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One-dimensional skew diffusions: explicit expressions of densities and resolvent kernel*

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Abstract: The study of skew diffusion is of primary concern for their implication in the modeling and simulation of diffusion phenomena in media with interfaces. First, we provide results on one-dimensional processes with discontinuous coefficients and their connections with the Feller theory of generators as well as the one of stochastic differential equations involving local time. Second, in view of developing new simulation techniques, we give a method to compute the density and the resolvent kernel of skew diffusions. Explicit closed-form are given for some particular cases.

MSC 2010 subject classifications: Primary Diffusion processes, Divergence form operators, SDE involving local time, resolvent kernel; secondary Laplace transform.

Keywords and phrases: Diffusion processes, Divergence form operators, SDE with local time, resolvent kernel, Laplace transform, Weyl-Titchmarsh-Kodaira theory.

Introduction

Despite being introduced in the '60, the topic of skew diffusions is now more and more popular for its practical applications in many domains where interfaces and discontinuous coefficients are encountered. Among others, let us cite geophysics, ecology, brain imaging, astrophysics, meteorology or molecular dynamics (See [29, 38] for a list of references).

There are several ways to construct a so-called one-dimensional skew diffusion. One could define it through a scale function and a speed measure. This construction was given for the Skew Brownian motion (SBM) by K. Itô and H.P. McKean [22]. An other formulation consider a skew diffusion as the unique strong solutions of stochastic differential equations (SDE) involving local time. This was used by J.M. Harrison and L.A. Shepp [20] for their construction of the SBM. Among others possibilities (see

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[27] for a full panel), N. Portenko choose to see the SBM as a process with a singular drift and give a purely analytic method for its construction (See *e.g.* [35]).

In the '80, J.-F. Le Gall generalized the notion of Stochastic Differential Equations (SDE) involving local time and provides us, among others, with strong existence and uniqueness results [25]. In the same time, J. Groh linked this type of SDEs through scale functions and speed measures [18].

These last studies were of great help for the understanding of diffusions processes whose infinitesimal generators are divergence form operators with piecewise regular coefficients. Indeed, in a series of work [8, 27, 28], ..., the authors show that these processes are solutions of SDEs involving local time. This endows their importance to understand the diffusion phenomenon in media with interfaces and therefore justifies the search for numerical simulation algorithms (See [29] and references within).

This article has three goals.

Firstly, it aims at completing the study of a large class of SDE with local time by characterizing their infinitesimal generators not only in term of divergence form operators but also of classical differential operators following the approach of W. Feller [12, 13]. Going back to Sturm-Liouville problem, which was a starting point for W. Feller, we characterize the domain of the operator. This provides us with an alternative formulation to the so-called *diffraction problem* (see *e.g.* [24]) with minimal regularity assumption on the coefficients. We also show a stability property of this class of functions under a family of piecewise regular one-to-one space transforms. Such a result was also obtained using the Itô-Tanaka formula in [11].

Secondly, using the Weyl-Titchmarsh-Kodaira theory ([23, 42, 44] and an analytic machinery, we provide a closed-form for the resolvent kernel of the operator in free space. Since this approach relies on a appropriate combination of known resolvents in free space, it can be easily generalized to deal with boundary conditions and solve related problems such as the elastic skew Brownian motion and the Feynman-Kac formula. This also leads to expressions for the Laplace transform of the local time at 0 or the occupation time of some segment or half-line. Although our framework presents similarities with the one developed by B. Gaveau *et al.* [17], we differ by our application of the Weyl-Titchmarsh-Kodaira theory. Besides, the drift term, which was our primarily purpose, can only be taken into account in [17] at the price of using limiting arguments on rather complex formulae.

We point out that obtaining explicit expressions for the resolvent or the density is of higher importance to set-up numerical schemes as in *e.g.* [29], since the dynamic of a process solution to a SDE with local time is difficult to apprehend.

Finally, we consider the skew Brownian motion with a possibly piecewise constant drift which takes two values b_+ on \mathbb{R}_+ and b_- on \mathbb{R}_- . For several practical situations, we give an explicit closed-form of the density. Although it seems hard to perform this inversion for general values of b_+ and b_- , we provide an explicit expression for a new case, called *constant Péclet*, which possesses a natural interpretation in fluid mechanics.

While the density of the SBM is easily found [43], a heavy probabilistic machinery was set up for the same purpose when some particular values of the drift are encountered [3, 10, 14, 16]. We consider our work as a first incursion in alternative methods to overcome the difficulties raised by approaches based on stochastic calculus.

Outline

In Section 1, we show the link between processes generated by divergence form operators, those defined in the sense of Feller through their scale functions and their speed measures, and SDE with local time. We also exhibits the behavior of the resolvent kernel. In Section 2, we provide a method for the explicit computation of the resolvent. Finally, in Section 3, densities are computed through the Laplace inversion of resolvents in some particular cases.

1. Operators, semi-groups and diffusion processes

This section is mostly theoretical and pursues multiple goals:

1. To identify the stochastic process X generated by a divergence form operator (in the sense defined by D. Stroock in [41]) with the one defined through its scale function and speed measure. This complements and extends the results obtained in [28].
2. For coefficients that are piecewise smooth with discontinuities of the first kind, to identify the domain of the generator of $X = (X_t)_{t \geq 0}$ and state a result on the regularity of the function given by the probabilistic representation $(t, x) \mapsto \mathbf{E}_x[f(X_t)]$. This extends the result provided in [33].
3. To obtain some analytic properties on the resolvent kernel associated to the process X .
4. To relate the process X when its coefficients are as in 2. above with a SDE involving local time in the sense defined by J.-F. Le Gall in [25].

When I is a union of disjoint open abutting intervals, we mean by f in $\mathcal{C}^k(I, \mathbb{R})$ a function of class \mathcal{C}^k in each interval composing I such that f and its derivatives up to order k have limits at the inner separation of the intervals.

We set $\overline{\mathbb{R}} = [-\infty, \infty]$ and for $\mathbb{K} = \mathbb{R}$ or \mathbb{C} ,

$$\mathcal{C}_b(\overline{\mathbb{R}}, \mathbb{K}) = \{f : \overline{\mathbb{R}} \mapsto \mathbb{K} \mid f \text{ is continuous and bounded}\},$$

$$\mathcal{C}_0(\overline{\mathbb{R}}, \mathbb{K}) = \{f \in \mathcal{C}_b(\overline{\mathbb{R}}, \mathbb{K}) \mid f(\pm\infty) = 0\}.$$

Any continuous function $f : \mathbb{R} \rightarrow \mathbb{K}$ vanishing at infinity is naturally identified with a function in $\mathcal{C}_0(\overline{\mathbb{R}}, \mathbb{K})$.

Let $0 < \lambda < \Lambda$ be two fixed real numbers. We consider the sets

$$\mathfrak{C} = \{\mathbf{a} = (a, \rho, b) \text{ measurable from } \mathbb{R}^3 \rightarrow \mathbb{R} \mid \lambda \leq a, \rho \leq \Lambda \text{ and } |b| \leq \Lambda\},$$

$$\mathfrak{S} = \{\mathbf{a} = (a, \rho, b) \in \mathfrak{C} \mid a, \rho, b \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})\}$$

$$\text{and } \mathfrak{P} = \left\{ (a, \rho, b) \in \mathfrak{C} \left| \begin{array}{l} a \in \mathcal{C}^1(\mathbb{R} \setminus J, \mathbb{R}), b, \rho \in \mathcal{C}(\mathbb{R} \setminus J, \mathbb{R}) \\ \text{where the set } J \text{ has no cluster points} \end{array} \right. \right\}.$$

For a piecewise smooth function $f : \mathbb{R} \rightarrow \mathbb{R}^d$, we denote by $\text{Dis}(f)$ the set of discontinuities of the first kind. When we extend this notation to $\mathbf{a} = (a, \rho, b)$, we obtain $\text{Dis}(\mathbf{a}) = \text{Dis}(a) \cup \text{Dis}(\rho) \cup \text{Dis}(b)$.

1.1. Diffusion process associated to a divergence-form operator

Here we construct a Feller process X associated to a divergence form operator $\frac{\rho}{2}\nabla(a\nabla\cdot) + b\nabla\cdot$ when a is discontinuous. Unlike the solutions of SDE, such a process is not always a semi-martingale. In Subsection 1.6, we however characterize the situations where X is solution to a SDE involving local time.

This section relies on [41] (See also [28]). We use the framework of the variational theory of PDE where the solutions are primary sought in subspaces of $L^2(\mathbb{R})$, the space of square integrable functions on \mathbb{R} . We base our construction on a Gaussian estimate on the fundamental solution.

To each $\mathbf{a} = (a, \rho, b) \in \mathfrak{C}$ is associated a family of bilinear form

$$E_\alpha(u, v) = \frac{1}{2} \int a(x) \nabla u(x) \nabla v(x) dx - \int b(x) \nabla u(x) v(x) \frac{dx}{\rho(x)} + \alpha \int u(x) v(x) \frac{dx}{\rho(x)}, \quad \forall (u, v) \in H^1(\mathbb{R}) \times H^1(\mathbb{R}),$$

where $H^1(\mathbb{R})$ is the Sobolev space of square of functions f with weak derivatives ∇f such that $f, \nabla f \in L^2(\mathbb{R})$.

Clearly, E_α is a Gårding form for $\alpha \geq \alpha_0$ large enough (depending only on λ and Λ). By an application of the Lax-Milgram, one may associate to $(E_\alpha)_{\alpha > 0}$ a resolvent $(G_\alpha)_{\alpha \geq \alpha_0}$ on $L^2(\mathbb{R})$.

When the coefficients \mathbf{a} are smooth, any solution u to the variational problem

$$E_\alpha(u, v) = \langle f, v \rangle_{1/\rho} \stackrel{\text{def}}{=} \int_{\mathbb{R}} f(x) v(x) \frac{dx}{\rho(x)}$$

is smooth when f is.

As a result, u becomes a classical solution to

$$(\alpha - L)u(x) = f(x) \text{ with } Lu(x) = \frac{1}{2}\rho(x)\nabla(a(x)\nabla u(x)) + b(x)\nabla u(x) \quad (1.1)$$

and the domain $\text{Dom}(L) = G_\alpha(L^2(\mathbb{R}))$ for any $\alpha > 0$. But it works even for non-smooth coefficients. Indeed, such a $(L, \text{Dom}(L))$ exists and satisfies

$$\langle (\alpha - L)u, v \rangle_{1/\rho} = E_\alpha(u, v), \quad \forall u \in \text{Dom}(L), \quad \forall v \in H^1(\mathbb{R}).$$

From the Hille-Yosida theorem, the operator L is the infinitesimal generator of a contraction semi-group $(P_t)_{t \geq 0}$. A result from J. Nash and J. Moser extended by D. Aronson [5] and D. Stroock [41] shows that there exists a density q with respect to the measure $dx/\rho(x)$ such that for any $f \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$,

$$u(t, x) = \int_{\mathbb{R}} q(t, x, y) f(y) \frac{dy}{\rho(y)}, \quad t > 0, \quad x \in \mathbb{R}$$

is a version of $P_t f(x) \in L^2(\mathbb{R})$. Moreover, q is Hölder continuous and satisfies a Gaussian upper bound. The main feature is that the constants appearing in the bound

and in the Hölder modulus of continuity depend only on λ and Λ . This proves the existence of a diffusion as well as some stability properties (See [9, 28, 40] where similar arguments are deployed).

Let us set

$$\mathfrak{p} = \{(t, x, y) \in \mathbb{R}^3 \mid t > 0, x \in \mathbb{R}, y \in \mathbb{R}\}. \quad (1.2)$$

Proposition 1.1. *To each element $\mathbf{a} = (a, \rho, b) \in \mathfrak{C}$ is associated a conservative Feller process $(X^{\mathbf{a}}, (\mathcal{F}_t), \mathbf{P}_x)_{t \geq 0, x \in \mathbb{R}}$ with continuous paths. Its transition function admits a density with respect to the Lebesgue measure denoted by $p(t, x, y)$ which satisfies a Gaussian upper bound and is jointly continuous on \mathfrak{p} defined by (1.2). Moreover, if a sequence $\{\mathbf{a}_n\}_{n \in \mathbb{N}}$ of elements of \mathfrak{C} converges pointwise to $\mathbf{a} \in \mathfrak{C}$, then $X^{\mathbf{a}_n}$ converges in distribution to $X^{\mathbf{a}}$ under \mathbf{P}_x for any starting point x .*

Of course, for this diffusion process $X^{\mathbf{a}}$,

$$P_t^{X^{\mathbf{a}}} f(x) \stackrel{\text{def}}{=} \mathbf{E}_x[f(X_t^{\mathbf{a}})] = \int_{\mathbb{R}} p(t, x, y) f(y) dy \stackrel{\text{a.e.}}{=} P_t f(x), \quad \forall t \geq 0, \quad (1.3)$$

$$\text{and } G_{\alpha}^{X^{\mathbf{a}}} f(x) \stackrel{\text{def}}{=} \int_0^{+\infty} e^{-\alpha t} \mathbf{E}_x[f(X_t^{\mathbf{a}})] dt \stackrel{\text{a.e.}}{=} G_{\alpha} f(x), \quad \forall \alpha > 0, \quad (1.4)$$

for any $x \in \mathbb{R}$ and $f \in L^2(\mathbb{R})$. Even if P_t and G_{α} are only operators from $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$, we now identify P_t and G_{α} to $P_t^{X^{\mathbf{a}}}$ and $G_{\alpha}^{X^{\mathbf{a}}}$. These operators $P_t^{X^{\mathbf{a}}}$ and $G_{\alpha}^{X^{\mathbf{a}}}$ also act on $\mathcal{C}_b(\mathbb{R}, \mathbb{R})$. Since $X^{\mathbf{a}}$ is a Feller process, $P_t^{X^{\mathbf{a}}}$ and $G_{\alpha}^{X^{\mathbf{a}}}$ also map $\mathcal{C}_0(\mathbb{R}, \mathbb{R})$ to $\mathcal{C}_0(\mathbb{R}, \mathbb{R})$.

1.2. The infinitesimal generator of $X^{\mathbf{a}}$

In the previous Subsection, a process $X^{\mathbf{a}}$ has been associated to $\mathbf{a} \in \mathfrak{C}$. The domain of its infinitesimal generator may be described as the range of the resolvent $G_{\alpha}(L^2(\mathbb{R}))$. Here we give an alternative description of the infinitesimal generator of $X^{\mathbf{a}}$ using the approach from W. Feller on scale functions and speed measures where the underlying Banach space is the one of continuous bounded functions [12, 13].

We point out that every one-dimensional regular diffusion process Y is characterized by its scale function S and its speed measure M (for classical references on this subject, see e.g. [6, 39, 31]). We also recall that the *speed measure* M is a right-continuous, increasing function, which can be identified with a measure on \mathbb{R} and the *scale function* S , which is unique up to an affine transformation, is a convex, continuous and increasing function.

For a continuous, increasing function $g : \mathbb{R} \rightarrow \mathbb{R}$, let us define

$$D_g^+ f(x) = \lim_{y \searrow x} \frac{f(y+) - f(x-)}{g(y) - g(x)} \quad \text{and} \quad D_g^- f(x) = \lim_{y \nearrow x} \frac{f(x+) - f(y-)}{g(y) - g(x)}. \quad (1.5)$$

From now, we only consider the case of a diffusion Y having \mathbb{R} as a state of space and possessing a continuous speed measure M . The infinitesimal generator of such a process Y associated to (S, M) is

$$Af = \frac{1}{2} D_M D_S f \quad \text{where} \quad D_S f(x) = D_S^+ f(x) = D_S^- f(x) \quad \text{and} \quad D_M = D_M^+ = D_M^-$$

whose domain is [12, Theorem 8.2] for $\bar{\mathbb{R}} = [-\infty, \infty]$,

$$\text{Dom}(A) = \{f \in \mathcal{C}_b(\bar{\mathbb{R}}, \mathbb{R}) \mid D_M D_S f \in \mathcal{C}_b(\bar{\mathbb{R}}, \mathbb{R})\}. \quad (1.6)$$

To $\mathbf{a} = (a, \rho, b) \in \mathfrak{C}$, we associate

$$h(x) = 2 \int_0^x \frac{b(y)}{a(y)\rho(y)} dy, \quad S(x) = \int_0^x s(y) dy \quad \text{where } s(x) = \frac{\exp(-h(x))}{a(x)} \quad (1.7)$$

$$\text{and } M(x) = \int_0^x m(y) dy \quad \text{where } m(x) = \frac{\exp(h(x))}{\rho(x)}. \quad (1.8)$$

Thus $D_M f(x) = m(x)^{-1} D^+ f(x)$ and $D_S f(x) = a(x) \exp(h(x)) D^+ f(x)$, where $D^+ f$ is the right derivative operator.

Proposition 1.2. *For $a \in \mathfrak{C}$, the infinitesimal generator of $X^{\mathbf{a}}$ is $A = \frac{1}{2} D_M D_S$ with (S, M) defined by (1.7)-(1.8) and $\text{Dom}(A)$ given by (1.6).*

Remark 1.1. The infinitesimal generator $\frac{1}{2} D_M D_S$ is the same for $(a, \rho, b) \in \mathfrak{C}$ as well as for $(\tilde{a}, \tilde{\rho}, 0)$ with $\tilde{a}(x) = a(x) \exp(h(x))$ and $\tilde{\rho}(x) = \exp(-h(x)) \rho(x)$. However $(\tilde{a}, \tilde{\rho}, 0)$ does not necessarily belong to \mathfrak{C} as the coefficients could decrease to 0 or increase to ∞ for large values of $|x|$. But, since h is continuous whatever b , this explains why the domain of A does not depend on the regularity of b .

Proof. Let $Y^{\mathbf{a}}$ be the diffusion process with infinitesimal generator A . When $\mathbf{a} \in \mathfrak{S}$, it is easily seen that $Y^{\mathbf{a}}$ is equal in distribution to $X^{\mathbf{a}}$ associated to \mathbf{a} using Proposition 1.1. From (1.7)-(1.8), we deduce that the pointwise convergence of \mathbf{a}_n to \mathbf{a} implies the one of the associated scale function and speed measure. Thus, $(Y^{\mathbf{a}_n})_{n \geq 0}$ converges in distribution to $Y^{\mathbf{a}}$ through the results of [15] on the convergence of the scale function and speed measure. Now Proposition 1.1 implies that $(X^{\mathbf{a}_n})_{n \geq 0}$ converges in distribution to $X^{\mathbf{a}}$. Hence $X^{\mathbf{a}}$ is equal in distribution to $Y^{\mathbf{a}}$ by using for \mathbf{a}_n a regularization of the coefficients. \square

1.3. The resolvent kernel

Let us assume that $\mathbf{a} \in \mathfrak{C}$. Throughout this section, we consider the scale function S and the speed measure M given by (1.7) and (1.8), and A the associated operator $A = \frac{1}{2} D_M D_S$ with domain given by (1.6).

For an interval I of \mathbb{R} , let $\mathcal{AC}(I, \mathbb{C})$ be the space of absolutely continuous functions from I to \mathbb{C} and $\mathcal{S}_{\mathbf{a}}(I, \mathbb{C})$ the space of functions u in $\mathcal{AC}(I, \mathbb{C})$ with $D_S u \in \mathcal{AC}(I, \mathbb{C})$. The next proposition concerns the existence and the regularity of solutions to $(\lambda - A)u = 0$. This proposition is the cornerstone of the Sturm-Liouville theory and its extension by H. Weyl, K. Kodaira and E.C. Titchmarsh using complex analysis.

Proposition 1.3. *For any $(\lambda, \alpha, \beta) \in \mathbb{C}^3$, there exists a unique function $u \in \mathcal{S}_{\mathbf{a}}(\mathbb{R}, \mathbb{C})$ that solves $(\lambda - A)u = 0$ with $u(0) = \alpha$ and $D_S u(0) = \beta$. Besides, $(\lambda, \alpha, \beta) \mapsto (u(x), D_S u(x))$ is holomorphic for any $x \in \mathbb{R}$ on \mathbb{C}^3 .*

Proof. For any $\lambda \in \mathbb{C}$, any solution $Z : \mathbb{R} \rightarrow \mathbb{C}^2$ to the first order differential equation

$$Z'(x) = \begin{bmatrix} 0 & s(x) \\ 2\lambda m(x) & 0 \end{bmatrix} Z(x), \quad Z(0) = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \in \mathbb{C}^2 \text{ for any } x \in \mathbb{R} \quad (1.9)$$

is such that $(\lambda - A)u(x) = 0$ when $Z(x) = (u(x), D_S u(x))$. Regarding the spatial regularity of the solution, let u and f be continuous functions from \mathbb{R} to \mathbb{C} such that $(\lambda - A)u = f$. Then integrating between x and y against the measure M leads to

$$\lambda \int_x^y u(z)m(z) dz - D_S u(x) + D_S u(y) = 2 \int_x^y f(z)m(z) dz. \quad (1.10)$$

In particular, $D_S u \in \mathcal{AC}(\mathbb{R}, \mathbb{C})$ and $u \in \mathcal{S}_{\mathbf{a}}(\mathbb{R}, \mathbb{C})$. □

The *Wronskian* of two functions u and v in $\mathcal{S}_{\mathbf{a}}(\mathbb{R}, \mathbb{C})$ is

$$\text{Wr}[u, v](x) \stackrel{\text{def}}{=} u(x)D_S v(x) - v(x)D_S u(x).$$

If u and v solve $(\lambda - A)u = 0$, then $\text{Wr}[u, v](x)$ is constant over x (See (1.14) below). The solutions u and v are independent if and only $\text{Wr}[u, v](x) \neq 0$. In this case, any solution to $(\lambda - A)w = 0$ on some interval I is a linear combination of u and v on I .

We introduce

$$\mathbb{H}^+ = \{\lambda \in \mathbb{C} \mid \text{Re}(\lambda) > 0\},$$

$$\mathfrak{h}^+ = \{\lambda \in \mathbb{C} \mid \text{Im}(\lambda) = 0, \text{Re}(\lambda) > 0\}, \text{ and } \bar{\mathfrak{h}}^- = \{\lambda \in \mathbb{C} \mid \text{Im}(\lambda) = 0, \text{Re}(\lambda) \leq 0\}.$$

Lemma 1.1. *Let $\tau_x = \inf\{t > 0 \mid X_t^{\mathbf{a}} = x\}$ be the first hitting time of $x \in \mathbb{R}$ of the process $X^{\mathbf{a}}$. For any $y < x$, the function*

$$\phi(\lambda, x) = \mathbf{E}_x[\exp(-\lambda\tau_y)]$$

is holomorphic on \mathbb{H}^+ . Besides, for any $\lambda \in \mathbb{H}^+$, $\phi(\lambda, x)$ is solution to $(\lambda - A)\phi(\lambda, x) = 0$ for $x \geq y$ and $\phi(\lambda, y) = 1$.

Proof. Whatever $\lambda \in \mathbb{H}^+$, $0 \leq |\phi(\lambda, x)| \leq \phi(\text{Re}(\lambda), x) \leq 1$ so that ϕ is well defined on \mathbb{H}^+ . In addition, from the Feynman-Kac formula [22, § 2.6],

$$(\lambda - A)\phi(\lambda, x) = 0, \quad \forall x \geq y.$$

We decide to show that $\phi(\lambda, x)$ can be developed as a power series on a neighborhood of any $\lambda \in \mathbb{H}^+$ since it is equivalent of being holomorphic for complex function.

Notice that

$$\phi(\lambda, x) = 1 - \lambda \mathbf{E}_x \left[\int_0^{\tau_0} \exp(-\lambda s) ds \right],$$

Now using the series representation of the exponential, for $\sigma \in \mathbb{C}$,

$$\begin{aligned} \phi(\lambda + \sigma, x) &= 1 - \lambda \mathbf{E}_x \left[\int_0^{\tau_y} \exp(-\lambda s) \exp(-\sigma s) \, ds \right] \\ &\quad - \sigma \mathbf{E}_x \left[\int_0^{\tau_y} \exp(-\lambda s) \exp(-\sigma s) \, ds \right] = \phi(\lambda, x) - \sum_{k \geq 1} \sigma^k c_k(\lambda, x) \\ &\quad \text{with } c_k(\lambda, x) = \mathbf{E}_x \left[\int_0^{\tau_y} \left(\lambda \frac{s^k}{k!} - \frac{s^{k-1}}{(k-1)!} \right) \exp(-\lambda s) \, ds \right]. \end{aligned} \quad (1.11)$$

Since $\int_0^{+\infty} s^k \exp(-\lambda s) \, ds = k!$ and $\tau_y \geq 0$, $|c_k(\lambda, x)| \leq 1 + 1/\lambda$ for each k . Hence, the series in the right-hand side of (1.11) converges absolutely for any $\sigma \in \mathbb{H}^+$, $|\sigma| < 1$ and the result is proved. \square

The function $\lambda \mapsto \phi(\lambda, x)$ is not necessarily analytic around $\lambda = 0$. Otherwise, it would mean that τ_y has finite moments of all orders. This is not true for example for the Brownian motion.

Proposition 1.4. *For any $\lambda \in \mathbb{H}^+$, the functions*

$$u^\nearrow(\lambda, x) = \mathbf{E}_x[\exp(-\lambda\tau_0)], \quad u^\searrow(\lambda, x) = \frac{1}{\mathbf{E}_0[\exp(-\lambda\tau_x)]} \text{ if } x < 0, \quad (1.12)$$

$$u^\nearrow(\lambda, x) = \frac{1}{\mathbf{E}_0[\exp(-\lambda\tau_x)]}, \quad u^\searrow(\lambda, x) = \mathbf{E}_x[\exp(-\lambda\tau_0)] \text{ if } x > 0 \quad (1.13)$$

are solutions to $(\lambda - A)u = 0$. For λ on the half-line \mathbb{H}^+ , $x \mapsto u^\nearrow(\lambda, x)$ is increasing from 0 to $+\infty$, $x \mapsto u^\searrow(\lambda, x)$ is decreasing from $+\infty$ to 0. Moreover, $u^\nearrow(\lambda, 0) = u^\searrow(\lambda, 0) = 1$. For $\lambda \in \mathbb{H}^+$, $|u(\lambda, x)|$ converges to 0 as $|x| \rightarrow \infty$. Besides, for any $x \in \mathbb{R}$, $\lambda \mapsto u(\lambda, x)$ and $\lambda \mapsto D_S u(\lambda, x)$ for $u = u^\nearrow, u^\searrow$ are holomorphic on \mathbb{H}^+ .

Proof. We follow the classical probabilistic construction of u^\nearrow and u^\searrow [6, 39] for $\lambda \in \mathbb{R}$, which extends to $\lambda \in \mathbb{H}^+$. In the order,

1. From the Feynman-Kac formula, u^\nearrow and u^\searrow are solutions to $(\lambda - A)u = 0$ with $u(0) = 1$.
2. Using the strong Markov property, $\mathbf{E}_x[e^{-\lambda\tau_y}] = \mathbf{E}_x[e^{-\lambda\tau_z}] \mathbf{E}_z[e^{-\lambda\tau_y}]$ for any $x \leq y \leq z$. From this, one easily checks that $\mathbf{E}_x[e^{-\lambda\tau_y}] = u(x)/u(z)$ for $u = u^\nearrow(\lambda, \cdot)$ or $u = u^\searrow(\lambda, \cdot)$. Since these functions are positive, Lemma 1.1 implies that they are holomorphic on \mathbb{H}^+ .
3. Using for a basis of solutions for $(\lambda - A)u = 0$ the functions u_1 and u_2 with $u_1(\lambda, 0) = D_S u_2(\lambda, 0) = 1$ and $D_S u_1(\lambda, 0) = u_2(\lambda, 0) = 0$, u^\nearrow may be written $u^\nearrow(\lambda, x) = u_1(\lambda, x) + \alpha(\lambda)u_2(\lambda, x)$. Since u_1 and u_2 are holomorphic from Proposition 1.3, $\alpha(\lambda)$ is meromorphic. Since $u^\nearrow(\cdot, x)$ is holomorphic on \mathbb{H}^+ from Lemma 1.1, α is necessarily holomorphic on \mathbb{H}^+ . With Proposition 1.3, $D_S u^\nearrow(\cdot, x)$ is holomorphic on \mathbb{H}^+ for any $x \in \mathbb{R}$ since α , $D_S u_1(\cdot, x)$ and $D_S u_2(\cdot, x)$ are holomorphic on \mathbb{H}^+ (actually, $\alpha(\lambda) = D_S u(\lambda, 0)$ from our choice of u_1 and u_2). The same argument holds for u^\searrow . We have then proved that $D_S u^\nearrow$ and $D_S u^\searrow$ are holomorphic on \mathbb{H}^+ .

4. For $\lambda \in \mathbb{h}^+$, it is easily checked that the functions u^{\nearrow} and u^{\searrow} defined by (1.12) and (1.13) are positive, monotone, and range respectively from 0 to ∞ and from ∞ to 0. \square

Corollary 1.1. *For $\lambda \in \mathbb{H}^+$, let u in $\mathcal{S}_a(\mathbb{R}_+, \mathbb{C})$ (resp. $\mathcal{S}_a(\mathbb{R}_-, \mathbb{C})$) be a solution to $(\lambda - A)u = 0$ on \mathbb{R}_+ (resp. on \mathbb{R}_-) which vanishes at $+\infty$ (resp. $-\infty$) and satisfies $u(0) = 0$. Then $u = 0$.*

Proof. For a continuous, bounded function f on \mathbb{R}_+ , set

$$H_\lambda f(x) \stackrel{\text{def}}{=} \mathbf{E}_x \left[\int_0^{\tau_0} e^{-\lambda s} f(X_s^a) ds \right].$$

Using the strong Markov property of X^a ,

$$H_\lambda f(x) = G_\lambda^{X^a} f(x) - u^{\searrow}(\lambda, x) G_\lambda^{X^a} f(0).$$

With the Feynman-Kac formula, $(\lambda - A)H_\lambda f(x) = f(x)$ on \mathbb{R}_+ . In addition, $H_\lambda f(0) = 0$. Finally, H_λ maps bounded functions to bounded functions. Besides, if f vanishes at infinity, then $G_\lambda^{X^a} f$ and $u^{\searrow}(\lambda, \cdot)$ vanish at infinity, so does $H_\lambda f$. This proves that H_λ is a bounded inverse of $\lambda - A$ for $\lambda \in \mathbb{H}^+$. Therefore, there exists a unique solution to $(\lambda - A)u = f$ when f is bounded, continuous on \mathbb{R}_+ , and this solution vanishes at infinity when f does. Hence, for $f = 0$, $u = 0$. \square

Remark 1.2. Proposition 1.4 and Corollary 1.1 could have been treated by a purely analytic method with the theory of limit circle and limit points that was initiated by H. Weyl [44] (See also [23, 42]). It could be proved that u^{\nearrow} and u^{\searrow} are actually meromorphic on $\mathbb{C} \setminus \overline{\mathbb{h}^-}$.

The functions u^{\nearrow} and u^{\searrow} serve to build the resolvent. For our purpose, it is important that they are holomorphic on \mathbb{H}^+ so that we could identify the resolvent kernel with another analytic expression to recover the densities through Laplace inversion (See Proposition 2.1 below).

The holomorphic property is the key to prove results such as Proposition 1.5 below on the regularity of the density (See also [21]) as well as the existence of a diffusion process. This is not our purpose here. Therefore, we took benefit of the known probabilistic representation of the resolvent to refine our knowledge on the resolvent kernel.

Corollary 1.2. *The map $\lambda \mapsto \text{Wr}[u^{\searrow}, u^{\nearrow}](\lambda, x)$ is holomorphic on \mathbb{H}^+ and vanishes only on the closed half-line $\overline{\mathbb{h}^-}$.*

Proof. First, $\text{Wr}[u^{\searrow}, u^{\nearrow}](\lambda, x)$ does not depend on x . Second, from the very definition of u^{\nearrow} and u^{\searrow} , $\text{Wr}[u^{\searrow}, u^{\nearrow}](\lambda, x) \neq 0$ for $x \in \mathbb{h}^+$ and is holomorphic in λ . Third, $\text{Wr}[u^{\searrow}, u^{\nearrow}](\lambda, x) = 0$ for some $\lambda \in \mathbb{C}$ if and only if $u^{\nearrow} = \alpha u^{\searrow}$ for $\alpha \in \mathbb{C} \setminus \{0\}$. Necessarily, $\alpha = 1$ since $u^{\nearrow}(\lambda, 0) = u^{\searrow}(\lambda, 0)$. Using an integration by parts on an interval $[a, b]$, for any $\phi, \psi \in \text{Dom}(A)$,

$$\int_a^b [\phi(x)A\psi(x) - \psi(x)A\phi(x)]m(x) dx = \frac{1}{2} \text{Wr}[\phi, \psi](b) - \frac{1}{2} \text{Wr}[\phi, \psi](a). \quad (1.14)$$

Since $u^\succ(\lambda, 0)$ is solution to $(\bar{\lambda} - A)u = 0$, and when $u^\succ(\lambda, \cdot) = u^\succ(\lambda, \cdot)$, (1.14) with $\phi = u^\succ$ and $\psi = \overline{u^\succ}$ yields

$$2i \operatorname{Im} \left(\int_a^b \lambda u^\succ(\lambda, x) \overline{u^\succ(\lambda, x)} m(x) dx \right) = 0 \text{ or } \operatorname{Im}(\lambda) = 0.$$

Therefore, $\operatorname{Wr}[u^\succ, u^\succ](\lambda, x) = 0$ implies that $\operatorname{Im}(\lambda) = 0$. On the other hand, since $u^\succ(\lambda, \cdot)$ increases whereas $u^\succ(\lambda, \cdot)$ decreases for $\lambda \in \mathbb{H}^+$ while both functions are positive implies that $\operatorname{Wr}[u^\succ, u^\succ](\lambda, x) = 0$ only on $\overline{\mathbb{H}^-}$. \square

We give two examples that are of great importance later.

Example 1.1. For $s(x) = m(x) = 1$ so that $A = \frac{1}{2}D_x^2$, $u^\succ(\lambda, x) = \exp(\sqrt{2\lambda}x)$ and $u^\succ(\lambda, x) = \exp(-\sqrt{2\lambda}x)$. They are holomorphic on $\mathbb{C} \setminus \overline{\mathbb{H}^-}$. Besides, $\operatorname{Wr}[u^\succ, u^\succ](\lambda, x) = 0$ if and only if $\lambda = 0$.

Example 1.2. For $s(x) = \exp(-\gamma x)$ and $m(x) = \exp(\gamma x)$ so that $A = \frac{1}{2}D_x^2 + \gamma D_x$,

$$\begin{aligned} u^\succ(\gamma, \lambda, x) &= \exp((-\gamma + \sqrt{\gamma^2 + 2\lambda})x) \\ \text{and } u^\succ(\gamma, \lambda, x) &= \exp((-\gamma - \sqrt{\gamma^2 + 2\lambda})x). \end{aligned} \quad (1.15)$$

They are holomorphic on $\mathbb{C} \setminus \{\lambda \in \mathbb{C} \mid \operatorname{Im}(\lambda) = 0, \operatorname{Re}(\lambda) < -|\gamma|/2\}$. Besides, $\operatorname{Wr}[u^\succ, u^\succ](\lambda, x) = \sqrt{\gamma^2 + 2\lambda}$ and $\operatorname{Wr}[u^\succ, u^\succ](\lambda, x) = 0$ if and only if $\lambda = -\gamma^2/2$.

Let us set for $\lambda \in \mathbb{H}^+$ and the density m of the speed measure M ,

$$r(\lambda, x, y) = \frac{m(y)}{\operatorname{Wr}[u^\succ, u^\succ](\lambda, y)} \begin{cases} u^\succ(\lambda, x) u^\succ(\lambda, y) & \text{if } x < y, \\ u^\succ(\lambda, x) u^\succ(\lambda, y) & \text{if } x > y. \end{cases}$$

The resolvent $R_\lambda = G_\lambda^{X^a}|_{\mathcal{C}_0(\overline{\mathbb{R}}, \mathbb{R})}$ can be extended for $\lambda \in \mathbb{H}^+$ as a bounded (by $1/\operatorname{Re}(\lambda)$) operator. The resolvent is actually $(\lambda - A)^{-1}$ for $\lambda \in \mathbb{H}^+$. From standard results (See e.g. [6, Theorem 16.75] or [12]), for any $\lambda \in \mathbb{H}^+$,

$$R_\lambda f(x) = \int_{\mathbb{R}} r(\lambda, x, y) f(y) dy, \quad f \in \mathcal{C}_0(\overline{\mathbb{R}}, \mathbb{R}).$$

Exploiting (1.10), we easily see that $r(\lambda, x, y)$ satisfies

$$D_S r(\lambda, y-, y) - D_S r(\lambda, y+, y) = 2m(y), \quad \text{for any } y \in \mathbb{R}, \lambda \in \mathbb{H}^+. \quad (1.16)$$

The density $p(t, x, y)$ of the process X^a generated by $(A, \operatorname{Dom}(A))$ or equivalently by $(L, \operatorname{Dom}(L))$ is related to $r(\lambda, x, y)$ by the Laplace transform \mathcal{L} . The next proposition is due to H.P. McKean. It is actually valid for any process with generators of type $\frac{1}{2}D_M D_S$. Proposition's proof exploits the fact that $(A, \operatorname{Dom}(A))$ is a self-adjoint process to provide the existence of a density p which is analytic¹.

¹approach, L is self-adjoint and the semi-group $(P_t)_{t>0}$ is analytic and maps $L^2(\mathbb{R})$ to $\operatorname{Dom}(L)$. The non-trivial part is the existence of a density. In [34], H.P. McKean uses a spectral representation on the kernel of the resolvent to show the existence of a density transition function and analyse its properties, in particular to show that $p(t, \cdot, y)$ belongs to $\operatorname{Dom}(A^n)$ for any $n \in \mathbb{N}$. These two approaches are parallel, one at the level of the operators, the second on their kernels. Many of the properties of p exhibited in [34] are not quoted in the subsequent literature such as [6, 22].

Proposition 1.5 (H.P. McKean [34]). *The density $p(t, x, y)$ of the stochastic process $X^{\mathbf{a}}$ for $\mathbf{a} \in \mathfrak{C}$ satisfies $p(t, x, y) = \mathcal{L}^{-1}(r(\lambda, x, y))$. In addition, for any $n \geq 1$, $\partial_t^n p$ exists and is continuous on \mathbb{P} defined by (1.2). In addition, it satisfies $\partial_t^n p = A^n p$ on \mathbb{P} and $\partial_t^n p(t, \cdot, y)$ vanishes at infinity for any $t > 0, y \in \mathbb{R}$.*

Remark 1.3. The above theorem is done with $S(x) = x$. Nevertheless we could always reduce to this case (see also the results in Section 1.5).

1.4. Diffusion with piecewise regular coefficients

We now focus on the process associated to $\mathbf{a} \in \mathfrak{P}$, that is a Feller process with piecewise regular coefficients.

Notation 1.1. For $\mathbb{K} = \mathbb{R}$ or \mathbb{C} , given $\mathbf{a} \in \mathfrak{P}$ and I an interval of \mathbb{R} , we denote by $\mathcal{A}_{\mathbf{a}}(I, \mathbb{K})$ the set of functions $u \in \mathcal{C}(I, \mathbb{K}) \cap \mathcal{C}^2(I \setminus \text{Dis}(\mathbf{a}), \mathbb{K})$ such that

$$a(x-)u'(x-) = a(x+)u'(x+), \quad \forall x \in I \cap \text{Dis}(\mathbf{a}). \quad (1.17)$$

We show how close the domains of A and L are.

Proposition 1.6. For $\mathbf{a} \in \mathfrak{P}$,

$$\text{Dom}(A) \subset \mathcal{A}_{\mathbf{a}}(\mathbb{R}, \mathbb{R}) \text{ and } \mathcal{A}_{\mathbf{a}}(\mathbb{R}, \mathbb{R}) \cap \{f \in L^2(\mathbb{R}) \mid Af \in L^2(\mathbb{R})\} \subset \text{Dom}(L).$$

Besides, the latter inclusion is dense in $\text{Dom}(L)$ with respect to the L^2 -norm.

Proof. The inclusion $\text{Dom}(A) \subset \mathcal{A}_{\mathbf{a}}(\mathbb{R}, \mathbb{R})$ is straightforward using the fact that $\text{Dom}(A) \subset \mathcal{S}_{\mathbf{a}}(\mathbb{R}, \mathbb{R})$ and the continuity of m and s out of $\text{Dis}(\mathbf{a})$.

Let $u \in \mathcal{A}_{\mathbf{a}}(\mathbb{R}, \mathbb{R})$. The set $\mathbb{R} \setminus \text{Dis}(\mathbf{a})$ may be expressed as the disjoint union of intervals of type (x_i, x_{i+1}) with $\text{Dis}(\mathbf{a}) = \{x_i\}_{i \in J}$ with $x_i < x_{i+1}, i \in J \subset \mathbb{Z}$. Thus, for $\phi \in \mathcal{C}_c^1(\mathbb{R}, \mathbb{R})$,

$$\begin{aligned} 2 \int_{\mathbb{R}} Au(x)\phi(x) \frac{dx}{\rho(x)} &= \sum_{i \in \mathbb{Z}} \int_{x_i}^{x_{i+1}} e^{-h(x)} \left(\frac{1}{s(x)} u'(x) \right)' \phi(x) dx \\ &= \sum_{i \in \mathbb{Z}} \left(\frac{1}{s(x_{i+1}-)} u'(x_{i+1}-) \phi(x_{i+1}) e^{-h(x_{i+1})} - \frac{1}{s(x_i)} u'(x_i) \phi(x_i) e^{-h(x_i)} \right) \\ &\quad - \int_{\mathbb{R}} a(x) u'(x) \phi'(x) dx + 2 \int_{\mathbb{R}} b(x) u'(x) \phi(x) \frac{dx}{\rho(x)}. \end{aligned}$$

Using the interface condition (1.17) at x_i ,

$$\int_{\mathbb{R}} (\alpha - A)u(x)\phi(x) \frac{dx}{\rho(x)} = E_{\alpha}(u, \phi), \quad \forall \phi \in \mathcal{C}_c^1(\mathbb{R}, \mathbb{R}). \quad (1.18)$$

If $u \in L^2(\mathbb{R})$ is such that $Au \in L^2(\mathbb{R})$, it follows from (1.18) that $u = (\alpha - L)^{-1}(\alpha - A)u \in \text{Dom}(L)$.

The space $\mathcal{C}_c(\mathbb{R}, \mathbb{R})$ is dense in $L^2(\mathbb{R})$. For $f \in \mathcal{C}_c(\mathbb{R}, \mathbb{R})$, $u_A = (\alpha - A)^{-1}f \in \text{Dom}(A) \subset \mathcal{A}_{\mathbf{a}}(\mathbb{R}, \mathbb{R})$ and $u_L = (\alpha - L)^{-1}f \in \text{Dom}(L)$. With (1.18), we deduce that u_A is a version of u_L . This proves that $(\alpha - A)^{-1}(\mathcal{C}_c(\mathbb{R}, \mathbb{R})) \subset \text{Dom}(A) \subset \mathcal{A}_{\mathbf{a}}(\mathbb{R}, \mathbb{R})$ is dense in $\text{Dom}(L)$ since $\mathcal{C}_c(\mathbb{R}, \mathbb{R})$ is. \square

We characterize the resolvent kernel.

Proposition 1.7. *The resolvent kernel r is such that*

1. *For any $\lambda \in \mathbb{H}^+$, $y \in \mathbb{R}$, $x \mapsto r(\lambda, x, y)$ belongs to $\mathcal{A}_{\mathbf{a}}(\mathbb{R} \setminus \{y\}, \mathbb{C})$ and vanishes at infinity.*
2. *The map $r(\lambda, \cdot, \cdot)$ is continuous from \mathbb{R}^2 to \mathbb{C} for any $\lambda \in \mathbb{H}^+$ and $r(\cdot, x, y)$ is holomorphic on \mathbb{H}^+ for any $(x, y) \in \mathbb{R}^2$.*
3. *For any $y \notin \text{Dis}(\mathbf{a})$,*

$$D_x r(\lambda, y-, y) - D_x r(\lambda, y+, y) = \frac{2}{a(y)\rho(y)}. \quad (1.19)$$

Besides, if for some $\lambda \in \mathbb{H}^+$ and $y \in \mathbb{R}$, $g \in \mathcal{A}_{\mathbf{a}}(\mathbb{R} \setminus \{y\}, \mathbb{C})$ vanishes at infinity and satisfies $(\lambda - A)g(x) = 0$ for $x \neq y$ and (1.19) at y , then $g(x) = r(\lambda, x, y)$ for any $x \in \mathbb{R}$.

Proof. The direct part is an immediate consequence of Proposition 1.6 and (1.16).

Concerning the uniqueness, for any $y \notin \text{Dis}(\mathbf{a})$, the difference $u(x) = g(x) - r(\lambda, x, y)$ is solution to $(\lambda - A)u(x) = 0$ for any $x \notin \text{Dis}(\mathbf{a}) \cup \{y\}$ and vanishes at infinity. Because both g and $r(\lambda, \cdot, y)$ satisfy (1.19), u belongs to $\mathcal{A}_{\mathbf{a}}(\mathbb{R}, \mathbb{C})$. It also satisfies $(\lambda - A)u(x) = 0$ for $x \notin \text{Dis}(\mathbf{a})$. Since $\text{Dis}(\mathbf{a})$ is discrete and due to (1.10), $(\lambda - A)u(x) = 0$ for every $x \in \mathbb{R}$. As R_λ is a bounded operator for $\lambda \in \mathbb{H}^+$, u vanishes everywhere because it vanishes at infinity. \square

Using Proposition 1.5, we weaken the conditions of applications of Theorem 3.1 in [33].

Corollary 1.3. *If $f \in L^\infty(\mathbb{R})$, let us set*

$$u(t, x) = \int_{\mathbb{R}} p(t, x, y) f(y) dy = \mathbf{E}_x[f(X_t^{\mathbf{a}})].$$

For each $t > 0$, $u(t, x) \in \mathcal{A}_{\mathbf{a}}(\mathbb{R})$. Hence, it is a classical solution to $\partial_t u(t, x) = Lu(t, x)$ for $t > 0$, $x \in \mathbb{R} \setminus \text{Dis}(\mathbf{a})$ satisfying the transmission condition (1.17) at any point of $\text{Dis}(\mathbf{a})$. If in addition, $f \in L^2(\mathbb{R})$, then $u(t, x)$ is a variational solution in $L^2([0, T], H^1(\mathbb{R})) \cap C([0, T], L^2(\mathbb{R}))$ to $\partial_t u(t, x) = Lu(t, x)$.

1.5. Stability under space transforms

Let us consider a continuous, increasing one-to-one function ϕ and define for every $f : \mathbb{R} \rightarrow \mathbb{R}$ the transform $\phi_* f = f \circ \phi^{-1}$.

For a continuous increasing function g , it is easily checked that $D_{\phi_* g}(\phi_* f) = \phi_*(D_g f)$ when $D_g f$ or $D_{\phi_* g}(\phi_* f)$ exist and are continuous.

Now let us introduce the operator $A^\phi = \frac{1}{2} D_{\phi_* M} D_{\phi_* S}$ associated to the scale function $\phi_* S$ and the speed measure $\phi_* M$.

Clearly, $f \in \text{Dom}(A)$ if and only if $\phi_* f \in \text{Dom}(A^\phi)$ since

$$A^\phi(\phi_* f) = \frac{1}{2} D_{\phi_* M} D_{\phi_* S}(\phi_* f) = \frac{1}{2} \phi_*(D_M D_S f) = \phi_*(A f). \quad (1.20)$$

Besides, (f, F) satisfies $(\lambda - A)f = F$ if and only if (ϕ_*f, ϕ_*F) satisfies $(\lambda - A^\phi)\phi_*f = \phi_*F$.

By an identification of the domains, we easily found the infinitesimal generator of $\phi(X)$.

Proposition 1.8. *Let X be the process with infinitesimal generator A . Then $X^\phi = \phi(X)$ has infinitesimal generator A^ϕ .*

As a result, a large class of transforms keep the family $\{X^{\mathbf{a}}\}_{\mathbf{a} \in \mathfrak{C}}$ invariant.

Proposition 1.9. *Assume that ϕ is continuous, locally absolutely continuous on \mathbb{R} and that for some $\eta > 0$, $\eta < \phi'(x) < \eta^{-1}$ for almost every $x \in \mathbb{R}$. Let us set $\phi_\#g(x) = g \circ \phi^{-1} \cdot \phi' \circ \phi^{-1}$. Then for $\mathbf{a} = (a, \rho, b) \in \mathfrak{C}$, $\phi_\#\mathbf{a} \in \mathfrak{C}$ (the constants λ and Λ may be changed) and $\phi(X^{\mathbf{a}}) = X^{\phi_\#\mathbf{a}}$ where $\phi_\#\mathbf{a} = (\phi_\#a, \phi_\#\rho, \phi_\#b)$.*

Remark 1.4. This choice is such that $\widehat{g}(x)\nabla(\phi_*f(x)) = \phi_*(g\nabla f)(x)$ for almost every $x \in \mathbb{R}$.

Proof. From the conditions on ϕ , $\lim_{x \rightarrow \pm\infty} \phi(x) = \pm\infty$ so that ϕ is one-to-one. Moreover, we have seen in Proposition 1.8 that the scale function and the speed measure of X^ϕ are ϕ_*S and ϕ_*M . From a change of variable, ϕ_*S and ϕ_*M are indeed identified with the scale function and speed measure of $\phi_\#\mathbf{a}$. Thus, the result follows from Proposition 1.2. \square

Corollary 1.4. *If ϕ is continuous and piecewise of class \mathcal{C}^2 , where the points of discontinuity of ϕ' have no cluster points (hence absolutely continuous), and $0 < \eta < \phi'(x) < \eta^{-1}$ for almost every x . If $\mathbf{a} \in \mathfrak{P}$, then $\phi_\#\mathbf{a} \in \mathfrak{P}$.*

Remark 1.5. We could have also prove the Proposition 1.9 using the Itô-Tanaka formula as in [8, 11].

Remark 1.6. Using a well chosen transform is actually the key for reducing the problem to a simpler one. It was heavily used with different transforms in [28, 29] and in [32, 33].

The case we consider in the sequel is the following: if $\mathbf{a} \in \mathfrak{P}$ is piecewise continuous, and ϕ is piecewise linear, then \mathbf{a}^ϕ is also piecewise continuous.

1.6. Stochastic differential equations

Here we prove that a diffusion $X^{\mathbf{a}}$ associated to $\mathbf{a} \in \mathfrak{P}$ is the unique strong solution to a SDE involving local time. For this purpose, we introduce

$$\mathfrak{G} = \left\{ (\sigma, \nu) \left| \begin{array}{l} \sigma \text{ is of bounded variation over } \mathbb{R} \text{ and } \lambda \leq \sigma(x) \leq \Lambda, \forall x \in \mathbb{R} \\ \nu \text{ is a finite measure over } \mathbb{R} \text{ with } |\nu(\{x\})| < 1, \forall x \in \mathbb{R} \end{array} \right. \right\}.$$

The measure ν is associated to a function s of bounded variation through

$$\nu(dx) = \frac{s'(dx)}{s(x) + s(x-)}$$

where $s'(dx)$ is the measure associated to s .

Now let us consider the SDE given by

$$X_t = x + \int_0^t \sigma(X_s) dB_s + \int_{\mathbb{R}} \nu(dy) L_t^y(X) \quad (1.21)$$

where B is a Brownian motion and $L^x(X)$ the symmetric local time of X at x .

From [25, Theorem 2.3], this SDE has a unique strong solution. Now using [25, Proposition 2.2], we easily deduced that X is a diffusion process with scale function $S = \int s$ and speed measure with density $m(dx) = s(x)^{-1} \sigma^{-2}(x) dx$.

Considering $\mathbf{a} = (a, \rho, b) \in \mathfrak{A}$, the expressions (1.7)-(1.8) for the scale function and the speed measure leads us to associate to \mathbf{a} the function $\sigma(x) = \sqrt{a(x)\rho(x)}$ as well as the measure

$$\nu(dx) = \frac{b(x)}{a(x)\rho(x)} dx + \frac{a'(x)}{2a(x)} dx + \sum_{x \in \text{Dis}(\mathbf{a})} \frac{a(x+) - a(x-)}{a(x+) + a(x-)}, \quad (1.22)$$

where $a'(x)$ is the derivative of a on the intervals on which a is \mathcal{C}^1 and $a' = 0$ on the set of discontinuities of a of zero Lebesgue measure. Because ν is not necessarily a finite measure over \mathbb{R} , σ and ν do not necessarily belong to \mathfrak{G} . However, using a localization argument, there is no problem to set $(a, \rho, b) = (1, 1, 0)$ outside a given compact K so that ν becomes a finite measure over \mathbb{R} and then to let K growing.

Since a one-dimensional diffusion process is uniquely characterized by its scale function and speed measure, $X^{\mathbf{a}}$ is a semi-martingale solution to (1.21).

Proposition 1.10. *When $\mathbf{a} \in \mathfrak{A}$, $X^{\mathbf{a}}$ is the unique strong solution to (1.21) with ν given by (1.22) and $\sigma = \sqrt{a\rho}$.*

Remark 1.7. For $\mathbf{a} = (a, \rho, b)$ and $\kappa > 0$, let us set $\mathbf{a}_\kappa = (\kappa a, \kappa^{-1} \rho, b)$. With (1.22), it is easily seen that $X^{\mathbf{a}}$ and $X^{\mathbf{a}_\kappa}$ are solutions to the same SDE and equal in distribution.

For $\mathbf{a} \in \mathfrak{C}$, the diffusion $X^{\mathbf{a}}$ has been characterized as a diffusion with infinitesimal generator $(L, \text{Dom}(L))$ obtained through the quadratic form $(E_\alpha, \mathbb{H}^1(\mathbb{R}))$ as well as a diffusion $(A, \text{Dom}(A))$ with $A = \frac{1}{2} D_M D_S$ with (S, M) given by (1.7)-(1.8).

Proposition 1.11. *Let σ be piecewise \mathcal{C}^1 and ν be a finite measure of type*

$$\nu(dx) = d(x) dx + \sum_{i \in \mathbb{Z}} \alpha_{x_i} \delta_{x_i}, \quad |\alpha_i| \leq 1 - \epsilon, \quad i \in J \subset \mathbb{Z},$$

for an increasing family $\{x_i\}_{i \in J}$ with no cluster points, some $\epsilon > 0$, and a bounded function d . Then there exists $\mathbf{a} = (a, \rho, b) \in \mathfrak{A}$ such that $X^{\mathbf{a}}$ is solution to (1.21).

Proof. Select an interval $[x_i, x_{i+1}]$. Set $a_i(x) = \beta_i = 1$, $\rho_i(x) = \sigma(x)$ and $b(x) = d(x)a_i(x)\rho_i(x)$ for $x \in (x_i, x_{i+1})$. Now, at the endpoints x_{i+1} , find β_{i+1} such that

$$\frac{\beta_{i+1} - \beta_i}{\beta_{i+1} + \beta_i} = \alpha_i.$$

Set $a_{i+1}(x) = \beta_i$, $\rho_i(x) = \sigma(x)/\beta_i$ and $b(x) = d(x)a_i(x)\rho_i(x)$ for $x \in (x_{i+1}, x_{i+2})$. Hence, we construct (a, ρ, b) iteratively on the intervals (x_j, x_{j+1}) for $j > i$. The same construction could be performed for $j < i$ by going downward instead of upward. It is easily checked that $\mathbf{a} = (a, \rho, b) \in \mathfrak{A}$ with $\text{Dis}(\mathbf{a}) = \{x_i\}_{i \in J}$. \square

2. Resolvent in the presence of one discontinuity

2.1. A computation method

We now assume that the coefficients $\mathbf{a} = (a, \rho, b) \in \mathfrak{P}$ are discontinuous only at one point, say 0. We fix $y \neq 0$ and $\lambda \in \mathbb{C}$.

We denote by I the intervals $(-\infty, 0 \wedge y)$, $(0 \wedge y, y \vee 0)$ and $(y \vee 0, +\infty)$ and introduce the operator A_I whose coefficients \mathbf{a}_I are such that $\mathbf{a}_I = \mathbf{a}$ on I .

Let u_I^{\leftarrow} and u_I^{\rightarrow} be the two functions on $\mathbb{C} \times \mathbb{R}$ given by Proposition 1.4 for each A_I . When $y > 0$, we set

$$\begin{aligned} q_1(\lambda, x) &= u_{(-\infty, 0)}^{\leftarrow}(\lambda, x), \quad q_2(\lambda, x) = u_{(0, y)}^{\rightarrow}(\lambda, x), \\ q_3(\lambda, x) &= u_{(0, y)}^{\leftarrow}(\lambda, x) \text{ and } q_4(\lambda, x) = u_{(y, \infty)}^{\rightarrow}(\lambda, x), \end{aligned}$$

whereas, for $y < 0$,

$$\begin{aligned} q_1(\lambda, x) &= u_{(-\infty, y)}^{\leftarrow}(\lambda, x), \quad q_2(\lambda, x) = u_{(y, 0)}^{\rightarrow}(\lambda, x), \\ q_3(\lambda, x) &= u_{(y, 0)}^{\leftarrow}(\lambda, x) \text{ and } q_4(\lambda, x) = u_{(0, \infty)}^{\rightarrow}(\lambda, x). \end{aligned}$$

Proposition 2.1. *When $\mathbf{a} \in \mathfrak{P}$ is only discontinuous at 0, for each $y \in \mathbb{R} \setminus \{0\}$, such that for any $\lambda \in \mathbb{H}^+$ and any $x \in \mathbb{R}$,*

$$\begin{aligned} r(\lambda, x, y) &= k_1(\lambda, y) q_1(\lambda, x) \mathbb{1}(x < 0) + k_2(\lambda, y) q_2(\lambda, x) \mathbb{1}(x \in (0, y)) \\ &\quad + k_3(\lambda, y) q_3(\lambda, x) \mathbb{1}(x \in (0, y)) + k_4(\lambda, y) q_4(\lambda, x) \mathbb{1}(x \geq y), \end{aligned} \quad (2.1)$$

where the function $k = (k_1, k_2, k_3, k_4)$ is the solution

$$M(\lambda, y)k(\lambda, y)^T = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T \quad (2.2)$$

where for $y > 0$,

$$\begin{aligned} &M(\lambda, y) \\ &= \begin{bmatrix} q_1(\lambda, 0) & -q_2(\lambda, 0) & -q_3(\lambda, 0) & 0 \\ a(0-)'q_1'(\lambda, 0) & -a(0+)q_2'(\lambda, 0) & -a(0+)q_3'(\lambda, 0) & 0 \\ 0 & q_2(\lambda, y) & q_3(\lambda, y) & -q_4(\lambda, y) \\ 0 & \frac{\rho(y)a(y)}{2}q_2'(\lambda, y) & \frac{\rho(y)a(y)}{2}q_3'(\lambda, y) & -\frac{\rho(y)a(y)}{2}q_4'(\lambda, y) \end{bmatrix} \end{aligned}$$

while for $y < 0$,

$$\begin{aligned} &M(\lambda, y) \\ &= \begin{bmatrix} 0 & q_2(\lambda, 0) & q_3(\lambda, 0) & -q_4(\lambda, 0) \\ 0 & a(0-)q_2'(\lambda, 0) & a(0-)q_3'(\lambda, 0) & a(0+)q_4'(\lambda, 0) \\ q_1(\lambda, y) & -q_2(\lambda, y) & -q_3(\lambda, y) & 0 \\ \frac{\rho(y)a(y)}{2}q_1'(\lambda, y) & -\frac{\rho(y)a(y)}{2}q_2'(\lambda, y) & -\frac{\rho(y)a(y)}{2}q_3'(\lambda, y) & 0 \end{bmatrix}. \end{aligned}$$

Proof. When $\lambda \in \mathbb{C}$, any solution to $(\lambda - A)u = 0$ on some interval I should be sought as a linear combination of $u^{\nearrow}(\lambda, \cdot)$ and $u^{\searrow}(\lambda, \cdot)$ provided that $\text{Wr}[u^{\searrow}, u^{\nearrow}](\lambda) \neq 0$. This is the case for $\lambda \in \mathbb{H}^+$ (See Corollary 1.2). Our choice of $q_i(\lambda, \cdot)$ in the decomposition 2.1 ensures the correct vanishing behaviour at infinity for $\lambda \in \mathbb{H}^+$.

The system (2.2) is an algebraic transcription of the compatibility condition at 0 given by (1.17) and y given by (1.19) in Proposition 1.7. Hence, with any solution $k(\lambda, y)$ to (2.2), the right-hand-side of (2.1) is indeed equal to $r(\lambda, x, y)$ for any $\lambda \in \mathbb{H}^+$.

When $M(\lambda, y)$ is invertible for $y \in \mathbb{R}$ and λ in a domain \mathbb{H}^+ then $\lambda \mapsto M(\lambda, y)^{-1}$ is holomorphic on \mathbb{H}^+ since it is composed of sums, products and ratios of the terms of $M(\cdot, y)$ which are themselves holomorphic on \mathbb{H}^+ .

Let us prove now that $M(\lambda, y)$ is invertible for $\lambda \in \mathbb{H}^+$. Since $q_2(\lambda, y) = q_4(\lambda, y)$ and $q'_2(\lambda, y) = q'_4(\lambda, y)$ for any $\lambda \in \mathbb{C}$, the determinant of $M(\lambda, y)$ is for $y > 0$

$$\det M(\lambda, y) = \frac{\rho(y) \exp(-h(y))}{2} \text{Wr}[q_2, q_3](\lambda, y) \times \{a(0-)q'_1(\lambda, 0)q_2(\lambda, 0) - a(0+)q_1(\lambda, 0)q'_2(\lambda, 0)\}$$

where $\text{Wr}[q_2, q_3](\lambda, \cdot)$ denotes the Wronskian of $q_2(\lambda, \cdot)$ and $q_3(\lambda, \cdot)$ in x . The above equality follows from the relation $a(x)D_x = \exp(-h(x))D_S$.

From Corollary 1.2, $\text{Wr}[q_2, q_3](\lambda, y) \neq 0$ for any $\lambda \in \mathbb{H}^+$.

Let us consider the solutions u^{\nearrow} and u^{\searrow} associated to the coefficients \mathbf{a} given by Proposition 1.4. From our choice of q_1 and q_2 , $q_1(\lambda, 0) = q_2(\lambda, 0) = u^{\nearrow}(\lambda, 0) = u^{\searrow}(\lambda, 0) = 1$ and the functions q_1 and u^{\nearrow} vanish at $-\infty$ whereas q_2 and u^{\searrow} vanish at $+\infty$. With Corollary 1.1, then $u^{\nearrow}(\lambda, x) = q_1(\lambda, x)$ for $x \leq 0$ and $u^{\searrow}(\lambda, x) = q_2(\lambda, x)$ for $x \geq 0$. Thus,

$$a(0-)q'_1(\lambda, 0)q_2(\lambda, 0) - a(0+)q_1(\lambda, 0)q'_2(\lambda, 0) = \exp(-h(0)) \text{Wr}[u^{\searrow}, u^{\nearrow}](\lambda, 0).$$

This quantity cannot vanish on \mathbb{H}^+ according to Corollary 1.2. Thus, $\det M(\lambda, y) \neq 0$ for $y > 0$, $\lambda \in \mathbb{H}^+$ and $M(\lambda, y)$ is holomorphic on \mathbb{H}^+ . The case $y < 0$ is similar. \square

Remark 2.1. Of course, this approach is also feasible when the state space is bounded, one should consider up to 6 functions with the appropriate boundary conditions. This is useful for example to provide formulae related to the first exit time or first passage time as in [26, 2]. One could also use an expansion relying on eigenvalues and eigenfunctions. For practical purposes, especially for numerical simulation which was our original aims, a large number of eigenfunctions should be considered (See [19]). Actually, the eigenvalue formulation is best for estimating the density for large times.

2.2. Skew Brownian motion with a piecewise constant drift

We now apply the computation method developed in the previous section to skew diffusions with a piecewise constant drift. Therefore, let us consider the operator

$$L = \frac{\rho(x)}{2} \frac{d}{dx} \left(a(x) \frac{d}{dx} \right) + b(x) \frac{d}{dx} \quad (2.3)$$

with for some $\beta \in (0, 1)$, $\mathbf{a} = (a, \rho, b) = \widehat{\mathbf{a}}(\beta, b_+, b_-)$ defined in Table 1.

	for $x \geq 0$	for $x < 0$
$a(x)$	β	$1 - \beta$
$\rho(x)$	β^{-1}	$(1 - \beta)^{-1}$
$b(x)$	b_+	b_-

TABLE 1
Coefficients $\widehat{\mathbf{a}}(\beta, b_+, b_-) \in \mathfrak{P}$.

Remark 2.2. Out of 0, $\rho(x)a(x) = 1$. Thus, the associated diffusion process $X^{\mathbf{a}}$ is solution to

$$X_t^{\mathbf{a}} = x + B_t + (2\beta - 1)L_t^0(X^{\mathbf{a}}) + \int_0^t b(X_s^{\mathbf{a}}) ds \text{ for } t \geq 0. \quad (2.4)$$

For some constant $\gamma \in \mathbb{R}$, we consider the two functions u^{\nearrow} and u^{\searrow} given by (1.15) in Example 1.2 and remark that the Green function of $L_\gamma = \frac{1}{2}D_x^2 + \gamma D_x$ has a density with respect to the Lebesgue measure containing these functions and which is defined by

$$g(\gamma, \lambda, x, y) = \frac{1}{\sqrt{\gamma^2 + 2\lambda}} \begin{cases} e^{(-\gamma + \sqrt{\gamma^2 + 2\lambda})(x-y)} & \text{if } x < y, \\ e^{(-\gamma - \sqrt{\gamma^2 + 2\lambda})(x-y)} & \text{if } x \geq y. \end{cases} \quad (2.5)$$

Proposition 2.2. The resolvent kernel of L with coefficients $\widehat{\mathbf{a}}(\beta, b_+, b_-)$ is for $y \geq 0$ and $\lambda \in \mathbb{C} \setminus (-\infty, 0)$,

$$\begin{aligned} r(\lambda, x, y) = & g(b_+, \lambda, x, y)\mathbb{1}(x \geq 0) + A^{-+}(\lambda, y)g(b_-, \lambda, x, y)\mathbb{1}(x < 0) \\ & + A^{++}(\lambda, y)g(b_+, \lambda, x, -y)\mathbb{1}(x \geq 0) \end{aligned} \quad (2.6)$$

and for $y < 0$,

$$\begin{aligned} r(\lambda, x, y) = & g(b_-, \lambda, x, y)\mathbb{1}(x < 0) + A^{--}(\lambda, y)g(b_-, \lambda, x, -y)\mathbb{1}(x < 0) \\ & + A^{+-}(\lambda, y)g(b_+, \lambda, x, y)\mathbb{1}(x \geq 0) \end{aligned} \quad (2.7)$$

with

$$\begin{aligned} A^{-+}(\lambda, y) &= \Theta(\lambda, b_+, b_-)^{-1} 2\beta \sqrt{b_-^2 + 2\lambda} e^{(b_+ - b_- + \sqrt{b_-^2 + 2\lambda} - \sqrt{b_+^2 + 2\lambda})y}, \\ A^{++}(\lambda, y) &= \Theta(\lambda, b_+, b_-)^{-1} (\beta(\sqrt{b_+^2 + 2\lambda} - b_+) - (1 - \beta)(\sqrt{b_-^2 + 2\lambda} - b_-)) e^{2b_+y}, \\ A^{--}(\lambda, y) &= \Theta(\lambda, b_+, b_-)^{-1} (-\beta(\sqrt{b_+^2 + 2\lambda} + b_+) + (1 - \beta)(\sqrt{b_-^2 + 2\lambda} + b_-)) e^{2b_-y}, \\ A^{+-}(\lambda, y) &= \Theta(\lambda, b_+, b_-)^{-1} 2(\beta - 1)\sqrt{b_+^2 + 2\lambda} e^{(b_- - b_+ + \sqrt{b_-^2 + 2\lambda} - \sqrt{b_+^2 + 2\lambda})y}, \end{aligned}$$

where the common denominator is

$$\Theta(\lambda, b_+, b_-) = \beta(\sqrt{b_+^2 + 2\lambda} + b_+) + (1 - \beta)(\sqrt{b_-^2 + 2\lambda} - b_-). \quad (2.8)$$

Remark 2.3. Using the expressions of $g(b_{\pm}, \lambda, x, y)$ and $A^{\pm, \pm}(\lambda, y)$, the inverse transform \mathcal{L}^{-1} can be applied to each terms of the decompositions (2.6) and (2.7), as they are holomorphic whenever $\Theta(\lambda, b_+, b_-) \neq 0$ and $2\lambda \neq -b_{\pm}^2$ (only for $\lambda \in \mathbb{R}_-$) and decreasing to 0 as $|\lambda| \rightarrow \infty$.

Remark 2.4. When $x = 0$, (2.6) and (2.7) simplify to

$$r(\lambda, 0, y) = \begin{cases} 2\beta\Theta(\lambda, b_+, b_-)^{-1}g(b_+, \lambda, 0, y) & \text{for } y > 0, \\ 2(\beta - 1)\Theta(\lambda, b_+, b_-)^{-1}g(b_-, \lambda, 0, y) & \text{for } y < 0, \end{cases} \quad (2.9)$$

where $\Theta(\lambda, b_+, b_-)$ is given by (2.8).

Proof. Since $u^{\nearrow}(\gamma, \lambda, 0) = u^{\searrow}(\gamma, \lambda, 0) = 1$, we set

$$\begin{aligned} q_1(\lambda, x) &= v^{\nearrow}(b_-, \lambda, x), \quad q_3(\lambda, x) = v^{\nearrow}(b(y), \lambda, x), \\ q_2(\lambda, x) &= v^{\searrow}(b(y), \lambda, x) \text{ and } q_4(\lambda, x) = v^{\searrow}(b_+, \lambda, x). \end{aligned}$$

The solution of $M(\lambda, y)k(\lambda, y) = [0 \ 0 \ 0 \ 1]^T$ is for $y \geq 0$,

$$\begin{cases} k_1(\lambda, y) = \frac{-2\beta}{(\beta-1)(-b_+ + \sqrt{b_-^2 + 2\lambda}) + \beta(-b_+ - \sqrt{b_+^2 + 2\lambda})} e^{-(b_+ + \sqrt{b_+^2 + 2\lambda})y}, \\ k_2(\lambda, y) = \frac{(1-\beta)(-b_+ + \sqrt{b_-^2 + 2\lambda}) - \beta(-b_+ + \sqrt{b_+^2 + 2\lambda})}{\sqrt{b_+^2 + 2\lambda}[(\beta-1)(-b_+ + \sqrt{b_-^2 + 2\lambda}) + \beta(-b_+ - \sqrt{b_+^2 + 2\lambda})]} e^{-(b_+ + \sqrt{b_+^2 + 2\lambda})y}, \\ k_3(\lambda, y) = \frac{1}{\sqrt{b_+^2 + 2\lambda}} e^{-(b_+ + \sqrt{b_+^2 + 2\lambda})y}, \\ k_4(\lambda, y) = \frac{(1-\beta)(-b_+ + \sqrt{b_-^2 + 2\lambda}) - \beta(-b_+ + \sqrt{b_+^2 + 2\lambda})}{\sqrt{b_+^2 + 2\lambda}[(\beta-1)(-b_+ + \sqrt{b_-^2 + 2\lambda}) + \beta(-b_+ - \sqrt{b_+^2 + 2\lambda})]} e^{-(b_+ + \sqrt{b_+^2 + 2\lambda})y} \\ + \frac{1}{\sqrt{b_+^2 + 2\lambda}} e^{-(b_+ - \sqrt{b_+^2 + 2\lambda})y}, \end{cases}$$

while for $y < 0$,

$$\begin{cases} k_1(\lambda, y) = \frac{(1-\beta)(-b_- - \sqrt{b_-^2 + 2\lambda}) - \beta(-b_+ - \sqrt{b_+^2 + 2\lambda})}{\sqrt{b_-^2 + 2\lambda}[(\beta-1)(-b_- + \sqrt{b_-^2 + 2\lambda}) + \beta(-b_+ - \sqrt{b_+^2 + 2\lambda})]} e^{-(-b_- - \sqrt{b_-^2 + 2\lambda})y} \\ + \frac{1}{\sqrt{b_-^2 + 2\lambda}} e^{-(-b_- + \sqrt{b_-^2 + 2\lambda})y}, \\ k_2(\lambda, y) = \frac{1}{\sqrt{b_-^2 + 2\lambda}} e^{-(-b_- - \sqrt{b_-^2 + 2\lambda})y}, \\ k_3(\lambda, y) = \frac{(1-\beta)(-b_- - \sqrt{b_-^2 + 2\lambda}) - \beta(-b_+ - \sqrt{b_+^2 + 2\lambda})}{\sqrt{b_-^2 + 2\lambda}[(\beta-1)(-b_- + \sqrt{b_-^2 + 2\lambda}) + \beta(-b_+ - \sqrt{b_+^2 + 2\lambda})]} e^{-(-b_- - \sqrt{b_-^2 + 2\lambda})y} \\ k_4(\lambda, y) = \frac{2(\beta-1)}{(\beta-1)(-b_- + \sqrt{b_-^2 + 2\lambda}) + \beta(-b_+ - \sqrt{b_+^2 + 2\lambda})} e^{-(-b_- - \sqrt{b_-^2 + 2\lambda})y}. \end{cases}$$

Finally, it remains to plug the coefficients at their rightful places to obtain the solutions in each case. After some simplifications, we get (2.6) and (2.7). \square

2.3. Skew diffusion with piecewise constant coefficients

Let us now consider the diffusion whose infinitesimal generator is $(L, \text{Dom}(L))$ given by (2.3) and coefficients $\mathbf{a} = (a, \rho, b)$ are given in Table 2.

	for $x \geq 0$	for $x < 0$
$a(x)$	a_+	a_-
$\rho(x)$	ρ_+	ρ_-
$b(x)$	b_+	b_-

TABLE 2
Coefficients $\mathbf{a} = (a, \rho, b)$ constants on \mathbb{R}_+ and \mathbb{R}_- .

In order to set up the diffusion coefficient $\sqrt{a\rho}$ to 1, the one-to-one transform $\phi(x) = x/\sqrt{a(x)\rho(x)}$ is convenient. If $X^{\mathbf{a}}$ is the diffusion process associated to $\mathbf{a} = (a, \rho, b)$, then $X^{\phi_{\#}\mathbf{a}}$ is the diffusion process associated to

$$\phi_{\#}\mathbf{a} = \left(\frac{\sqrt{a}}{\sqrt{\rho}}, \frac{\sqrt{\rho}}{\sqrt{a}}, \frac{b}{\sqrt{a\rho}} \right).$$

With Remark 1.7, we can consider $\kappa = (\sqrt{a_+/\rho_+} + \sqrt{a_-/\rho_-})^{-1}$ and see that $X^{\phi_{\#}\mathbf{a}}$ has the same distribution as the SBM with parameter $\beta = \kappa\sqrt{a_+/\rho_+} \in (0, 1)$ and piecewise constant drift $\bar{b}(x) = b(x)/\sqrt{a(x)\rho(x)}$, that is $\phi_{\#}\mathbf{a} = \hat{\mathbf{a}}(\beta, b_+/\sqrt{a_+\rho_+}, b_-/\sqrt{a_-\rho_-})$. Hence, the computations of Section 2.1 could be applied to get the resolvent of $X^{\phi_{\#}\mathbf{a}}$. Now using a change of variable,

$$r_{\mathbf{a}}(\lambda, x, y) = \frac{r_{\phi_{\#}\mathbf{a}}(\lambda, \phi(x), \phi(y))}{\sqrt{a(y)\rho(y)}},$$

where $r_{\mathbf{a}}$ (resp. $r_{\phi_{\#}\mathbf{a}}$) is the resolvent kernel of $X^{\mathbf{a}}$ (resp. $X^{\phi_{\#}\mathbf{a}}$).

3. Some explicit expressions for the density

We introduce the functions

$$\begin{aligned} H(y) &= \mathbb{1}(y \geq 0), \quad H^-(y) = \mathbb{1}(y < 0) = \mathbb{1}(y \in \mathbb{R}) - H(y), \\ \text{sgn}(y) &= \mathbb{1}(y \geq 0) - \mathbb{1}(y < 0), \\ \text{erf}(x) &= \frac{2}{\sqrt{\pi}} \int_0^x e^{-v^2} dv \quad \text{and} \quad \text{erfc}(x) = 1 - \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_x^{+\infty} e^{-v^2} dv. \end{aligned}$$

3.1. The Skew Brownian Motion

The SBM of parameter β is given by the choice of the coefficients $\hat{\mathbf{a}}(\beta, 0, 0)$. Thus,

$$A^{-+}(\lambda, y) = 2\beta, \quad A^{++}(\lambda, y) = 2\beta - 1, \quad A^{- -}(\lambda, y) = 1 - 2\beta, \quad A^{--}(\lambda, y) = 1 - 2\beta.$$

To recover the density of the SBM, we need to invert the Green function of the Laplace operator since none of the $A^{\pm\pm}(\lambda, y)$ depend on λ . It is well known that it gives the Gaussian kernel.

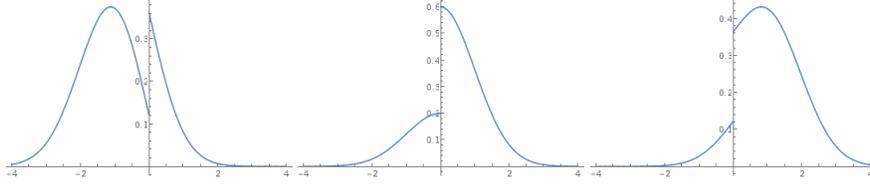


FIG 1. The density $p(t, x, y)$ of the SBM for $\beta = \frac{3}{4}$ at time $t = 1$ for the initial positions: $x = -1$, $x = 0$ and $x = 1$ (from left to right).

More precisely, from [1],

$$\mathcal{L}^{-1}\left(\frac{1}{\sqrt{\lambda}} e^{-k\sqrt{\lambda}}\right) = \frac{1}{\sqrt{\pi t}} e^{-\frac{k^2}{4t}}, \quad k \geq 0, \quad (3.1)$$

and

$$\mathcal{L}^{-1}(f(c\lambda + d)) = \frac{1}{c} e^{-\frac{d}{c}t} \mathcal{L}^{-1}(f)\left(\frac{t}{c}\right), \quad c > 0. \quad (3.2)$$

Thus, after an application of (3.1)-(3.2) to (2.6) and (2.7) as well as some rewriting

$$p(t, x, y) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(y-x)^2}{2t}} + \operatorname{sgn}(y)(2\beta - 1) \frac{1}{\sqrt{2\pi t}} e^{-\frac{(|y|+|x|)^2}{2t}}.$$

This density was obtained in [43] through a probabilistic argument. Some plots are given in Figure 1.

The cumulative distribution function $P : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \mapsto (0, 1)$ of p is

$$P(t, x, y) = \frac{1}{2} \left[1 + \operatorname{erf}\left(\frac{y-x}{\sqrt{2t}}\right) \right] - \frac{2\beta-1}{2} \left[1 - \operatorname{sgn}(y) \operatorname{erf}\left(\frac{y+\operatorname{sgn}(y)|x|}{\sqrt{2t}}\right) \right].$$

3.2. The Skew Brownian Motion with a constant drift

The density of the SBM with a constant drift, which corresponds to $\widehat{\mathbf{a}}(\beta, b, b)$ for some $b \neq 0$ was computed in [3, 4, 10, 16] with different probabilistic arguments.

If b is constant, then in Proposition 2.2,

$$\begin{aligned} A^{-+}(\lambda, y) &= \frac{2\beta\sqrt{b^2+2\lambda}}{\sqrt{b^2+2\lambda}+b(2\beta-1)}, \quad A^{++}(\lambda, y) = e^{2by} \frac{(2\beta-1)(\sqrt{b^2+2\lambda}-b)}{\sqrt{b^2+2\lambda}+b(2\beta-1)}, \\ A^{--}(\lambda, y) &= e^{2by} \frac{(1-2\beta)(\sqrt{b^2+2\lambda}+b)}{\sqrt{b^2+2\lambda}+b(2\beta-1)}, \\ \text{and } A^{+-}(\lambda, y) &= \frac{2(\beta-1)\sqrt{b^2+2\lambda}}{\sqrt{b^2+2\lambda}+b(2\beta-1)}. \end{aligned}$$

Let us compute the Laplace inverse of $A^{\pm\pm}(\lambda, y)g(b, \lambda, x, y)$. From [1],

$$\mathcal{L}^{-1}\left(\frac{e^{-k\sqrt{\lambda}}}{d+\sqrt{\lambda}}\right) = \frac{1}{\sqrt{\pi t}} e^{-\frac{k^2}{4t}} - de^{dk} e^{d^2t} \operatorname{erfc}\left(d\sqrt{t} + \frac{k}{2\sqrt{t}}\right), \quad k \geq 0, \quad (3.3)$$

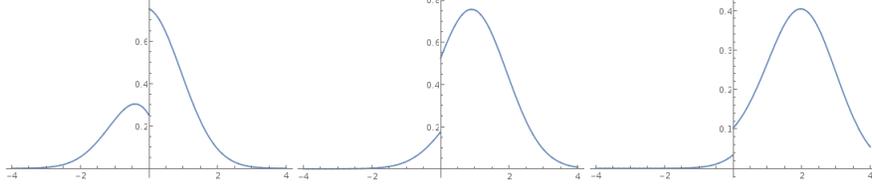


FIG 2. The density $p(t, x, y)$ of the SBM for $\beta = \frac{3}{4}$ with a constant drift $b = 2$ at time $t = 1$ for the initial positions: $x = -1$, $x = 0$ and $x = 1$ (from left to right).

and

$$\mathcal{L}^{-1} \left(\frac{e^{-k\sqrt{\lambda}}}{\sqrt{\lambda}(d + \sqrt{\lambda})} \right) = e^{dk} e^{d^2 t} \operatorname{erfc} \left(d\sqrt{t} + \frac{k}{2\sqrt{t}} \right), \quad k \geq 0. \quad (3.4)$$

Thus, after using (3.1)-(3.2) and (3.3)-(3.4) as well as some recombination (see Figure 2),

$$\begin{aligned} p(t, x, y) &= \frac{1}{\sqrt{2\pi t}} e^{-\frac{(bt - (y-x))^2}{2t}} + \operatorname{sgn}(y)(2\beta - 1) \frac{1}{\sqrt{2\pi t}} e^{-\frac{(|y|+|x|)^2}{2t}} e^{-b^2 \frac{t}{2}} e^{b(y-x)} \\ &\quad + \operatorname{sgn}(y)(H^-(y) - \beta)b(2\beta - 1)e^{b(2\beta-1)(|y|+|x|)} e^{b(y-x)} e^{2b^2\beta(\beta-1)t} \\ &\quad \times \operatorname{erfc} \left(b(2\beta - 1)\sqrt{\frac{t}{2}} + \frac{|y| + |x|}{\sqrt{2t}} \right). \end{aligned}$$

We recover the density obtained in [3, 10] up to a conversion of erfc to its probabilistic counterpart.

3.3. The Bang-Bang Skew Brownian Motion

The Bang-Bang SBM is the diffusion corresponding to the choice of the coefficients $\hat{\mathbf{a}}(\beta, b, -b)$ for some $b \neq 0$. This process was introduced in [14].

With this choice of coefficients, in Proposition 2.2,

$$\begin{aligned} A^{-+}(\lambda, y) &= e^{2by} \frac{2\beta\sqrt{b^2 + 2\lambda}}{b + \sqrt{b^2 + 2\lambda}}, \quad A^{++}(\lambda, y) = e^{2by} \frac{(2\beta - 1)\sqrt{b^2 + 2\lambda} - b}{b + \sqrt{b^2 + 2\lambda}}, \\ A^{--}(\lambda, y) &= e^{-2by} \frac{-(2\beta - 1)\sqrt{b^2 + 2\lambda} - b}{b + \sqrt{b^2 + 2\lambda}} \quad \text{and} \quad A^{+-}(\lambda, y) = e^{-2by} \frac{2(\beta - 1)\sqrt{b^2 + 2\lambda}}{b + \sqrt{b^2 + 2\lambda}}. \end{aligned}$$

Using (3.1)-(3.2) and (3.3)-(3.4), the density associated to the coefficients $\hat{\mathbf{a}}(\beta, b, -b)$ is (see Figure 3)

$$\begin{aligned} p(t, x, y) &= \frac{1}{\sqrt{2\pi t}} e^{-\frac{(bt - \operatorname{sgn}(y)(y-x))^2}{2t}} \\ &\quad + \operatorname{sgn}(y)(2\beta - 1) \frac{1}{\sqrt{2\pi t}} e^{-\frac{(|y|+|x|)^2}{2t}} e^{-b^2 \frac{t}{2}} e^{b(|y|-|x|)} \\ &\quad + \operatorname{sgn}(y)(H^-(y) - \beta)be^{2b|y|} \operatorname{erfc} \left(b\sqrt{\frac{t}{2}} + \frac{|y| + |x|}{\sqrt{2t}} \right). \end{aligned}$$

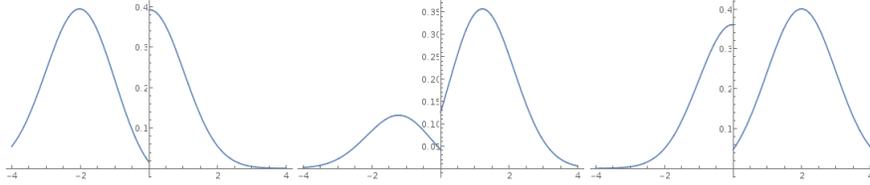


FIG 3. The density $p(t, x, y)$ of the SBM for $\beta = \frac{3}{4}$ with a Bang bang drift $b = 2$ at time $t = 1$ for the initial positions $x = -1$, $x = 0$ and $x = 1$ (from left to right).

This is the expression given in [14] up to a conversion of erfc into its probabilistic counterpart.

3.4. The constant Péclet case

Computing the transition density in all situations seems to be a very difficult problem. There are however other cases where simplifications occur. Among these situations, we consider the case where b_+ and b_- are linked by

$$b_- = \mu b_+ \text{ with } \mu = \frac{\beta}{1-\beta} \text{ or equivalently } (1-\beta)b_- = \beta b_+, \beta \neq \frac{1}{2}. \quad (3.5)$$

Remark 3.1. This assumption on b_+ and b_- is natural. If we consider the diffusion X^a with piecewise constant coefficients \mathbf{a} as in Section 2.3, for $\phi(x) = x/\sqrt{a(x)\rho(x)}$, then $\phi(X)$ is a SBM with a piecewise constant drift. Is it easily checked that (3.5) is satisfied when

$$\frac{b_+}{\rho_+} = \frac{b_-}{\rho_-}.$$

When $a = 1$, the ratio b/ρ is called the *Péclet number*. It is a dimensionless quantity which plays a very important role in fluid mechanics by characterizing the effect of the convection against the diffusion and vice versa.

When (3.5) holds,

$$\Theta(\lambda, b_+, \mu b_-) = \beta \sqrt{b_+^2 + 2\lambda} + (1-\beta) \sqrt{b_-^2 + 2\lambda}.$$

Hence,

$$\begin{aligned} \frac{1}{\Theta(\lambda, b_+, \mu b_-)} &= \frac{\beta \sqrt{b_+^2 + 2\lambda} - (1-\beta) \sqrt{b_-^2 + 2\lambda}}{\beta^2 (b_+^2 + 2\lambda) - (1-\beta)^2 (b_-^2 + 2\lambda)} \\ &= \frac{\beta \sqrt{b_+^2 + 2\lambda} - (1-\beta) \sqrt{b_-^2 + 2\lambda}}{2\lambda(2\beta - 1)}. \end{aligned}$$

Proposition 3.1. For any $t \geq 0$ and $\widehat{\mathbf{a}}(\beta, b, \mu b)$,

$$\begin{aligned} p(t, 0, y) &= \frac{2(H^-(y) - \beta)^2}{(2\beta - 1)\sqrt{2\pi t}} e^{-\frac{(b(x)^2 t - y)^2}{2t}} \\ &+ \frac{b(x)(H^-(y) - \beta)^2}{2(2\beta - 1)} \left(\operatorname{erfc}\left(\frac{y}{\sqrt{2t}} - b(x)\sqrt{\frac{t}{2}}\right) - e^{2b(x)y} \operatorname{erfc}\left(\frac{y}{\sqrt{2t}} + b(x)\sqrt{\frac{t}{2}}\right) \right) \\ &- \frac{\beta(1 - \beta)e^{b(x)y}}{2(2\beta - 1)\pi} \int_0^t e^{-\frac{b(x)^2 \tau}{2}} \int_0^{b^2(-x) - b^2(x)} \sin(y\sqrt{r}) \sqrt{b^2(-x) - b^2(x) - r} e^{-\frac{r\tau}{2}} dr d\tau \\ &\frac{\beta(1 - \beta)e^{b(x)y}}{2(2\beta - 1)\pi} \int_0^t e^{-\frac{(b(-x))^2 \tau}{2}} \int_0^{+\infty} \cos(y\sqrt{r + b^2(-x) - b^2(x)}) \sqrt{r} e^{-\frac{r\tau}{2}} dr d\tau. \end{aligned}$$

Proof. First,

$$\begin{aligned} \mathcal{L}^{-1}(r)(t, y) &= \frac{\beta^2 e^{b_+ y}}{2\beta - 1} \mathcal{L}^{-1}\left(\frac{\sqrt{b_+^2 + 2\lambda}}{\lambda} e^{-\sqrt{b_+ + 2\lambda}y}\right)(t, y) \mathbb{1}(y \geq 0) \\ &- \frac{\beta(1 - \beta)e^{b_+ y}}{2\beta - 1} \mathcal{L}^{-1}\left(\frac{\sqrt{b_-^2 + 2\lambda}}{\lambda} e^{-\sqrt{b_+ + 2\lambda}y}\right)(t, y) \mathbb{1}(y \geq 0) \\ &- \frac{\beta(1 - \beta)e^{b_- y}}{2\beta - 1} \mathcal{L}^{-1}\left(\frac{\sqrt{b_+^2 + 2\lambda}}{\lambda} e^{\sqrt{b_- + 2\lambda}y}\right)(t, y) \mathbb{1}(y < 0) \\ &+ \frac{(1 - \beta)^2 e^{b_- y}}{2\beta - 1} \mathcal{L}^{-1}\left(\frac{\sqrt{b_-^2 + 2\lambda}}{\lambda} e^{\sqrt{b_- + 2\lambda}y}\right)(t, y) \mathbb{1}(y < 0). \end{aligned}$$

By remarking that

$$\begin{aligned} \mathcal{L}^{-1}\left(\frac{\sqrt{b_+^2 + 2\lambda}}{\lambda} e^{-\sqrt{b_+ + 2\lambda}y}\right)(t, y) &= \mathcal{L}^{-1}\left(\frac{b_+^2}{\lambda \sqrt{b_+^2 + 2\lambda}} e^{-\sqrt{b_+ + 2\lambda}y}\right)(t, y) \\ &+ 2 \mathcal{L}^{-1}\left(\frac{1}{\sqrt{b_+^2 + 2\lambda}} e^{-\sqrt{b_+ + 2\lambda}y}\right)(t, y) \mathbb{1}(y \geq 0), \end{aligned}$$

and using formulas (3.1), (3.2) and

$$\begin{aligned} \mathcal{L}^{-1}\left(\frac{e^{-ak\sqrt{s}}}{(s - d)\sqrt{s}}\right) &= \frac{e^{dt}}{2d} \sqrt{d} e^{-ak\sqrt{d}} \operatorname{erfc}\left(\frac{ak}{2\sqrt{t}} - \sqrt{dt}\right) \\ &- \frac{e^{dt}}{2d} \sqrt{d} e^{ak\sqrt{d}} \operatorname{erfc}\left(\frac{ak}{2\sqrt{t}} + \sqrt{dt}\right) \end{aligned}$$

which can be found in [30],

$$\begin{aligned} \mathcal{L}^{-1} \left(\frac{\sqrt{b_+^2 + 2\lambda}}{\lambda} e^{-\sqrt{b_+ + 2\lambda}y} \right) (t, y) &= \frac{1}{2} b_+ e^{-b_+ y} \operatorname{erfc} \left(\frac{y}{\sqrt{2t}} - b_+ \sqrt{\frac{t}{2}} \right) \\ &\quad - \frac{1}{2} b_+ e^{b_+ y} \operatorname{erfc} \left(\frac{y}{\sqrt{2t}} + b_+ \sqrt{\frac{t}{2}} \right) + \frac{\sqrt{2}}{\sqrt{\pi t}} e^{-\frac{b_+^2 t}{2}} e^{-\frac{y^2}{2t}}. \end{aligned}$$

Then, using Lemma 3.1,

$$\begin{aligned} \mathcal{L}^{-1} \left(\frac{\sqrt{b_-^2 + 2\lambda}}{\lambda} e^{-\sqrt{b_+ + 2\lambda}y} \right) (t, y) &= \\ &\quad \frac{1}{2\pi} \int_0^t e^{-\frac{b_+^2}{2}\tau} \int_0^{b_-^2 - b_+^2} \sin(y\sqrt{r}) \sqrt{b_-^2 - b_+^2 - r} e^{-\frac{r\tau}{2}} dr d\tau \\ &\quad - \frac{1}{2\pi} \int_0^t e^{-\frac{b_+^2}{2}\tau} \int_0^{+\infty} \cos(y\sqrt{r + b_-^2 - b_+^2}) \sqrt{r} e^{-\frac{(r + b_-^2 - b_+^2)\tau}{2}} dr d\tau. \end{aligned}$$

Finally, when $y \geq 0$,

$$\begin{aligned} \mathcal{L}^{-1}(r)(t, y) &= \frac{b_+ \beta^2}{2(2\beta - 1)} \operatorname{erfc} \left(\frac{y}{\sqrt{2t}} - b_+ \sqrt{\frac{t}{2}} \right) \mathbb{1}(y \geq 0) \\ &\quad - \frac{b_+ \beta^2 e^{2b_+ y}}{2(2\beta - 1)} \operatorname{erfc} \left(\frac{y}{\sqrt{2t}} + b_+ \sqrt{\frac{t}{2}} \right) \mathbb{1}(y \geq 0) + \frac{2\beta^2}{(2\beta - 1)\sqrt{2\pi t}} e^{-\frac{(b_+ t - y)^2}{2t}} \\ &\quad - \frac{\beta(1 - \beta) e^{b_+ y}}{2(2\beta - 1)\pi} \int_0^t e^{-\frac{b_+^2}{2}\tau} \int_0^{b_-^2 - b_+^2} \sin(y\sqrt{r}) \sqrt{b_-^2 - b_+^2 - r} e^{-\frac{r\tau}{2}} dr d\tau \mathbb{1}(y \geq 0) \\ &\quad + \frac{\beta(1 - \beta) e^{b_+ y}}{2(2\beta - 1)\pi} \int_0^t e^{-\frac{b_+^2}{2}\tau} \int_0^{+\infty} \cos(y\sqrt{r + b_-^2 - b_+^2}) \sqrt{r} e^{-\frac{r\tau}{2}} dr d\tau \mathbb{1}(y \geq 0). \end{aligned}$$

The result proved using a symmetry argument when $y < 0$. \square

conclusion

We have proposed a new method to compute the resolvent kernel of skew diffusions. In some cases, it leads to explicit expressions of their density.

Our approach is different from the one of B. Gaveau *et al.* [17]. Drift term can be taken in consideration, even when the drift is discontinuous. Not only known expressions are easily recovered but also new ones are computed. Hence, our approach overcomes the difficulties of a purely stochastic one.

At first glance, this approach seems to be rooted in the Feller theory of one-dimensional diffusions. Some generalizations are possible in particular cases of multi-dimensional diffusions. In addition, it leads to the development of new simulation techniques for

the diffusions relying on exponential time steps instead of constant time step. This way, the expressions of the resolvent kernel are used as they are simpler than the ones of the density. These two directions of research are currently under investigation.

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