mathbf{a} \in \mathcal{V}_h$. Then set $r_h := \sum_{\mathbf{a} \in \mathcal{V}_h} \psi_{\mathbf{a}} r_h^{\mathbf{a}}$.

The functions $r_h^{\mathbf{a}}$ are discrete Riesz projections of the local residual with hat-weighted test functions. Note that $r_h^{\mathbf{a}} \notin V$ (when extended by zero outside of $\omega_{\mathbf{a}}$) but $\psi_{\mathbf{a}} r_h^{\mathbf{a}} \in H_0^1(\omega_{\mathbf{a}})$, whence the sum r_h indeed belongs to the space V .

Remark 4.5 (Conforming local Dirichlet problems). *One can also solve homogeneous local Dirichlet problems, typically on the simplest nonoverlapping partition of Ω given by the mesh \mathcal{T}_h . Let $K \in \mathcal{T}_h$ and let X_h^K stand for an arbitrary finite-dimensional subspace of $H_0^1(K)$. Typically, for $u_h \in \mathbb{P}_p(\mathcal{T}_h)$, we choose $X_h^K := \mathbb{P}_{\min\{d+1, p+1\}}(K) \cap H_0^1(K)$. For each $K \in \mathcal{T}_h$, define $r_h^K \in X_h^K$ by*

$$(\nabla r_h^K, \nabla v_h)_K = \langle \text{Res}(u_h, \lambda_h), v_h \rangle_{V', V} \quad \forall v_h \in X_h^K.$$

Then set $r_h := \sum_{K \in \mathcal{T}_h} r_h^K$. Now each r_h^K is the discrete Riesz projection of the local residual; as $r_h^K \in V$ (when extended by zero), $r_h \in V$ follows immediately. Our numerical results, however, reveal a superiority of Definition 4.4.

4.4 Poincaré–Friedrichs and stability inequalities

We will finally need two important inequalities. Recall first the scaled *Poincaré–Friedrichs* inequality

$$\|v\|_{\omega_{\mathbf{a}}} \leq C_{\text{PF}, \omega_{\mathbf{a}}} h_{\omega_{\mathbf{a}}} \|\nabla v\|_{\omega_{\mathbf{a}}} \quad \forall v \in H_*^1(\omega_{\mathbf{a}}),$$

corresponding to (2.4) on the patches $\omega_{\mathbf{a}}$, with a slightly different notation, where the constant $C_{\text{PF}, \omega_{\mathbf{a}}}$ is uniformly bounded by positive constants independent of the mesh size h . Then the constant $C_{\text{cont}, \text{PF}} := \max_{\mathbf{a} \in \mathcal{V}_h} \{1 + C_{\text{PF}, \omega_{\mathbf{a}}} h_{\omega_{\mathbf{a}}} \|\nabla \psi_{\mathbf{a}}\|_{\infty, \omega_{\mathbf{a}}}\}$ only depends on the mesh regularity parameter $\kappa_{\mathcal{T}}$ and can be fully estimated from above, see the discussion in [22, proof of Lemma 3.12]. It follows from Carstensen and Funken [13, Theorem 3.1] or Braess *et al.* [7, Section 3] that

$$\|\nabla(\psi_{\mathbf{a}} v)\|_{\omega_{\mathbf{a}}} \leq C_{\text{cont}, \text{PF}} \|\nabla v\|_{\omega_{\mathbf{a}}} \quad \forall v \in H_*^1(\omega_{\mathbf{a}}), \forall \mathbf{a} \in \mathcal{V}_h. \quad (4.5)$$

Second, a fundamental result is that the reconstructions (4.3) under Assumptions 4.1 and 4.2 satisfy the following *stability* property, see Braess *et al.* [7, Theorem 7] and Ern and Vohralík [22, Corollary 3.16]:

$$\|\psi_{\mathbf{a}} \nabla u_h + \sigma_h^{\mathbf{a}}\|_{\omega_{\mathbf{a}}} \leq C_{\text{st}} \sup_{v \in H_*^1(\omega_{\mathbf{a}}); \|\nabla v\|_{\omega_{\mathbf{a}}} = 1} \langle \text{Res}(u_h, \lambda_h), \psi_{\mathbf{a}} v \rangle_{V', V}. \quad (4.6)$$

Importantly, the constant $C_{\text{st}} > 0$ again only depends on $\kappa_{\mathcal{T}}$; a computable upper bound on C_{st} is in [22, Lemma 3.23] shown to be $\frac{\|\psi_{\mathbf{a}} \nabla u_h + \sigma_h^{\mathbf{a}}\|_{\omega_{\mathbf{a}}}}{\|\nabla r_h^{\mathbf{a}}\|_{\omega_{\mathbf{a}}}}$ for $r_h^{\mathbf{a}}$ of Definition 4.4.

4.5 Dual norm of the residual equivalences

We will now prove the following crucial result for the reconstructions of Definitions 4.3 and 4.4:

Theorem 4.6 (Residual equivalences). *Let $(u_h, \lambda_h) \in V \times \mathbb{R}$ verifying Assumptions 4.1 and 4.2 be arbitrary. Let σ_h be constructed via Definition 4.3 and r_h via Definition 4.4. Then*

$$\frac{\langle \text{Res}(u_h, \lambda_h), r_h \rangle_{V', V}}{\|\nabla r_h\|} \leq \|\text{Res}(u_h, \lambda_h)\|_{-1} \leq \|\nabla u_h + \sigma_h\|, \quad (4.7a)$$

$$\|\nabla u_h + \sigma_h\| \leq (d+1)C_{\text{st}}C_{\text{cont,PF}}\|\text{Res}(u_h, \lambda_h)\|_{-1}. \quad (4.7b)$$

Proof. Let $v \in V$ with $\|\nabla v\| = 1$ be fixed. Using definition (2.7), adding and subtracting $(\sigma_h, \nabla v)$, and employing the Green theorem and the equilibrium property (4.1b) yields

$$\begin{aligned} \langle \text{Res}(u_h, \lambda_h), v \rangle_{V', V} &= \lambda_h(u_h, v) - (\nabla u_h, \nabla v) = (\lambda_h u_h - \nabla \cdot \sigma_h, v) - (\nabla u_h + \sigma_h, \nabla v) \\ &= -(\nabla u_h + \sigma_h, \nabla v). \end{aligned}$$

Thus, definition (2.8) of the dual norm of the residual and the Cauchy–Schwarz inequality yield the upper bound in (4.7a).

The lower bound in (4.7a) is trivial from (2.8): we estimate from below the supremum by taking one particular function $v = r_h \in V$. Importantly, this bound is positive (except for a pathological case where $r_h^{\mathbf{a}} = 0$ for all $\mathbf{a} \in \mathcal{V}_h$); indeed, from Definition 4.4, we have

$$\frac{\langle \text{Res}(u_h, \lambda_h), r_h \rangle_{V', V}}{\|\nabla r_h\|} = \frac{\sum_{\mathbf{a} \in \mathcal{V}_h} \langle \text{Res}(u_h, \lambda_h), \psi_{\mathbf{a}} r_h^{\mathbf{a}} \rangle_{V', V}}{\|\nabla r_h\|} = \frac{\sum_{\mathbf{a} \in \mathcal{V}_h} \|\nabla r_h^{\mathbf{a}}\|_{\omega_{\mathbf{a}}}^2}{\|\nabla r_h\|},$$

and, additionally,

$$\frac{\langle \text{Res}(u_h, \lambda_h), r_h \rangle_{V', V}}{\|\nabla r_h\|} \geq \frac{\left\{ \sum_{\mathbf{a} \in \mathcal{V}_h} \|\nabla r_h^{\mathbf{a}}\|_{\omega_{\mathbf{a}}}^2 \right\}^{\frac{1}{2}}}{(d+1)^{\frac{1}{2}} C_{\text{cont,PF}}},$$

which can be seen as below.

To prove (4.7b), we proceed as in Ern and Vohralík [22, Lemmas 3.22 and 3.23], while treating the weak norm $\|\text{Res}(u_h, \lambda_h)\|_{-1}$ as in Ciarlet and Vohralík [16, Theorems 4.7 and 5.1]. For a vertex $\mathbf{a} \in \mathcal{V}_h$, recall definition (4.4) of the space $H_*^1(\omega_{\mathbf{a}})$ and define a continuous version of $r_h^{\mathbf{a}}$ from Definition 4.4 as: find $r_{(h)}^{\mathbf{a}} \in H_*^1(\omega_{\mathbf{a}})$ such that

$$(\nabla r_{(h)}^{\mathbf{a}}, \nabla v)_{\omega_{\mathbf{a}}} = \langle \text{Res}(u_h, \lambda_h), \psi_{\mathbf{a}} v \rangle_{V', V} \quad \forall v \in H_*^1(\omega_{\mathbf{a}}).$$

Remark that $\|\nabla r_{(h)}^{\mathbf{a}}\|_{\omega_{\mathbf{a}}}$ equals to the supremum on the right-hand side of (4.6). Let us set $r_{(h)} := \sum_{\mathbf{a} \in \mathcal{V}_h} \psi_{\mathbf{a}} r_{(h)}^{\mathbf{a}} \in V$ and remark that $r_{(h)}|_K = (\sum_{\mathbf{a} \in \mathcal{V}_K} \psi_{\mathbf{a}} r_{(h)}^{\mathbf{a}})|_K$, where \mathcal{V}_K denotes the set of vertices of the element $K \in \mathcal{T}_h$. Then

$$\begin{aligned} \|\nabla r_{(h)}\|^2 &= \sum_{K \in \mathcal{T}_h} \left\| \sum_{\mathbf{a} \in \mathcal{V}_K} (\nabla(\psi_{\mathbf{a}} r_{(h)}^{\mathbf{a}}))|_K \right\|_K^2 \leq (d+1) \sum_{K \in \mathcal{T}_h} \sum_{\mathbf{a} \in \mathcal{V}_K} \|\nabla(\psi_{\mathbf{a}} r_{(h)}^{\mathbf{a}})\|_K^2 \\ &= (d+1) \sum_{\mathbf{a} \in \mathcal{V}_h} \|\nabla(\psi_{\mathbf{a}} r_{(h)}^{\mathbf{a}})\|_{\omega_{\mathbf{a}}}^2 \leq (d+1)C_{\text{cont,PF}}^2 \sum_{\mathbf{a} \in \mathcal{V}_h} \|\nabla r_{(h)}^{\mathbf{a}}\|_{\omega_{\mathbf{a}}}^2, \end{aligned}$$

as each simplex has $(d+1)$ vertices; in the last estimate, we have used the inequality (4.5). Crucially, the norms of $\nabla r_{(h)}^{\mathbf{a}}$ are connected to the residual via

$$\sum_{\mathbf{a} \in \mathcal{V}_h} \|\nabla r_{(h)}^{\mathbf{a}}\|_{\omega_{\mathbf{a}}}^2 = \sum_{\mathbf{a} \in \mathcal{V}_h} \langle \text{Res}(u_h, \lambda_h), \psi_{\mathbf{a}} r_{(h)}^{\mathbf{a}} \rangle_{V', V} = \langle \text{Res}(u_h, \lambda_h), r_{(h)} \rangle_{V', V},$$

which can be further bounded by $\|\text{Res}(u_h, \lambda_h)\|_{-1} \|\nabla r_{(h)}\|$ and implies

$$\sum_{\mathbf{a} \in \mathcal{V}_h} \|\nabla r_{(h)}^{\mathbf{a}}\|_{\omega_{\mathbf{a}}}^2 \leq (d+1)C_{\text{cont,PF}}^2 \|\text{Res}(u_h, \lambda_h)\|_{-1}^2.$$

Now using the definition of σ_h by the patchwise contributions $\sigma_h^{\mathbf{a}}$ gives as above

$$\|\nabla u_h + \sigma_h\|^2 \leq (d+1) \sum_{\mathbf{a} \in \mathcal{V}_h} \|\psi_{\mathbf{a}} \nabla u_h + \sigma_h^{\mathbf{a}}\|_{\omega_{\mathbf{a}}}^2,$$

so that estimate (4.6) leads to (4.7b). \square

5 Representation of the residual and eigenvalue bounds

In order to arrive at fully computable estimates from the results of the previous sections, we are left in this section to give computable bounds on the $L^2(\Omega)$ -norm of the Riesz representation of the residual $\|\mathbf{z}_{(h)}\|$ and to discuss available estimates on λ_1, λ_2 .

5.1 Bounds on the Riesz representation of the residual

The following is a simple consequence of the Poincaré–Friedrichs inequality (2.4) and of (2.10):

Lemma 5.1 (Poincaré–Friedrichs bound on $\|\mathbf{z}_{(h)}\|$). *Let $(u_h, \lambda_h) \in V \times \mathbb{R}$ be arbitrary and define the Riesz residual representation $\mathbf{z}_{(h)} \in V$ by (2.9). Then*

$$\|\mathbf{z}_{(h)}\| \leq \frac{1}{\sqrt{\lambda_1}} \|\text{Res}(u_h, \lambda_h)\|_{-1}. \quad (5.1)$$

Let $\zeta_{(h)} \in V$ be the solution of the Laplace source problem $-\Delta \zeta_{(h)} = \mathbf{z}_{(h)}$, i.e.,

$$(\nabla \zeta_{(h)}, \nabla v) = (\mathbf{z}_{(h)}, v) \quad \forall v \in V. \quad (5.2)$$

Denote by $V_h := \mathbb{P}_1(\mathcal{T}_h) \cap V$ the lowest-order conforming finite element space, i.e., the span of $\psi_{\mathbf{a}}$ over all $\mathbf{a} \in \mathcal{V}_h^{\text{int}}$, and by h the maximal element diameter over all $K \in \mathcal{T}_h$. We may find the following Aubin–Nitsche duality trick, cf. Brenner and Scott [8, Theorem 5.4.8], Grisvard [27, Theorem 4.3.1.4], Blum and Dobrowolski [5], and the references therein, useful in particular for convex two-dimensional domains:

Lemma 5.2 (Elliptic regularity bound on $\|\mathbf{z}_{(h)}\|$). *Let assumptions of Lemma 5.1 be satisfied. Let in addition Assumption 4.1 hold and assume that the solution $\zeta_{(h)}$ of (5.2) belongs to the space $H^{1+\delta}(\Omega)$, $0 < \delta \leq 1$, so that the approximation estimate*

$$\inf_{v_h \in V_h} \|\nabla(\zeta_{(h)} - v_h)\| \leq C_I h^\delta |\zeta_{(h)}|_{H^{1+\delta}(\Omega)} \quad (5.3)$$

and the stability estimate

$$|\zeta_{(h)}|_{H^{1+\delta}(\Omega)} \leq C_S \|\mathbf{z}_{(h)}\| \quad (5.4)$$

are satisfied. Then

$$\|\mathbf{z}_{(h)}\| \leq C_I C_S h^\delta \|\text{Res}(u_h, \lambda_h)\|_{-1}. \quad (5.5)$$

Proof. By (5.2), (2.9), and the Galerkin orthogonality of Assumption 4.1

$$\|\mathbf{z}_{(h)}\|^2 = (\nabla \zeta_{(h)}, \nabla \mathbf{z}_{(h)}) = (\nabla(\zeta_{(h)} - \zeta_h), \nabla \mathbf{z}_{(h)})$$

where $\zeta_h \in V_h$ is arbitrary. Thus the Cauchy–Schwarz inequality and estimates (5.3) with (5.4) give $\|\mathbf{z}_{(h)}\|^2 \leq C_I C_S h^\delta \|\mathbf{z}_{(h)}\| \|\nabla \mathbf{z}_{(h)}\|$. Now the characterization (2.10) finishes the proof. \square

Remark 5.3 (Constants C_I and C_S). *Let Ω be a convex polygon in two space dimensions. Then it is classical that $\zeta_{(h)} \in H^2(\Omega)$ and $|\zeta_{(h)}|_{H^2(\Omega)} = \|\Delta \zeta_{(h)}\| = \|\mathbf{z}_{(h)}\|$, so that $\delta = 1$ and $C_S = 1$, see Grisvard [27, Theorem 4.3.1.4]. In this situation, calculable bounds on C_I can be found in Liu and Kikuchi [38], see also Liu and Oishi [39, Section 2] and the references therein; in particular, for a mesh formed by isosceles right-angled triangles, $C_I \leq \frac{0.493}{\sqrt{2}}$.*

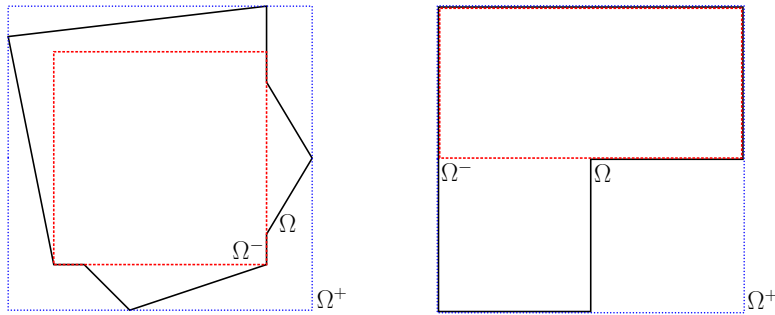


Figure 1: Inclusions for a nonconvex polygon (left) and for an L-shaped domain (right)

5.2 Estimates of eigenvalues via domain inclusion

We finally discuss lower (and upper) bounds on the exact eigenvalues λ_1 and λ_2 . One way to achieve this is by the following well-known *inclusion* result, which is a straightforward consequence of the min-max principle for eigenvalues of self-adjoint operators, see, e.g., Gilbarg and Trudinger [25], Babuška and Osborn [2], Boffi [6], or Trefethen and Betcke [50]:

$$\begin{aligned} \Omega \subset \Omega^+ &\Rightarrow \lambda_k \geq \lambda_k(\Omega^+), \\ \Omega \supset \Omega^- &\Rightarrow \lambda_k \leq \lambda_k(\Omega^-), \end{aligned} \quad \forall k \geq 1, \quad (5.6)$$

where $\lambda_k(\Omega^+)$ is the k -th eigenvalue on Ω^+ and similarly for $\lambda_k(\Omega^-)$. Naturally, the idea is to choose for Ω^+ and Ω^- domains where the exact eigenvalues are known, like a rectangular d -parallelepiped or a d -sphere, see [50] and the references therein. Then we can in particular verify Assumption 3.1 via the computable condition

$$\lambda_h < \lambda_2(\Omega^+) \quad \Rightarrow \quad \lambda_h < \lambda_2$$

and employ the bound

$$1 - \frac{\lambda_h}{\lambda_2(\Omega^+)} \leq 1 - \frac{\lambda_h}{\lambda_2}. \quad (5.7)$$

Below, relying on $\lambda_1 \leq \lambda_h$, we will also use the lower bound

$$1 - \frac{\lambda_h}{\lambda_2(\Omega^+)} \leq 1 - \frac{\lambda_1}{\lambda_2}, \quad (5.8)$$

which may be alternatively replaced by $1 - \frac{\lambda_1(\Omega^-)}{\lambda_2(\Omega^+)} \leq 1 - \frac{\lambda_1}{\lambda_2}$ (when positive).

Two examples are given in Figure 1. First, a general nonconvex polygon Ω is contained in the square $\Omega^+ := (-1, 1)^2$ and contains the square $\Omega^- := (-0.7, 0.7)^2$. The eigenvalues on a square of size H being $\pi^2(k^2 + l^2)/H^2$, $k, l = 1, \dots, \infty$, we have for $\lambda_2(\Omega^+) = 5\pi^2/4$. It is thus sufficient to check whether $\lambda_h < 1.25\pi^2$ in this case. This should happen very soon, as λ_h approximates λ_1 and we know that $\lambda_1 \leq \lambda_1(\Omega^-) = 2\pi^2/1.4^2 \approx 1.02\pi^2$ (the maximal ratio $h_{\Omega^+}^+/h_{\Omega^-}^-$ for $\lambda_1(\Omega^-) \leq \lambda_2(\Omega^+)$ turns here out to be approximately 1.5811388301). For an L-shaped domain $\Omega := (-1, 1)^2 \setminus [0, 1] \times [-1, 0]$, we know from [50] that $\lambda_1 \approx 9.6397238440$. We find again $\lambda_2(\Omega^+) = 5\pi^2/4 \approx 12.3370055014$, so that $\lambda_h < \lambda_2(\Omega^+)$ should occur soon. Note, however, that $\lambda_2(\Omega^+) = \lambda_1(\Omega^-)$, so that here the alternative to (5.8) cannot be used.

6 Guaranteed and fully computable upper and lower bounds

We combine here the different results of the previous sections to derive the actual guaranteed and fully computable bounds for eigenvalues (in Section 6.1) and eigenvectors (in Section 6.2). Possible improvements when the corresponding source problem is elliptic-regular are summarized in Section 6.3, and a thorough discussion of the results is provided in Section 6.4.

6.1 Eigenvalues

We first tackle the question of upper and lower bounds for the first eigenvalue λ_1 , relying on the generic bounds (3.6), (3.10), and (5.1):

Theorem 6.1 (Guaranteed bounds for the first eigenvalue). *Let $\underline{\lambda}_2 > 0$ and $\underline{\lambda}_1 > 0$ be such that $\underline{\lambda}_2 \leq \lambda_2$ and $\underline{\lambda}_1 \leq \lambda_1$; recall from Section 5.2 that we can always take $\underline{\lambda}_2 := \underline{\lambda}_2(\Omega^+)$ and $\underline{\lambda}_1 := \underline{\lambda}_1(\Omega^+)$ for a domain Ω^+ in \mathbb{R}^d such that $\Omega \subset \Omega^+$. Let (u_h, λ_h) be any element of $V \times \mathbb{R}^+$ verifying Assumption 3.4, and the inequality*

$$\lambda_h < \underline{\lambda}_2. \quad (6.1)$$

Let next Assumptions 4.1 and 4.2 hold and let σ_h be constructed following Definition 4.3 and r_h following Definition 4.4. Let us finally assume that the inequalities

$$\bar{\beta}_h := \frac{1}{\sqrt{\underline{\lambda}_1}} \left(1 - \frac{\lambda_h}{\underline{\lambda}_2}\right)^{-1} \|\nabla u_h + \sigma_h\| < 1, \quad (6.2a)$$

$$\bar{\alpha}_h := \sqrt{2 \left(1 - \sqrt{1 - \bar{\beta}_h^2}\right)} \leq |\Omega|^{-\frac{1}{2}} |(u_h, 1)| \quad (6.2b)$$

hold. Then

$$\lambda_h - \eta^2 \leq \lambda_1 \leq \lambda_h - \tilde{\eta}^2 \quad (6.3)$$

with

$$\eta^2 := \left(1 - \frac{\lambda_h}{\underline{\lambda}_2}\right)^{-2} \left(1 - \frac{\bar{\alpha}_h^2}{4}\right)^{-1} \|\nabla u_h + \sigma_h\|^2, \quad (6.4a)$$

$$\tilde{\eta}^2 := \frac{1}{2} \left(1 - \frac{\lambda_h}{\underline{\lambda}_2}\right) \left(1 - \frac{\bar{\alpha}_h^2}{4}\right) \frac{\lambda_1}{2} \left(\sqrt{1 + \frac{4 \langle \text{Res}(u_h, \lambda_h), r_h \rangle_{V',V}^2}{\|\nabla r_h\|^2}} - 1 \right). \quad (6.4b)$$

Proof. Observe first that the two conditions in (6.2) imply that Assumption 3.2 is fulfilled since $\beta_h \leq \bar{\beta}_h$ and consequently $\alpha_h \leq \bar{\alpha}_h$, by applying (5.1), (5.7), and the upper bound in (4.7a). Then, the lower bound claim in (6.3) follows by combining the upper bounds in (3.6), (3.10), and (4.7a) together with $\alpha_h \leq \bar{\alpha}_h$ and (5.7). The upper bound claim in (6.3) starts by the lower bound in (3.6). We then need to bound $\|\nabla(u_1 - u_h)\|^2$ from below. From (3.10) in combination with (5.6), we have

$$\|\text{Res}(u_h, \lambda_h)\|_{-1}^2 \leq \frac{\|\nabla(u_1 - u_h)\|^4}{\underline{\lambda}_1} + \|\nabla(u_1 - u_h)\|^2.$$

Developing this quadratic inequality as in the proof of Lemma 3.3 results in

$$\|\nabla(u_1 - u_h)\|^2 \geq \frac{\lambda_1}{2} \left(\sqrt{1 + \frac{4 \|\text{Res}(u_h, \lambda_h)\|_{-1}^2}{\underline{\lambda}_1}} - 1 \right). \quad (6.5)$$

Then the lower bound in (4.7a), (5.8), and the fact that $\alpha_h \leq \bar{\alpha}_h$ finish the proof. \square

6.2 Eigenvectors

We now summarize our estimate on the energy error in the approximation of the first eigenvector, as well as its efficiency and robustness:

Theorem 6.2 (Guaranteed and robust bound for the first eigenvector error). *Let the assumptions of Theorem 6.1 be verified. Then the energy error can be bounded via*

$$\|\nabla(u_1 - u_h)\| \leq \eta, \quad (6.6a)$$

where η is defined by (6.4a), and the estimator is efficient as

$$\|\nabla u_h + \sigma_h\|^2 \leq (d+1)^2 C_{\text{st}}^2 C_{\text{cont,PF}}^2 \left(\frac{\|\nabla(u_1 - u_h)\|^2}{\lambda_1} + 1 \right) \|\nabla(u_1 - u_h)\|^2. \quad (6.6b)$$

Proof. The upper bound (6.6a) follows upon combining the upper bounds in estimates (3.10) and (4.7a) together with (5.7) and $\alpha_h \leq \bar{\alpha}_h$. The lower bound (6.6b) is a consequence of (4.7b) and of the lower bound in (3.10). \square

6.3 Improvements under elliptic regularity

Relying on the more precise but possibly more restrictive bounds (3.7), (3.11), and (5.5), the following variants of Theorems 6.1 and 6.2 hold:

Theorem 6.3 (Elliptic regularity bounds for the first eigenvalue). *Let the assumptions of Theorem 6.1 be satisfied, except of (6.2). Assume that the solution $\zeta_{(h)}$ of problem (5.2) belongs to $H^{1+\delta}(\Omega)$, $0 < \delta \leq 1$, so that (5.3) and (5.4) hold. Let*

$$\bar{\gamma}_h := \left(1 - \frac{\lambda_h}{\lambda_2}\right)^{-1} C_1 C_S h^\delta,$$

and suppose that

$$\bar{\beta}_h := \bar{\gamma}_h \|\nabla u_h + \sigma_h\| < 1, \quad (6.7a)$$

$$\bar{\alpha}_h := \sqrt{2 \left(1 - \sqrt{1 - \bar{\gamma}_h^2 \|\nabla u_h + \sigma_h\|^2}\right)} \leq |\Omega|^{-\frac{1}{2}} |(u_h, 1)|. \quad (6.7b)$$

Then

$$\lambda_h - \eta^2 \leq \lambda_1 \leq \lambda_h - \tilde{\eta}^2 \quad (6.8)$$

with

$$\eta^2 := (1 + 4\lambda_h \bar{\gamma}_h^2) \|\nabla u_h + \sigma_h\|^2, \quad (6.9a)$$

$$\tilde{\eta}^2 := -2\lambda_h \bar{\gamma}_h^2 \|\nabla u_h + \sigma_h\|^2 + \frac{\lambda_1}{2} \left(\sqrt{1 + \frac{4}{\lambda_1} \frac{\langle \text{Res}(u_h, \lambda_h), r_h \rangle_{V',V}^2}{\|\nabla r_h\|^2}} - 1 \right). \quad (6.9b)$$

Proof. Observe first that the conditions (6.7) imply Assumption 3.2 by using (5.5), (5.7), and (4.7a), and since $\alpha_h \leq \bar{\alpha}_h$. Then we first arrive at the intermediary lower bound in (6.12a), stated below in Corollary 6.5, by combining the upper bounds in (3.6), (3.11), and (4.7a). Recalling that $\bar{\alpha}_h \leq \sqrt{2} \bar{\beta}_h$ from $\bar{\beta}_h < 1$ and (6.7b),

$$\lambda_h \bar{\alpha}_h^2 \leq 2\lambda_h \bar{\gamma}_h^2 \|\nabla u_h + \sigma_h\|^2 \quad (6.10)$$

immediately follows, and hence the lower estimate in (6.8) is true. The upper estimate combines the lower bounds (3.7), (6.5), and (4.7a) with the facts that $\lambda_1 \leq \lambda_h$ and $\alpha_h \leq \bar{\alpha}_h$ to first arrive at the upper bound in (6.12a) of Corollary 6.5 with (6.10). \square

Theorem 6.4 (Elliptic regularity bound for the first eigenvector error). *Let the assumptions of Theorem 6.3 be verified. Then the energy error can be bounded via*

$$\|\nabla(u_1 - u_h)\| \leq \eta, \quad (6.11)$$

where η is defined by (6.9a), and this estimator is efficient via (6.6b).

Proof. Combine (3.11) with the upper bound (4.7a), $\alpha_h \leq \bar{\alpha}_h$, and (6.10). \square

It follows from the proofs of Theorems 6.3 and 6.4 that still a little sharper bounds, with, however, somewhat less visible structure, are:

Corollary 6.5 (Improved bounds to (6.8) and (6.11)). *Let the assumptions of Theorem 6.3 be verified. Then*

$$\lambda_h - \eta^2 \leq \lambda_1 \leq \lambda_h - \tilde{\eta}^2, \quad (6.12a)$$

$$\|\nabla(u_1 - u_h)\| \leq \eta, \quad (6.12b)$$

with

$$\eta^2 := \|\nabla u_h + \boldsymbol{\sigma}_h\|^2 + 2\lambda_h \bar{\alpha}_h^2, \quad (6.13a)$$

$$\tilde{\eta}^2 := -\lambda_h \bar{\alpha}_h^2 + \frac{\lambda_1}{2} \left(\sqrt{1 + \frac{4}{\lambda_1} \frac{\langle \text{Res}(u_h, \lambda_h), r_h \rangle_{V', V}^2}{\|\nabla r_h\|^2}} - 1 \right). \quad (6.13b)$$

6.4 Comments

We collect in this section various comments about the results of Theorems 6.1–6.4. We start by a discussion of the assumptions:

Remark 6.6 (Admissibility condition (6.1)). *Condition (6.1) is the only crucial assumption we need for the validity of our estimates. It can be typically fully verified using the lower bounds on the continuous eigenvalue λ_2 discussed in Section 5.2. This is in particular an improvement over the situation in [36, Remark 3.2] or [48, condition (3.6)] which request λ_1 to be the relatively closest eigenvalue to λ_h .*

Remark 6.7 (Minimal resolution conditions (6.2) and (6.7)). *Both conditions in (6.2) can be fully checked for the given mesh and approximate solution. Moreover, whereas the term $\frac{1}{\sqrt{\lambda_1}} \left(1 - \frac{\lambda_h}{\lambda_2}\right)^{-1}$ of $\bar{\beta}_h$ in (6.2a) is roughly stable, $\|\nabla u_h + \boldsymbol{\sigma}_h\|$ goes to zero while refining (adaptively) the mesh or increasing (adaptively) the approximation polynomial degree. A similar “fine-resolution satisfaction” property holds for the condition (6.2b). Numerical experiments of Section 8 indicate that both conditions are satisfied already for very coarse approximations. It is equally straightforward to verify (6.7) whenever all δ , C_I , and C_S are known. This is in particular the case when Ω is a convex polygon in two space dimensions, see the discussion in Remark 5.3.*

We next discuss the eigenvalue bounds of Theorems 6.1 and 6.3:

Remark 6.8 (Upper bounds in (6.3) and (6.8)). *Note that the use of $\tilde{\eta}^2$ given by (6.4b) or (6.9b) allows one to sharpen the usual estimate $\lambda_1 \leq \lambda_h$, valid for any Galerkin (variational) method.*

Remark 6.9 (Convergence order of the bounds (6.3)). *The guaranteed lower bound in (6.3) contains the squared term $\|\nabla u_h + \boldsymbol{\sigma}_h\|^2$, whereas one only has $\|\nabla u_h + \boldsymbol{\sigma}_h\|$ in [48, Theorems 3.4 and 4.3], and similarly in [36, Remark 3.2]. This is a crucial improvement from a suboptimal to the optimal convergence order, where λ_h goes to λ_1 with the square of the eigenvector energy error. The upper bound in (6.3) likewise behaves optimally. Recalling that the parameter $\bar{\alpha}_h$ tends to zero with increasing resolution, so that $1 - \frac{\bar{\alpha}_h^2}{4} \rightarrow 1$, it is only the multiplicative constant in the convergence speed given by the factor $1 - \frac{\lambda_h}{\lambda_2}$ which may cause a (stable) overestimation, in particular influenced by the sharpness of the bounds of Section 5.2.*

Remark 6.10 (Sharpness and comparison of the bounds in (6.3) and (6.8)). *As discussed in Remark 6.9, Theorem 6.1 may lead to an overestimation with a factor $1 - \frac{\lambda_h}{\lambda_2}$. Its advantage is that it holds without any regularity/convexity/dimension assumption and all the quantities appearing are known. The factor $1 - \frac{\lambda_h}{\lambda_2}$ is removed in Theorem 6.3 that we can in particular easily apply in two space dimensions when Ω is convex. Then $\delta = 1$, $C_S = 1$, and calculable bounds on C_I are known, see the discussion in Remark 5.3. Note that $\bar{\gamma}_h = \mathcal{O}(h)$ in this case, so that $\lambda_h \bar{\gamma}_h^2$ is only a higher-order correction term. From another perspective, $\bar{\alpha}_h \leq \sqrt{2} \bar{\beta}_h$ converges by one order faster than $\|\nabla u_h + \boldsymbol{\sigma}_h\|$ from (6.7), so that (6.13a) reflects the (by one order) faster a priori convergence of $\|u_1 - u_h\|$ in comparison with that of $\|\nabla(u_1 - u_h)\|$.*

Remark 6.11 (Dependence on the mesh maximal diameter h). *The maximal element diameter h is not present at all in Theorem 6.1 and it does not necessarily need to tend to zero in Theorem 6.3: it only appears as a multiplicative factor of the principal estimator $\|\nabla u_h + \boldsymbol{\sigma}_h\|$. (Note in this respect that $h \rightarrow 0$ may not be needed in a posteriori refinement strategies/convergence proofs.) This stands in contrast to previous guaranteed results like [14, Theorem 3.2] or [39, Theorem 4.3].*

We finally discuss the eigenvector bounds of Theorems 6.2 and 6.4:

Remark 6.12 (Polynomial-degree robustness). Noting $\frac{\|\nabla(u_1 - u_h)\|^2}{\lambda_1} \leq \frac{2(\lambda_1 + \lambda_h)}{\lambda_1}$ (in fact this term becomes negligible with mesh refinement/increasing the polynomial degree), we conclude that the result of Theorem 6.2 is fully robust with respect to the polynomial degree p of u_h : all the constants in the comparison between the error $\|\nabla(u_1 - u_h)\|$ and the estimate featuring $\|\nabla u_h + \sigma_h\|$ are independent of p . A similar claim holds for Theorem 6.4: note that $(1 + 4\lambda_h \bar{\gamma}_h^2)$ is bounded/tends to one.

Remark 6.13 (Error localization and mesh adaptivity). Since $\|\nabla u_h + \sigma_h\|^2 = \sum_{K \in \mathcal{T}_h} \|\nabla u_h + \sigma_h\|_K^2$, a crucial advantage of the estimators in Theorems 6.2 and 6.4 is that these local contributions can be used in adaptive mesh refinement based on marking strategies. This is tightly linked to Remark 6.11.

Remark 6.14 (hp adaptivity). The approximation polynomial degree p was considered fixed here. Extension to variable polynomial degree is straightforward following Dolejší et al. [19] and immediately suggests hp adaptive refinement of both the computational mesh and the polynomial degree.

Remark 6.15 (Cheaper flux reconstruction). Choosing, for $u_h \in \mathbb{P}_p(\mathcal{T}_h)$, $\mathbf{V}_h \times Q_h$ the Raviart–Thomas–Nédélec space of order p instead of $(p + 1)$ requested in Assumption 4.2 still gives σ_h with (4.1b) and thus (6.6a)/(6.11). The polynomial-degree robustness (6.6b) may then, however, be lost in theory (not expected in practice).

Remark 6.16 (Reusing the lower bound for λ_1). It shall be noted that $\lambda_h - \eta^2$ is itself a lower bound of λ_1 . One can therefore start with a first initial lower bound $\underline{\lambda}_1$ of λ_1 and compute $\lambda_h - \eta^2$. In a second step, one sets $\underline{\lambda}_1 := \lambda_h - \eta^2$ and recomputes the estimates (6.4a)–(6.4b) (and similarly in Theorem 6.3) a second time. This is, however, of limited practical interest as the effectivities of the bounds appear to be more sensitive to the sharpness of the second lower bound $\underline{\lambda}_2$.

Remark 6.17 (Practical improvements). The sharpness of the lower bounds $\underline{\lambda}_1$ and $\underline{\lambda}_2$ impacts the effectivities of the a posteriori error estimates in particular in Theorems 6.1 and 6.2. A practical strategy can be applied whenever the linear algebra toolbox used to compute (λ_h, u_h) also provides bounds/efficient approximations $\lambda_{h,2}$ to λ_2 , see, e.g., Saad [47] and the references therein. One simply replaces $\underline{\lambda}_1$ and $\underline{\lambda}_2$ by the numerically computed values λ_h (or $\lambda_h - \eta^2$, cf. above) and $\lambda_{h,2}$. In this setting, Theorems 6.1 and 6.2 may no longer hold, but good results are expected in practice.

7 Application to conforming finite elements

We verify in this section the conditions of the application of our results for the conforming finite element method.

Let $V_h := \mathbb{P}_p(\mathcal{T}_h) \cap V$ for a given polynomial degree $p \geq 1$. In the finite element method, one looks for $(u_h, \lambda_h) \in V_h \times \mathbb{R}^+$ with $\|u_h\| = 1$ and $(u_h, 1) > 0$ such that, cf. [2, 6] and the references therein,

$$(\nabla u_h, \nabla v_h) = \lambda_h (u_h, v_h) \quad \forall v_h \in V_h. \quad (7.1)$$

Thus, the condition $\|\nabla u_h\|^2 = \lambda_h$ follows upon taking $v_h = u_h$ in (7.1), and Assumption 3.4 is satisfied. Condition (6.1) needs to be checked, see the discussion in Section 5.2 and Section 6.4. Definition (7.1) directly implies Assumption 4.1, whereas Assumption 4.2 is fulfilled by choosing the order of the mixed finite element reconstruction spaces $\mathbf{V}_h \times Q_h$ equal to $p + 1$. Thus, upon sufficient mesh/polynomial degree initial refinement leading to (6.2) or (6.7), the results of Theorems 6.1–6.4 hold.

8 Numerical experiments

We numerically illustrate in this section our estimates on two test cases in \mathbb{R}^2 . We have implemented them in the FreeFem++ code [29] with $p = 1$ and follow the advice in Remark 6.15 and use the Raviart–Thomas–Nédélec space of degree $p = 1$ for the flux equilibration. We do not observe any asymptotic loss of the effectivity of the estimates because of that.

	N	h	ndof	$\underline{\lambda}_2 - \lambda_h$ (6.1)	$\bar{\beta}_h$ (6.7a)	$ \Omega ^{-\frac{1}{2}} (u_h, 1) - \bar{\alpha}_h$ (6.7b)
$\underline{\lambda}_1 = 1.5\pi^2$ $\underline{\lambda}_2 = 4.5\pi^2$	2	0.71	9	12.41 (✓)	3.50 (×)	Nan (×)
	3	0.47	16	19.04 (✓)	1.01 (×)	Nan (×)
	4	0.35	25	21.5 (✓)	0.49 (✓)	0.30 (✓)
$\underline{\lambda}_1 = 0.5\pi^2$ $\underline{\lambda}_2 = 3\pi^2$	2	0.71	9	-2.39 (×)	Nan (×)	Nan (×)
	3	0.47	16	4.23 (✓)	3.03 (×)	Nan (×)
	4	0.35	25	6.74 (✓)	1.04 (×)	Nan (×)
	5	0.28	36	7.89 (✓)	0.56 (✓)	0.22 (✓)

Table 1: [Structured mesh, unit square] Validation of the assumptions (6.1) and (6.7) (Nan for $\bar{\beta}_h < 0$ or $\bar{\beta}_h > 1$)

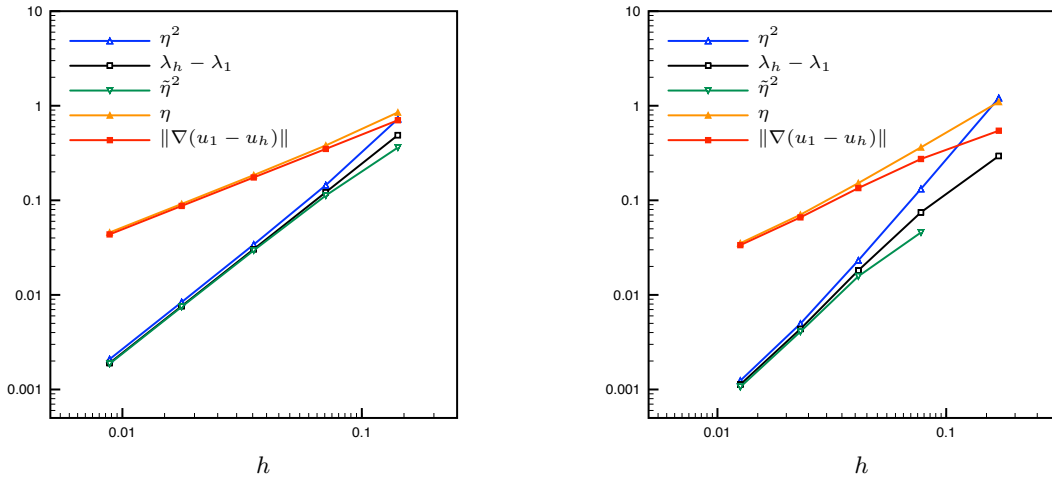


Figure 2: [Unit square] Error in the eigenvalue and eigenvector approximation, its lower bound (eigenvalue only) and its upper bound for the choice $\underline{\lambda}_1 = 1.5\pi^2$, $\underline{\lambda}_2 = 4.5\pi^2$; sequence of structured (left) and unstructured but quasi-uniform (right) meshes

8.1 Unit square

We start by testing the framework on a geometry where everything is known explicitly: the unit square $\Omega = (0, 1)^2$. Here, the first and second eigenvalues are $\lambda_1 = 2\pi^2$ and $\lambda_2 = 5\pi^2$, respectively, see Section 5.2. In consequence, we can easily choose different $\underline{\lambda}_1 \leq \lambda_1$ and $\underline{\lambda}_2 \leq \lambda_2$ and analyze the results with respect to these choices. The first eigenfunction is given by $u_1(x, y) = \sin(\pi x) \sin(\pi y)$. We can here apply the refined theory of Section 6.3 since the domain is convex and consider Corollary 6.5 using the constants $C_S = 1$ and $\delta = 1$ given in Remark 5.3.

8.1.1 Structured mesh

Considering a structured mesh, we first illustrate in Table 1 how quickly the computable conditions (6.1), (6.7) are satisfied under a uniform mesh refinement. We take $C_I = \frac{0.493}{\sqrt{2}}$ following Remark 5.3 and consider $N = 2, 4, 6, 8$ subdivisions of each boundary of Ω for the two choices $\underline{\lambda}_1 = 1.5\pi^2$, $\underline{\lambda}_2 = 4.5\pi^2$ and $\underline{\lambda}_1 = 0.5\pi^2$, $\underline{\lambda}_2 = 3\pi^2$, respectively. Note that the finite element space on the coarsest mesh such that all conditions are satisfied contains 25, resp. 36, degrees of freedom only. Indeed, it turns out that all our conditions are rather mild.

Next, we analyze the performance of the eigenvalue and eigenvector estimates of Corollary 6.5. Figure 2 (left) illustrates the convergence of the error $\lambda_h - \lambda_1$ as well as of its lower and upper bounds $\tilde{\eta}^2$, η^2 given by (6.13a)–(6.13b). We also plot the energy error in the eigenfunction $\|\nabla(u_1 - u_h)\|$ and its upper bound η . This test confirms that the convergence rates for the lower and upper bounds are the same as those of the error in the approximation of the eigenvalue and of the eigenvector.

N	h	ndof	λ_1	λ_h	$\lambda_h - \eta^2$	$\lambda_h - \tilde{\eta}^2$	$I_{\lambda,\text{eff}}^{\text{lb}}$	$I_{\lambda,\text{eff}}^{\text{ub}}$	$E_{\lambda,\text{rel}}$	$I_{u,\text{eff}}^{\text{ub}}$
10	0.1414	121	19.7392	20.2284	19.5054	19.8667	1.35	1.48	1.84E-02	1.21
20	0.0707	441	19.7392	19.8611	19.7164	19.7486	1.08	1.19	1.63E-03	1.09
40	0.0354	1,681	19.7392	19.7696	19.7356	19.7401	1.03	1.12	2.28E-04	1.06
80	0.0177	6,561	19.7392	19.7468	19.7384	19.7393	1.02	1.10	4.56E-05	1.05
160	0.0088	25,921	19.7392	19.7411	19.7390	19.7392	1.02	1.10	1.01E-05	1.05

Table 2: [Structured mesh, unit square] Lower and upper bounds of the exact eigenvalue λ_1 , the effectivity indices, and the relative eigenvalue error; case $\underline{\lambda}_1 = 1.5\pi^2$, $\underline{\lambda}_2 = 4.5\pi^2$

N	h	ndof	λ_1	λ_h	$\lambda_h - \eta^2$	$\lambda_h - \tilde{\eta}^2$	$I_{\lambda,\text{eff}}^{\text{lb}}$	$I_{\lambda,\text{eff}}^{\text{ub}}$	$E_{\lambda,\text{rel}}$	$I_{u,\text{eff}}^{\text{ub}}$
10	0.1698	143	19.7392	20.0336	18.8265	–	–	4.10	–	2.02
20	0.0776	523	19.7392	19.8139	19.6820	19.7682	1.63	1.77	4.37E-03	1.33
40	0.0413	1,975	19.7392	19.7573	19.7342	19.7416	1.15	1.28	3.75E-04	1.13
80	0.0230	7,704	19.7392	19.7436	19.7386	19.7395	1.07	1.14	4.56E-05	1.07
160	0.0126	30,666	19.7392	19.7403	19.7391	19.7393	1.06	1.10	1.01E-05	1.05

Table 3: [Unstructured mesh, unit square] Lower and upper bounds of the exact eigenvalue λ_1 , the effectivity indices, and the relative eigenvalue error; case $\underline{\lambda}_1 = 1.5\pi^2$, $\underline{\lambda}_2 = 4.5\pi^2$

We present in Table 2 precise numbers of the lower and upper bounds $\lambda_h - \eta^2 \leq \lambda_1 \leq \lambda_h - \tilde{\eta}^2$ for the exact eigenvalue λ_1 , the effectivity indices of the lower and upper bounds $\tilde{\eta}^2 \leq \lambda_h - \lambda_1 \leq \eta^2$ of the error $\lambda_h - \lambda_1$, and the effectivity index of the upper bound $\|\nabla(u_1 - u_h)\| \leq \eta$, given respectively by

$$I_{\lambda,\text{eff}}^{\text{lb}} := \frac{\lambda_h - \lambda_1}{\tilde{\eta}^2}, \quad I_{\lambda,\text{eff}}^{\text{ub}} := \frac{\eta^2}{\lambda_h - \lambda_1}, \quad I_{u,\text{eff}}^{\text{ub}} := \frac{\eta}{\|\nabla(u_1 - u_h)\|}. \quad (8.1)$$

We observe rather sharp results, and this also for the relative error

$$E_{\lambda,\text{rel}} := 2 \frac{(\lambda_h - \tilde{\eta}^2) - (\lambda_h - \eta^2)}{(\lambda_h - \tilde{\eta}^2) + (\lambda_h - \eta^2)}. \quad (8.2)$$

8.1.2 Unstructured mesh

Consider now a sequence of unstructured quasi-uniform meshes, obtained by an initial partition of each boundary edge into N intervals. Since for coarse discretizations, unstructured meshes resemble the structured ones, the conditions (6.1), (6.7) are satisfied similarly as in Table 1.

The convergence plots for this case are presented in Figure 2 (right), showing a similar behavior as for the structured meshes. This time, we use the upper bound of C_I according to [38, Eqn. (46)]: $C_I = 0.493 \max_{K \in \mathcal{T}_h} \frac{1 + |\cos(\theta_K)|}{\sin(\theta_K)} \sqrt{\frac{\nu_+(\alpha_K, \theta_K)}{2}} \frac{h_K^{[38]}}{h_K}$. We refer to [38] for the definition of $h_K^{[38]}$ and other notation. We observe in Table 3 that the results are similar to structured meshes; in particular the case of $\underline{\lambda}_1 = 0.5\pi^2$, $\underline{\lambda}_2 = 3\pi^2$ is less sensitive to the unstructured mesh (not presented).

8.2 L-shaped domain

We next consider the L-shaped domain $\Omega := (-1, 1)^2 \setminus [0, 1] \times [-1, 0]$ where $\lambda_1 \approx 9.6397238440$ is known to high accuracy [50]. By domain inclusion, as explained in Section 5.2, we obtain the lower bounds $\underline{\lambda}_1 = \pi^2/2$ and $\underline{\lambda}_2 = 5\pi^2/4$.

8.2.1 Unstructured mesh with quasi-uniform refinement

We first consider a sequence of unstructured quasi-uniform meshes, with N elements partitioning the edges of Ω of length 2 and $N/2$ elements the edges of length 1. Table 4 illustrates whether the conditions (6.1), (6.2) are satisfied or not depending on N . Compared to the square domain we observe that a finer mesh is

N	h	ndof	$\lambda_2 - \lambda_h$ (6.1)	$\bar{\beta}_h$ (6.2a)	$ \Omega ^{-\frac{1}{2}} (u_h, 1) - \bar{\alpha}_h$ (6.2b)
4	0.75	22	-1.02 (×)	Nan (×)	Nan (×)
6	0.47	41	1.06 (✓)	7.62 (×)	Nan (×)
22	0.14	453	2.55 (✓)	0.94 (✓)	-0.37 (×)
30	0.10	826	2.61 (✓)	0.70 (✓)	0.03 (✓)

Table 4: [Unstructured mesh, L-shaped domain] Validation of the assumptions (6.1), (6.2) for $\lambda_1 = \pi^2/2$ and $\lambda_2 = 5\pi^2/4$ (Nan for $\bar{\beta}_h < 0$ or $\bar{\beta}_h > 1$)

N	h	ndof	λ_1	λ_h	$\lambda_h - \eta^2$	$\lambda_h - \tilde{\eta}^2$	$I_{\lambda, \text{eff}}^{\text{lb}}$	$I_{\lambda, \text{eff}}^{\text{ub}}$	$E_{\lambda, \text{rel}}$	$I_{u, \text{eff}}^{\text{ub}}$
30	0.1038	826	9.63972	9.72744	6.88126	9.72064	12.90	32.45	3.42E-01	5.72
60	0.0608	3,154	9.63972	9.66968	8.81618	9.66705	11.39	28.49	9.21E-02	5.38
120	0.0299	12,747	9.63972	9.65032	9.35716	9.64937	11.08	27.65	3.07E-02	5.32
240	0.0164	49,119	9.63972	9.64367	9.53508	9.64331	11.03	27.51	1.13E-02	5.49
360	0.0104	114,806	9.63972	9.64192	9.58128	9.64173	11.08	27.55	6.29E-03	5.40

Table 5: [Unstructured mesh, L-shaped domain] Lower and upper bound of the exact eigenvalue λ_1 , the effectivity indices, and the relative eigenvalue error

needed until all conditions are satisfied. On the other hand, since the solution is of low regularity, a finer mesh is naturally needed to get some minimal accuracy with a uniform mesh refinement.

Figure 3 (left) illustrates the convergence of $\lambda_h - \lambda_1$ and of its lower and upper bounds $\tilde{\eta}^2$, η^2 given by (6.4a)–(6.4b) of Theorem 6.1. The energy error $\|\nabla(u_1 - u_h)\|$ and its upper bound η of Theorem 6.2 are given likewise, with details in Table 5. All the theoretical results are nicely confirmed, with, however, somewhat worse effectivity indices because of the multiplicative factor $1 - \frac{\lambda_h}{\lambda_2}$ as discussed in Remark 6.10.

8.2.2 Adaptive refinement

We finally propose an adaptive refinement strategy using the local character of the eigenvector estimator as explained in Remark 6.13. We have used the Dörfler marking with $\theta = 0.6$ and the newest vertex bisection mesh refinement using the Matlab iFEM implementation, see [15].

Figure 3 (middle) illustrates the error in the eigenvalue and the eigenvector and its lower (for eigenvalues only) and upper bounds using (6.4a)–(6.4b) of Theorem 6.1 and (6.6a) of Theorem 6.2. The optimal convergence rates are indicated by dashed lines. The initial mesh is structured with 1,281 degrees of freedom so that the conditions (6.1), (6.2a), and (6.2b) are all satisfied. Figure 3 (right) reports the same results using the practical but less rigorous strategy outlined in Remark 6.17. The initial (structured) mesh with 176 degrees of freedom again satisfies all (6.1), (6.2a), and (6.2b). Tables 6 and 7 then present more details.

Not surprisingly, we observe the benefit of the adaptive mesh refinement as we deal with nonsmooth eigenfunctions. Indeed, we obtain an accuracy in the eigenvalue approximation of 0.0106 with 12,747 degrees of freedom for a quasi-uniform mesh, compared to a precision of 0.00936 with 3,445 degrees of freedom on the adapted mesh. This happens independently of the not very good effectivity indices; applying the practical strategy of Remark 6.17 indeed decreases this overestimation.

8.3 Discussion

We summarize here some conclusions of the numerical tests, which are all in a full agreement with our theory:

- o The computationally verifiable conditions are satisfied on relative coarse meshes using few degrees of freedom.
- o The estimates are guaranteed and efficient (converge with same speed as the error).
- o The asymptotic behavior of the estimates is not sensitive with respect to the choices λ_1 and λ_2 for lower bounds of the first and second eigenvalues, respectively.

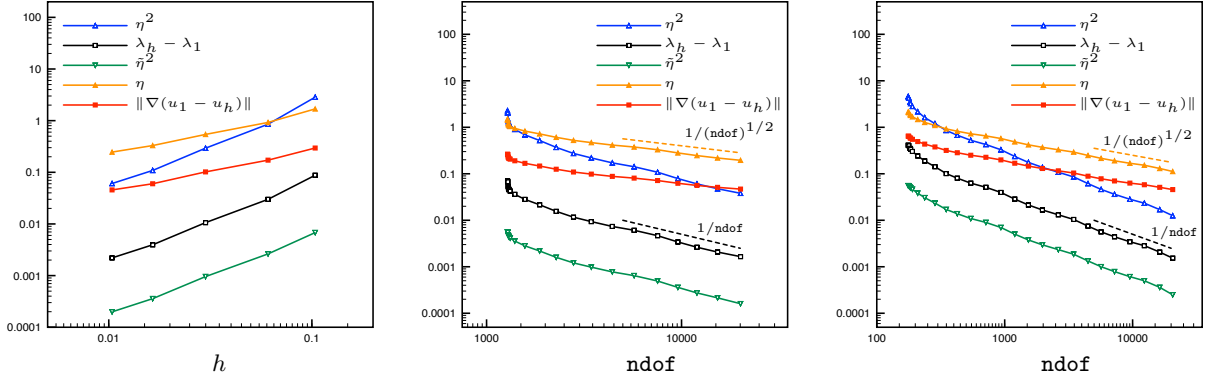


Figure 3: [Unstructured and adaptive mesh refinement, L-shaped domain] Error in the eigenvalue and eigenvector approximation, its lower bound (eigenvalue only) and its upper bound for a quasi-uniform refinement (left) using $\lambda_1 = \pi^2/2$, $\lambda_2 = 5\pi^2/4$, adaptive refinement using $\lambda_1 = \pi^2/2$, $\lambda_2 = 5\pi^2/4$ (middle), and the practical strategy of Remark 6.17 with $\lambda_1 = \lambda_h$, $\lambda_2 = \lambda_{h,2}$ (right)

Level	ndof	λ_1	λ_h	$\lambda_h - \eta^2$	$\lambda_h - \tilde{\eta}^2$	$I_{\lambda, \text{eff}}^{\text{lb}}$	$I_{\lambda, \text{eff}}^{\text{ub}}$	$E_{\lambda, \text{rel}}$	$I_{u, \text{eff}}^{\text{ub}}$
2	1,282	9.63972	9.70858	7.56083	9.70303	12.39	31.19	2.48E-01	5.62
6	1,294	9.63972	9.68971	8.35342	9.68509	10.83	26.73	1.48E-01	5.19
10	1,396	9.63972	9.67581	8.77643	9.67225	10.12	24.92	9.71E-02	4.98
14	2,792	9.63972	9.65137	9.37756	9.65016	9.63	23.51	2.87E-02	4.80
18	7,538	9.63972	9.64438	9.53634	9.64389	9.44	23.19	1.12E-02	4.60
22	20,071	9.63972	9.64137	9.60336	9.64122	10.30	23.01	3.93E-03	4.16

Table 6: [Adaptive mesh refinement, L-shaped domain] Lower and upper bound of the exact eigenvalue λ_1 , the effectivity indices, and the relative eigenvalue error

- o The quality of the estimates is conceptually not degraded when applied to unstructured and highly graded meshes (only shape regularity is required).
- o For convex Ω , the elliptic regularity estimates of Section 6.3 give much sharper results.
- o For nonconvex domains, the strategy of Remark 6.17 provides improved effectivity indices, using a not completely rigorous but more precise choice of λ_1 and λ_2 .

We finally compare our results with some existing ones from [14, 39]. For the unit square, Table 2 reveals that our estimates are sharper compared to the estimates presented in [14], see Figure 6.2 therein. For the L-shaped domain and uniformly refined meshes, we obtain better results than those presented in [14] (Figure 6.4), not efficient in this case. Compared to the results presented in [39] (Table 5.5), we observe that our lower bound $\lambda_h - \eta^2$ of the exact eigenvalue is a little less sharp, whereas the upper bound $\lambda_h - \tilde{\eta}^2$ is not present in [39]. Recall also from Section 1 that our estimates are much cheaper here than those of [39] (there is no auxiliary eigenvalue problem to solve). For adaptive meshes, we observe that our efficiency as measured in [14] by

$$\frac{\frac{1}{2}((\lambda_h - \tilde{\eta}^2) - (\lambda_h - \eta^2))}{|\lambda_1 - \frac{1}{2}(\lambda_h - \tilde{\eta}^2 + \lambda_h - \eta^2)|} = \frac{\frac{1}{2}(\eta^2 - \tilde{\eta}^2)}{|\lambda_1 - \lambda_h + \frac{1}{2}(\tilde{\eta}^2 + \eta^2)|} \quad (8.3)$$

is approaching 1.086 which is much better than in [14] (Figure 6.5). However, this index is not really fair in our opinion: as an example, applying the strategy of Remark 6.17 reduces the effectivity indices $I_{\lambda, \text{eff}}^{\text{ub}}$, $I_{\lambda, \text{eff}}^{\text{lb}}$, and $I_{u, \text{eff}}^{\text{ub}}$ but increases the index (8.3) (it now tends to 1.266). Recall that an important advantage of our estimates is that they also control the eigenvector error and enable adaptive mesh refinement.

Level	ndof	λ_1	λ_h	$\lambda_h - \eta^2$	$\lambda_h - \tilde{\eta}^2$	$I_{\lambda, \text{eff}}^{\text{lb}}$	$I_{\lambda, \text{eff}}^{\text{ub}}$	$E_{\lambda, \text{rel}}$	$I_{u, \text{eff}}^{\text{ub}}$
1	176	9.63972	10.0518	5.43638	9.99630	7.43	11.20	5.91E-01	3.33
6	190	9.63972	9.94166	7.12891	9.89532	6.52	9.32	3.25E-01	3.04
11	426	9.63972	9.72012	9.04493	9.70628	5.81	8.40	7.05E-02	2.90
16	1,533	9.63972	9.66102	9.48546	9.65725	5.65	8.24	1.79E-02	2.87
21	5,671	9.63972	9.64535	9.59920	9.64435	5.61	8.20	4.69E-03	2.75
26	20,587	9.63972	9.64125	9.62872	9.64101	6.14	8.19	1.28E-03	2.45

Table 7: [Adaptive mesh refinement, L-shaped domain] Lower and upper bound of the exact eigenvalue λ_1 , the effectivity indices, and the relative eigenvalue error; practical improvement of Remark 6.17

9 Concluding remarks

The present theory is only valid for conforming approximations $u_h \in V$ with $\lambda_h > 0$, where the constraints $\|u_h\| = 1$ and $\|\nabla u_h\|^2 = \lambda_h$ are satisfied. The extension to the nonconforming case $u_h \notin V$, for possibly zero (or even negative) λ_h , and with the above conditions violated is treated in the continuation of this work [11]. This allows one, on the one hand, to treat nonconforming, discontinuous Galerkin, or mixed finite element discretizations, and, on the other hand, to take into account the error in the solution of the algebraic systems (7.1) and devise adaptive stopping criteria.

Finally, problem (2.1) is the simplest in a hierarchy: our next goals are to extend the present theory to (2.1) with a linear potential (reaction) term, and finally to consider nonlinear Schrödinger equations. It appears that guaranteed eigenvector upper bounds can also be derived in this case, see Cancès *et al.* [10, Theorem 3.1] and [12], provided that guaranteed lower bounds for the first eigenvalue of a linearized problem are known. This was actually our motivation here. Finally, also for nonlinear eigenvalue problems, fully adaptive strategies with dynamic stopping criteria are of our interest, where some of the first results are summarized in [21, 10, 12].

References

- [1] M. G. ARMENTANO AND R. G. DURÁN, *Asymptotic lower bounds for eigenvalues by nonconforming finite element methods*, Electron. Trans. Numer. Anal., 17 (2004), pp. 93–101.
- [2] I. BABUŠKA AND J. OSBORN, *Eigenvalue problems*, in Handbook of numerical analysis, Vol. II, Handb. Numer. Anal., II, North-Holland, Amsterdam, 1991, pp. 641–787.
- [3] I. BABUŠKA AND T. STROUBOULIS, *The finite element method and its reliability*, Numerical Mathematics and Scientific Computation, The Clarendon Press Oxford University Press, New York, 2001.
- [4] N. W. BAZLEY AND D. W. FOX, *Lower bounds for eigenvalues of Schrödinger’s equation*, Phys. Rev. (2), 124 (1961), pp. 483–492.
- [5] H. BLUM AND M. DOBROWOLSKI, *On finite element methods for elliptic equations on domains with corners*, Computing, 28 (1982), pp. 53–63.
- [6] D. BOFFI, *Finite element approximation of eigenvalue problems*, Acta Numer., 19 (2010), pp. 1–120.
- [7] D. BRAESS, V. PILLWEIN, AND J. SCHÖBERL, *Equilibrated residual error estimates are p-robust*, Comput. Methods Appl. Mech. Engrg., 198 (2009), pp. 1189–1197.
- [8] S. C. BRENNER AND L. R. SCOTT, *The mathematical theory of finite element methods*, vol. 15 of Texts in Applied Mathematics, Springer, New York, third ed., 2008.
- [9] F. BREZZI AND M. FORTIN, *Mixed and hybrid finite element methods*, vol. 15 of Springer Series in Computational Mathematics, Springer-Verlag, New York, 1991.
- [10] E. CANCÈS, G. DUSSON, Y. MADAY, B. STAMM, AND M. VOHRALÍK, *A perturbation-method-based a posteriori estimator for the planewave discretization of nonlinear Schrödinger equations*, C. R. Math. Acad. Sci. Paris, 352 (2014), pp. 941–946.

- [11] ———, *Guaranteed and robust a posteriori bounds for Laplace eigenvalues and eigenvectors: a unified framework*. In preparation, 2015.
- [12] ———, *A perturbation-method-based post-processing for the planewave discretization of Kohn–Sham models*. HAL Preprint 01140818, 2015.
- [13] C. CARSTENSEN AND S. A. FUNKEN, *Fully reliable localized error control in the FEM*, SIAM J. Sci. Comput., 21 (1999/00), pp. 1465–1484.
- [14] C. CARSTENSEN AND J. GEDICKE, *Guaranteed lower bounds for eigenvalues*, Math. Comp., 83 (2014), pp. 2605–2629.
- [15] L. CHEN, *iFEM: an innovative finite element methods package in MATLAB*, tech. report, University of California at Irvine, 2009.
- [16] P. CIARLET, JR. AND M. VOHRALÍK, *Robust a posteriori error control for transmission problems with sign changing coefficients using localization of dual norms*. HAL Preprint 01148476, 2015.
- [17] P. G. CIARLET, *The Finite Element Method for Elliptic Problems*, vol. 4 of Studies in Mathematics and its Applications, North-Holland, Amsterdam, 1978.
- [18] P. DESTUYNDER AND B. MÉTIVET, *Explicit error bounds in a conforming finite element method*, Math. Comp., 68 (1999), pp. 1379–1396.
- [19] V. DOLEJŠÍ, A. ERN, AND M. VOHRALÍK, *hp-refinement strategies using polynomial-degree-robust estimates*. HAL Preprint 01165187, 2015.
- [20] R. G. DURÁN, C. PADRA, AND R. RODRÍGUEZ, *A posteriori error estimates for the finite element approximation of eigenvalue problems*, Math. Models Methods Appl. Sci., 13 (2003), pp. 1219–1229.
- [21] G. DUSSON AND Y. MADAY, *A posteriori analysis of a non-linear Gross–Pitaevskii type eigenvalue problem*. HAL Preprint 00903715, 2013.
- [22] A. ERN AND M. VOHRALÍK, *Polynomial-degree-robust a posteriori estimates in a unified setting for conforming, nonconforming, discontinuous Galerkin, and mixed discretizations*, SIAM J. Numer. Anal., 53 (2015), pp. 1058–1081.
- [23] G. E. FORSYTHE, *Asymptotic lower bounds for the fundamental frequency of convex membranes*, Pacific J. Math., 5 (1955), pp. 691–702.
- [24] D. W. FOX AND W. C. RHEINBOLDT, *Computational methods for determining lower bounds for eigenvalues of operators in Hilbert space*, SIAM Rev., 8 (1966), pp. 427–462.
- [25] D. GILBARG AND N. S. TRUDINGER, *Elliptic partial differential equations of second order*, vol. 224 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Springer-Verlag, Berlin, second ed., 1983.
- [26] F. GOERISCH AND Z. Q. HE, *The determination of guaranteed bounds to eigenvalues with the use of variational methods. I*, in Computer arithmetic and self-validating numerical methods (Basel, 1989), vol. 7 of Notes Rep. Math. Sci. Engrg., Academic Press, Boston, MA, 1990, pp. 137–153.
- [27] P. GRISVARD, *Elliptic problems in nonsmooth domains*, vol. 24 of Monographs and Studies in Mathematics, Pitman (Advanced Publishing Program), Boston, MA, 1985.
- [28] L. GRUBIŠIĆ AND J. S. OVAL, *On estimators for eigenvalue/eigenvector approximations*, Math. Comp., 78 (2009), pp. 739–770.
- [29] F. HECHT, *New development in freefem++*, J. Numer. Math., 20 (2012), pp. 251–265.
- [30] V. HEUVELINE AND R. RANNACHER, *A posteriori error control for finite approximations of elliptic eigenvalue problems*, Adv. Comput. Math., 15 (2001), pp. 107–138.

- [31] J. HU, Y. HUANG, AND Q. LIN, *Lower bounds for eigenvalues of elliptic operators: by nonconforming finite element methods*, J. Sci. Comput., 61 (2014), pp. 196–221.
- [32] J. HU, Y. HUANG, AND Q. SHEN, *The lower/upper bound property of approximate eigenvalues by nonconforming finite element methods for elliptic operators*, J. Sci. Comput., 58 (2014), pp. 574–591.
- [33] T. KATO, *On the upper and lower bounds of eigenvalues*, J. Phys. Soc. Japan, 4 (1949), pp. 334–339.
- [34] J. R. KUTTLER AND V. G. SIGILLITO, *Bounding eigenvalues of elliptic operators*, SIAM J. Math. Anal., 9 (1978), pp. 768–778.
- [35] ———, *Estimating eigenvalues with a posteriori/a priori inequalities*, vol. 135 of Research Notes in Mathematics, Pitman (Advanced Publishing Program), Boston, MA, 1985.
- [36] Y. A. KUZNETSOV AND S. I. REPIN, *Guaranteed lower bounds of the smallest eigenvalues of elliptic differential operators*, J. Numer. Math., 21 (2013), pp. 135–156.
- [37] M. G. LARSON, *A posteriori and a priori error analysis for finite element approximations of self-adjoint elliptic eigenvalue problems*, SIAM J. Numer. Anal., 38 (2000), pp. 608–625.
- [38] X. LIU AND F. KIKUCHI, *Analysis and estimation of error constants for P_0 and P_1 interpolations over triangular finite elements*, J. Math. Sci. Univ. Tokyo, 17 (2010), pp. 27–78.
- [39] X. LIU AND S. OISHI, *Verified eigenvalue evaluation for the Laplacian over polygonal domains of arbitrary shape*, SIAM J. Numer. Anal., 51 (2013), pp. 1634–1654.
- [40] Y. MADAY AND A. T. PATERA, *Numerical analysis of a posteriori finite element bounds for linear functional outputs*, Math. Models Methods Appl. Sci., 10 (2000), pp. 785–799.
- [41] C. B. MOLER AND L. E. PAYNE, *Bounds for eigenvalues and eigenvectors of symmetric operators*, SIAM J. Numer. Anal., 5 (1968), pp. 64–70.
- [42] M. PLUM, *Guaranteed numerical bounds for eigenvalues*, in Spectral theory and computational methods of Sturm-Liouville problems (Knoxville, TN, 1996), vol. 191 of Lecture Notes in Pure and Appl. Math., Dekker, New York, 1997, pp. 313–332.
- [43] W. PRAGER AND J. L. SYNGE, *Approximations in elasticity based on the concept of function space*, Quart. Appl. Math., 5 (1947), pp. 241–269.
- [44] R. RANNACHER, A. WESTENBERGER, AND W. WOLLNER, *Adaptive finite element solution of eigenvalue problems: balancing of discretization and iteration error*, J. Numer. Math., 18 (2010), pp. 303–327.
- [45] S. I. REPIN, *A posteriori estimates for partial differential equations*, vol. 4 of Radon Series on Computational and Applied Mathematics, Walter de Gruyter GmbH & Co. KG, Berlin, 2008.
- [46] J. E. ROBERTS AND J.-M. THOMAS, *Mixed and hybrid methods*, in Handbook of Numerical Analysis, Vol. II, North-Holland, Amsterdam, 1991, pp. 523–639.
- [47] Y. SAAD, *Numerical methods for large eigenvalue problems*, Algorithms and Architectures for Advanced Scientific Computing, Manchester University Press, Manchester; Halsted Press [John Wiley & Sons, Inc.], New York, 1992.
- [48] I. ŠEBESTOVÁ AND T. VEJCHODSKÝ, *Two-sided bounds for eigenvalues of differential operators with applications to Friedrichs, Poincaré, trace, and similar constants*, SIAM J. Numer. Anal., 52 (2014), pp. 308–329.
- [49] G. STILL, *Computable bounds for eigenvalues and eigenfunctions of elliptic differential operators*, Numer. Math., 54 (1988), pp. 201–223.
- [50] L. N. TREFETHEN AND T. BETCKE, *Computed eigenmodes of planar regions*, in Recent advances in differential equations and mathematical physics, vol. 412 of Contemp. Math., Amer. Math. Soc., Providence, RI, 2006, pp. 297–314.

- [51] R. VERFÜRTH, *A posteriori error estimates for nonlinear problems. Finite element discretizations of elliptic equations*, Math. Comp., 62 (1994), pp. 445–475.
- [52] H. F. WEINBERGER, *Upper and lower bounds for eigenvalues by finite difference methods*, Comm. Pure Appl. Math., 9 (1956), pp. 613–623.
- [53] Y. YANG, J. HAN, H. BI, AND Y. YU, *The lower/upper bound property of the Crouzeix–Raviart element eigenvalues on adaptive meshes*, J. Sci. Comput., 62 (2015), pp. 284–299.