



**HAL**  
open science

# Packing and covering the balanced complete bipartite multigraph with cycles and stars

Hung-Chih Lee

► **To cite this version:**

Hung-Chih Lee. Packing and covering the balanced complete bipartite multigraph with cycles and stars. *Discrete Mathematics and Theoretical Computer Science*, 2014, Vol. 16 no. 3 (3), pp.189–202. 10.46298/dmtcs.2091 . hal-01188905

**HAL Id: hal-01188905**

**<https://inria.hal.science/hal-01188905>**

Submitted on 31 Aug 2015

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Packing and covering the balanced complete bipartite multigraph with cycles and stars

Hung-Chih Lee\*

Department of Information Technology, Ling Tung University, Taichung, Taiwan

received 24<sup>th</sup> Nov. 2013, accepted 16<sup>th</sup> Sep. 2014.

Let  $C_k$  denote a cycle of length  $k$  and let  $S_k$  denote a star with  $k$  edges. For multigraphs  $F$ ,  $G$  and  $H$ , an  $(F, G)$ -decomposition of  $H$  is an edge decomposition of  $H$  into copies of  $F$  and  $G$  using at least one of each. For  $L \subseteq H$  and  $R \subseteq rH$ , an  $(F, G)$ -packing (resp.  $(F, G)$ -covering) of  $H$  with leave  $L$  (resp. padding  $R$ ) is an  $(F, G)$ -decomposition of  $H - E(L)$  (resp.  $H + E(R)$ ). An  $(F, G)$ -packing (resp.  $(F, G)$ -covering) of  $H$  with the largest (resp. smallest) cardinality is a maximum  $(F, G)$ -packing (resp. minimum  $(F, G)$ -covering), and its cardinality is referred to as the  $(F, G)$ -packing number (resp.  $(F, G)$ -covering number) of  $H$ . In this paper, we determine the packing number and the covering number of  $\lambda K_{n,n}$  with  $C_k$ 's and  $S_k$ 's for any  $\lambda$ ,  $n$  and  $k$ , and give the complete solution of the maximum packing and the minimum covering of  $\lambda K_{n,n}$  with 4-cycles and 4-stars for any  $\lambda$  and  $n$  with all possible leaves and paddings.

**Keywords:** complete bipartite multigraph, cycle, star, packing, covering

## 1 Introduction

For positive integers  $m$  and  $n$ ,  $K_{m,n}$  denotes the complete bipartite graph with parts of sizes  $m$  and  $n$ . If  $m = n$ , the complete bipartite graph is referred to as *balanced*. A  $k$ -cycle, denoted by  $C_k$ , is a cycle of length  $k$ . A  $k$ -star, denoted by  $S_k$ , is the complete bipartite graph  $K_{1,k}$ . A  $k$ -path, denoted by  $P_k$ , is a path with  $k$  vertices. For a graph  $H$  and a positive integer  $\lambda$ , we use  $\lambda H$  to denote the multigraph obtained from  $H$  by replacing each edge  $e$  by  $\lambda$  edges each having the same endpoints as  $e$ . When  $\lambda = 1$ ,  $1H$  is simply written as  $H$ .

Let  $F$ ,  $G$ , and  $H$  be multigraphs. A *decomposition* of  $H$  is a set of edge-disjoint subgraphs of  $H$  whose union is  $H$ . An  $(F, G)$ -*decomposition* of  $H$  is a decomposition of  $H$  into copies of  $F$  and  $G$  using at least one of each. If  $H$  has an  $(F, G)$ -decomposition, we say that  $H$  is  $(F, G)$ -*decomposable* and write  $(F, G)|H$ . If  $H$  does not admit an  $(F, G)$ -decomposition, two natural questions arise:

- (1) What is the minimum number of edges needed to be removed from the edge set of  $H$  so that the resulting graph is  $(F, G)$ -decomposable, and what does the collection of removed edges look like?

\*Email: birdy@teamail.ltu.edu.tw. Supported by the Ministry of Science and Technology of Taiwan.

- (2) What is the minimum number of edges needed to be added to the edge set of  $H$  so that the resulting graph is  $(F, G)$ -decomposable, and what does the collection of added edges look like?

These questions are respectively called the maximum packing problem and the minimum covering problem of  $H$  with  $F$  and  $G$ .

Let  $F, G$ , and  $H$  be multigraphs. For  $L \subseteq H$  and  $R \subseteq rH$ , an  $(F, G)$ -packing of  $H$  with leave  $L$  is an  $(F, G)$ -decomposition of  $H - E(L)$ , and an  $(F, G)$ -covering with padding  $R$  is an  $(F, G)$ -decomposition of  $H + E(R)$ . For an  $(F, G)$ -packing  $\mathcal{P}$  of  $H$  with leave  $L$ , if  $|\mathcal{P}|$  is as large as possible (so that  $|L|$  is as small as possible), then  $\mathcal{P}$  and  $L$  are referred to as a *maximum  $(F, G)$ -packing* and a *minimum leave*, respectively. Moreover, the cardinality of the maximum  $(F, G)$ -packing of  $H$  is called the  *$(F, G)$ -packing number* of  $H$ , denoted by  $p(H; F, G)$ . For an  $(F, G)$ -covering  $\mathcal{C}$  of  $H$  with padding  $R$ , if  $|\mathcal{C}|$  is as small as possible (so that  $|R|$  is as small as possible), then  $\mathcal{C}$  and  $R$  are referred to as a *minimum covering* and a *minimum padding*, respectively. Moreover, the cardinality of the minimum  $(F, G)$ -covering of  $H$  is called the  *$(F, G)$ -covering number* of  $H$ , denoted by  $c(H; F, G)$ . Clearly, an  $(F, G)$ -decomposition of  $H$  is a maximum  $(F, G)$ -packing with leave the empty graph, and also a minimum  $(F, G)$ -covering with padding the empty graph.

Recently, decomposition into a pair of graphs has attracted a fair share of interest. Abueida and Daven [3] investigated the problem of  $(K_k, S_k)$ -decomposition of the complete graph  $K_n$ . Abueida and Daven [4] investigated the problem of the  $(C_4, E_2)$ -decomposition of several graph products where  $E_2$  denotes two vertex disjoint edges. Abueida and O'Neil [7] settled the existence problem for  $(C_k, S_{k-1})$ -decomposition of the complete multigraph  $\lambda K_n$  for  $k \in \{3, 4, 5\}$ . Priyadharsini and Muthusamy [12, 13] gave necessary and sufficient conditions for the existence of  $(G_n, H_n)$ -decompositions of  $\lambda K_n$  and  $\lambda K_{n,n}$  where  $G_n, H_n \in \{C_n, P_n, S_{n-1}\}$ . A *graph-pair*  $(G, H)$  of order  $m$  is a pair of non-isomorphic graphs  $G$  and  $H$  on  $m$  non-isolated vertices such that  $G \cup H$  is isomorphic to  $K_m$ . Abueida and Daven [2] and Abueida, Daven and Roblee [5] completely determined the values of  $n$  for which  $\lambda K_n$  admits a  $(G, H)$ -decomposition where  $(G, H)$  is a graph-pair of order 4 or 5. Abueida, Clark and Leach [1] and Abueida and Hampson [6] considered the existence of decompositions of  $K_n - F$  for the graph-pair of order 4 and 5, respectively, where  $F$  is a Hamiltonian cycle, a 1-factor, or almost 1-factor. Furthermore, Shyu [14] investigated the problem of decomposing  $K_n$  into paths and stars with  $k$  edges, giving a necessary and sufficient condition for  $k = 3$ . In [15, 16], Shyu considered the existence of a decomposition of  $K_n$  into paths and cycles with  $k$  edges, giving a necessary and sufficient condition for  $k \in \{3, 4\}$ . Shyu [17] investigated the problem of decomposing  $K_n$  into cycles and stars with  $k$  edges, settling the case  $k = 4$ . In [18], Shyu considered the existence of a decomposition of  $K_{m,n}$  into paths and stars with  $k$  edges, giving a necessary and sufficient condition for  $k = 3$ . Recently, Lee [9] and Lee and Lin [10] established necessary and sufficient conditions for the existence of  $(C_k, S_k)$ -decompositions of the complete bipartite graph and the complete bipartite graph with a 1-factor removed, respectively. However, much less work has been done on the problem of packing and covering graphs with a pair of graphs. Abueida and Daven [3] obtained the maximum packing and the minimum covering of the complete graph  $K_n$  with  $(K_k, S_k)$ . Abueida and Daven [2] and Abueida, Daven and Roblee [5] gave the maximum packing and the minimum covering of  $K_n$  and  $\lambda K_n$  with  $G$  and  $H$ , respectively, where  $(G, H)$  is a graph-pair of order 4 or 5. In this paper, we determine the packing number and the covering number of  $\lambda K_{n,n}$  with  $k$ -cycles and  $k$ -stars for any  $\lambda, n$  and  $k$ , and give the complete solution of the maximum packing and the minimum covering of  $\lambda K_{n,n}$  with 4-cycles and 4-stars for any  $\lambda$  and  $n$  with all possible leaves and paddings.

## 2 Preliminaries

In this section we first collect some needed terminology and notation, and then present a result which is useful for our discussions to follow.

Let  $G$  be a multigraph. The *degree* of a vertex  $x$  of  $G$ , denoted by  $\deg_G x$ , is the number of edges incident with  $x$ . The vertex of degree  $k$  in  $S_k$  is the *center* of  $S_k$  and any vertex of degree 1 is an *endvertex* of  $S_k$ . For  $W \subseteq V(G)$ , we use  $G[W]$  to denote the subgraph of  $G$  induced by  $W$ . Furthermore,  $\mu(uv)$  denotes the number of edges of  $G$  joining  $u$  and  $v$ ,  $(v_1, \dots, v_k)$  and  $v_1 \dots v_k$  denote the  $k$ -cycle and the  $k$ -path through vertices  $v_1, \dots, v_k$  in order, respectively, and  $(x; y_1, \dots, y_k)$  denotes the  $k$ -star with center  $x$  and endvertices  $y_1, \dots, y_k$ . When  $G_1, G_2, \dots, G_t$  are multigraphs, not necessarily disjoint, we write  $G_1 \cup G_2 \cup \dots \cup G_t$  or  $\bigcup_{i=1}^t G_i$  for the graph with vertex set  $\bigcup_{i=1}^t V(G_i)$  and edge set  $\bigcup_{i=1}^t E(G_i)$ . When the edge sets are disjoint,  $G = \bigcup_{i=1}^t G_i$  expresses the decomposition of  $G$  into  $G_1, G_2, \dots, G_t$ . Given an  $S_k$ -decomposition of  $G$ , a *central function*  $c$  from  $V(G)$  to the set of non-negative integers is defined as follows. For each  $v \in V(G)$ ,  $c(v)$  is the number of  $k$ -stars in the decomposition whose center is  $v$ .

The following result is essential to our proof.

**Proposition 2.1** (Hoffman [8]) *For a positive integer  $k$ , a multigraph  $H$  has an  $S_k$ -decomposition with central function  $c$  if and only if*

- (i)  $k \sum_{v \in V(H)} c(v) = |E(H)|$ ,
- (ii) for all  $x, y \in V(H)$ ,  $\mu(xy) \leq c(x) + c(y)$ ,
- (iii) for all  $S \subseteq V(H)$ ,  $k \sum_{v \in S} c(v) \leq \varepsilon(S) + \sum_{x \in S, y \in V(H) - S} \min\{c(x), \mu(xy)\}$ .

where  $\varepsilon(S)$  denotes the number of edges of  $H$  with both ends in  $S$ .

In the sequel of the paper,  $(A, B)$  denotes the bipartition of  $\lambda K_{n,n}$ , where  $A = \{a_0, a_1, \dots, a_{n-1}\}$  and  $B = \{b_0, b_1, \dots, b_{n-1}\}$ .

## 3 Packing numbers and covering numbers

In this section the packing number and the covering number of the balanced complete bipartite multigraph with  $k$ -cycles and  $k$ -stars are determined. We begin with a criterion for decomposing the complete bipartite graph into  $k$ -cycles.

**Proposition 3.1** (Sotteau [19]) *For positive integers  $m, n$ , and  $k$ , the graph  $K_{m,n}$  is  $C_k$ -decomposable if and only if  $m, n$ , and  $k$  are even,  $k \geq 4$ ,  $\min\{m, n\} \geq k/2$ , and  $k$  divides  $mn$ .*

Let  $K_{m,n}^*$  denote the symmetric complete bipartite digraph with parts of size  $m$  and  $n$ , and let  $\overrightarrow{C}_k$  denote the directed  $k$ -cycle.

**Proposition 3.2** (Sotteau [19]) *For positive integers  $m, n$ , and  $k$ , the digraph  $K_{m,n}^*$  is  $\overrightarrow{C}_k$ -decomposable if and only if  $k$  is even,  $k \geq 4$ ,  $\min\{m, n\} \geq k/2$ , and  $k$  divides  $2mn$ .*

Removing the directions from the arcs of directed cycles in a  $\overrightarrow{C}_k$ -decomposition of  $K_{m,n}^*$ , we obtain the following result by Proposition 3.2.

**Lemma 3.3** For positive integers  $m, n$ , and  $k$ , the multigraph  $2K_{m,n}$  is  $C_k$ -decomposable if  $k$  is even,  $k \geq 4$ ,  $\min\{m, n\} \geq k/2$ , and  $k$  divides  $2mn$ .

**Lemma 3.4** Let  $\lambda, k, m$ , and  $n$  be positive integers with  $\lambda m \equiv \lambda n \equiv k \equiv 0 \pmod{2}$  and  $\min\{m, n\} \geq k/2 \geq 2$ . If  $m$  or  $n$  is divisible by  $k$ , then  $\lambda K_{m,n}$  is  $C_k$ -decomposable.

**Proof:** Since  $\lambda K_{m,n}$  is isomorphic to  $\lambda K_{n,m}$ , it suffices to show that the result holds for  $k \mid m$ . If  $\lambda$  is odd, then  $m$  and  $n$  are even from the assumption  $\lambda m \equiv \lambda n \equiv 0 \pmod{2}$ . Since  $k$  divides  $mn$ , Proposition 3.1 implies that  $K_{m,n}$  is  $C_k$ -decomposable. If  $\lambda$  is even, then  $2K_{m,n} \mid \lambda K_{m,n}$ . Since  $k$  divides  $2mn$ ,  $2K_{m,n}$  is  $C_k$ -decomposable by Lemma 3.3. Hence  $\lambda K_{m,n}$  is  $C_k$ -decomposable.  $\square$

**Lemma 3.5** If  $k$  is a positive even integer with  $k \geq 4$ , then  $\lambda K_{k,k}$  is  $(C_k, S_k)$ -decomposable.

**Proof:** Note that  $\lambda K_{k,k} = \lambda K_{k,k-2} \cup \lambda K_{k,2}$ . By Lemma 3.4,  $\lambda K_{k,k-2}$  is  $C_k$ -decomposable. Trivially,  $\lambda K_{k,2}$  is  $C_k$ -decomposable. Therefore,  $\lambda K_{k,k}$  is  $(C_k, S_k)$ -decomposable.  $\square$

**Lemma 3.6** Let  $k$  be a positive even integer and let  $n$  be a positive integer with  $4 \leq k < n < 2k$ . If  $\lambda(n-k)^2 < k$ , then  $\lambda K_{n,n}$  has a  $(C_k, S_k)$ -packing with leave  $\lambda K_{n-k,n-k}$  and a  $(C_k, S_k)$ -covering with padding  $P_{k-\lambda(n-k)^2+1}$ .

**Proof:** Let  $n = k + r$ . The assumption  $k < n < 2k$  implies  $0 < r < k$ . We first give the required packing. Note that

$$\lambda K_{n,n} = \lambda K_{k,k} \cup \lambda K_{k,r} \cup \lambda K_{r,k} \cup \lambda K_{r,r}.$$

By Lemma 3.4,  $\lambda K_{k,k}$  has a  $C_k$ -decomposition  $\mathcal{D}_1$ . Trivially,  $\lambda K_{k,r}$  and  $\lambda K_{r,k}$  have  $S_k$ -decompositions  $\mathcal{D}_2$  and  $\mathcal{D}_3$ , respectively. Thus  $\bigcup_{i=1}^3 \mathcal{D}_i$  is a  $(C_k, S_k)$ -packing of  $\lambda K_{n,n}$  with leave  $\lambda K_{r,r}$ , as desired.

Now we give the required covering. Let  $s = \lambda r^2$ . Let  $A_0 = \{a_0, a_1, \dots, a_{\lfloor (s-1)/2 \rfloor}\}$ ,  $A_1 = A - A_0$ ,  $B_0 = \{b_0, b_1, \dots, b_{k-1}\}$  and  $B_1 = B - B_0$ . Define a  $k$ -cycle  $C$  and a  $(k-s+1)$ -path  $P$  as follows:

$$C = (b_0, a_0, b_1, a_1, \dots, b_{k/2-1}, a_{k/2-1})$$

$$P = \begin{cases} b_0 a_{k/2-1} b_{k/2-1} a_{k/2-2} \dots b_{(s+1)/2} a_{(s-1)/2} & \text{if } s \text{ is odd,} \\ b_0 a_{k/2-1} b_{k/2-1} a_{k/2-2} \dots a_s/2 b_{s/2} & \text{if } s \text{ is even.} \end{cases}$$

Let

$$H = \lambda K_{n,n} - E(C) + E(P).$$

Note that  $V(H) = V(\lambda K_{n,n})$ ,  $|E(H)| = \lambda n^2 - k + (k-s) = \lambda n^2 - \lambda r^2 = \lambda k(k+2r)$ , and  $\mu(uv) \leq \lambda$  for all  $u, v \in V(H)$ . Furthermore, for  $H' = H[A \cup B_0]$ , we have

$$\deg_{H'} v = \begin{cases} \lambda k - 2 & \text{if } v \in A_0 - \{a_{\lfloor (s-1)/2 \rfloor}\}, \\ \lambda k - \rho & \text{if } v = a_{\lfloor (s-1)/2 \rfloor}, \\ \lambda k & \text{if } v \in A_1, \end{cases}$$

where  $\rho = 1$  if  $s$  is odd, and  $\rho = 2$  if  $s$  is even. Define a function  $c : V(H) \rightarrow \mathbb{N}$  as follows:

$$c(v) = \begin{cases} 0 & \text{if } v \in B_0, \\ \lambda & \text{otherwise.} \end{cases}$$

Now we show that there exists an  $S_k$ -decomposition of  $H$  with central function  $c$  by Proposition 2.1.

First,  $k \sum_{v \in V(H)} c(v) = k\lambda(k+2r) = |E(H)|$ . This proves (i). Next, if  $u, v \in B_0$ , then  $c(u) + c(v) = 0 = \mu(uv)$ ; otherwise,  $c(u) + c(v) \geq \lambda \geq \mu(uv)$ . This proves (ii). Finally, for  $S \subseteq V(H)$  and  $i \in \{0, 1\}$ , let  $S \cap A_i = X_i$  and  $S \cap B_i = Y_i$ . Moreover, let  $X = X_0 \cup X_1$  and  $Y = Y_0 \cup Y_1$ . Define a set  $T$  of ordered pairs of vertices as follows:

$$T = \{(u, v) | u \in X, v \in B_1 - Y_1 \text{ or } u \in X_1, v \in B_0 - Y_0 \text{ or } u \in Y_1, v \in A - X\}.$$

Note that

$$k \sum_{w \in S} c(w) = k\lambda(|X| + |Y_1|), \quad (1)$$

$$\varepsilon(S) = \lambda(|X||Y_1| + |X_1||Y_0|) + \sum_{u \in X_0, v \in Y_0} \mu(uv), \quad (2)$$

and for  $u \in S$  and  $v \in V(H) - S$

$$\min\{c(u), \mu(uv)\} = \begin{cases} \lambda & \text{if } (u, v) \in T, \\ \mu(uv) & \text{if } u \in X_0, v \in B_0 - Y_0, \\ 0 & \text{otherwise.} \end{cases} \quad (3)$$

For  $S \subseteq V(H)$ , let

$$g(S) = \varepsilon(S) + \sum_{u \in S, v \in V(H) - S} \min\{c(u), \mu(uv)\} - k \sum_{w \in S} c(w).$$

Note that

$$\begin{aligned} & \sum_{u \in X_0, v \in Y_0} \mu(uv) + \sum_{u \in X_0, v \in B_0 - Y_0} \mu(uv) \\ &= \sum_{u \in X_0, v \in B_0} \mu(uv) \\ &= \begin{cases} |X_0|(\lambda k - 2) & \text{if } a_{\lfloor (s-1)/2 \rfloor} \notin X_0, \\ |X_0|(\lambda k - 2) + 2 - \rho & \text{if } a_{\lfloor (s-1)/2 \rfloor} \in X_0. \end{cases} \end{aligned}$$

By (1)–(3) and  $|X_0| + |X_1| = |X|$ , we have

$$\begin{aligned} g(S) &= \lambda(|X||Y_1| + |X_1||Y_0|) + \sum_{u \in X_0, v \in Y_0} \mu(uv) \\ &\quad + \lambda(|X|(r - |Y_1|) + |X_1|(k - |Y_0|) + |Y_1|(k + r - |X|)) \\ &\quad + \sum_{u \in X_0, v \in B_0 - Y_0} \mu(uv) - k\lambda(|X| + |Y_1|) \\ &= \begin{cases} \lambda(r|X| + |Y_1|(r - |X|)) - 2|X_0| & \text{if } a_{\lfloor (s-1)/2 \rfloor} \notin X_0, \\ \lambda(r|X| + |Y_1|(r - |X|)) - 2|X_0| + 2 - \rho & \text{if } a_{\lfloor (s-1)/2 \rfloor} \in X_0. \end{cases} \end{aligned}$$

If  $a_{\lfloor (s-1)/2 \rfloor} \notin X_0$ , then  $|X_0| \leq \lfloor (s-1)/2 \rfloor$ , which implies  $-2|X_0| \geq -s$ . If  $a_{\lfloor (s-1)/2 \rfloor} \in X_0$ , then  $|X_0| \leq \lfloor (s-1)/2 \rfloor + 1$ , which implies  $-2|X_0| + 2 - \rho \geq -2\lfloor (s-1)/2 \rfloor - \rho = -2(s-\rho)/2 - \rho = -s$ . Thus for  $|X| \geq r$ , we have

$$\begin{aligned} g(S) &\geq \lambda(r|X| - |Y_1|(|X| - r)) - s \\ &= \lambda(r|X| - |Y_1|(|X| - r)) - \lambda r^2 \\ &= \lambda(|X| - r)(r - |Y_1|) \\ &\geq 0. \end{aligned}$$

If  $\lambda r = 1$  and  $|X| < r$ , then  $|X_0| = |X| = 0$ , which implies  $-2|X_0| = -\lambda r|X_0|$ . If  $\lambda r \geq 2$ , then  $-2|X_0| \geq -\lambda r|X_0|$ . Note that  $2 - \rho \geq 0$ . Hence for  $|X| < r$ , we have

$$\begin{aligned} g(S) &\geq \lambda(r|X| + |Y_1|(r - |X|)) - \lambda r|X_0| \\ &= \lambda(r|X_1| + |Y_1|(r - |X|)) \\ &\geq 0. \end{aligned}$$

This settles (iii) and completes the proof.  $\square$

Before going on, the following results are needed.

**Proposition 3.7** (Ma et al. [11]) *For positive integers  $k$  and  $n$ , the graph obtained by deleting a 1-factor from  $K_{n,n}$  is  $C_k$ -decomposable if and only if  $n$  is odd,  $k$  is even,  $4 \leq k \leq 2n$ , and  $n(n-1)$  is divisible by  $k$ .*

**Lemma 3.8** *If  $\lambda$  and  $p$  are positive integers and  $k$  is a positive even integer with  $k \geq 4$ , then there exist  $\lambda pk/2 - p$  edge-disjoint  $k$ -cycles in  $\lambda K_{k/2, pk}$  (also in  $\lambda K_{pk, k/2}$ ).*

**Proof:** It suffices to show that the result holds for  $\lambda K_{k/2, pk}$ . If  $\lambda$  or  $k/2$  is even, then by Lemma 3.4 there exists a  $C_k$ -decomposition  $\mathcal{D}$  of  $\lambda K_{k/2, pk}$  with  $|\mathcal{D}| = \lambda pk/2$ , in which  $k$ -cycles are edge-disjoint. If  $k/2$  is odd, then by Proposition 3.7 there exists a  $C_k$ -decomposition  $\mathcal{D}'$  of  $K_{k/2, k/2} - I$  with  $|\mathcal{D}'| = (k-2)/4$ , where  $I$  is a 1-factor of  $K_{k/2, k/2}$ . Since  $K_{k/2, pk}$  can be decomposed into  $2p$  copies of  $K_{k/2, k/2}$ , there exist  $2p|\mathcal{D}'| = pk/2 - p$  edge-disjoint  $k$ -cycles in  $K_{k/2, pk}$ . For odd  $\lambda$  with  $\lambda \geq 3$ ,  $\lambda K_{k/2, k} = (\lambda-1)K_{k/2, k} \cup K_{k/2, k}$ . By Lemma 3.4 there exists a  $C_k$ -decomposition  $\mathcal{D}''$  of  $(\lambda-1)K_{k/2, pk}$  with  $|\mathcal{D}''| = (\lambda-1)pk/2$ . Hence there exist  $(\lambda-1)pk/2 + pk/2 - p = \lambda pk/2 - p$  edge-disjoint  $k$ -cycles in  $\lambda K_{k/2, pk}$ .  $\square$

**Lemma 3.9** *Let  $\lambda$  and  $r$  be positive integers and let  $k$  be a positive even integer with  $k \geq 4$  and  $r < k$ . If  $t = \lfloor \lambda r^2/k \rfloor$ , then there exist  $\lceil t/2 \rceil$  edge-disjoint  $k$ -cycles in  $\lambda K_{k/2, k}$ . Moreover, if  $\lambda \geq 2$  or  $r \leq k-2$  and  $\lambda r^2 \geq k$ , then  $\lfloor t/2 \rfloor + 1 \leq \lambda r/2$  and there exist  $\lfloor t/2 \rfloor + 1$  edge-disjoint  $k$ -cycles in  $\lambda K_{k/2, k}$ .*

**Proof:** Since  $r < k$ , we have  $t < \lambda r$ . Thus  $t+1 \leq \lambda r$ ; in turn,  $\lceil t/2 \rceil \leq (t+1)/2 \leq \lambda r/2 < \lambda k/2$ , which implies  $\lceil t/2 \rceil \leq \lambda k/2 - 1$ . By Lemma 3.8, there exist  $\lceil t/2 \rceil$  edge-disjoint  $k$ -cycles in  $\lambda K_{k/2, k}$ . When  $\lambda r^2 = k$ , the result is trivial. When  $\lambda r^2 > k$ , we have  $r > 2/\sqrt{\lambda}$  since  $k \geq 4$ . For  $\lambda \geq 2$ ,

$$\frac{\lambda r^2}{k} \leq \frac{\lambda r^2}{r+1} = \lambda r - \frac{\lambda}{1+1/r} < \lambda r - \frac{2\lambda}{2+\sqrt{\lambda}} < \lambda r - \frac{4}{2+\sqrt{2}}.$$

For  $r \leq k - 2$ ,

$$\frac{\lambda r^2}{k} \leq \frac{\lambda r^2}{r+2} = \lambda r - \frac{2\lambda}{1+2/r} < \lambda r - \frac{2\lambda}{1+\sqrt{\lambda}} < \lambda r - 1.$$

Therefore,  $t = \lfloor \lambda r^2/k \rfloor \leq \lambda r - 2$ . In turn,  $\lfloor t/2 \rfloor + 1 \leq t/2 + 1 \leq \lambda r/2$  for  $\lambda \geq 2$  or  $r \leq k - 2$ . It implies  $\lfloor t/2 \rfloor + 1 < \lambda k/2$ . Hence  $\lfloor t/2 \rfloor + 1 \leq \lambda k/2 - 1$  for  $\lambda \geq 2$  or  $r \leq k - 2$ . This assures us that there exist  $\lfloor t/2 \rfloor + 1$  edge-disjoint  $k$ -cycles in  $\lambda K_{k/2,k}$  by Lemma 3.8.  $\square$

**Lemma 3.10** *Let  $k$  be a positive even integer and let  $n$  be a positive integer with  $4 \leq k < n < 2k$ . If  $\lambda(n-k)^2 \geq k$ , then  $\lambda K_{n,n}$  has a  $(C_k, S_k)$ -packing  $\mathcal{P}$  with  $|\mathcal{P}| = \lfloor \lambda n^2/k \rfloor$  and a  $(C_k, S_k)$ -covering  $\mathcal{C}$  with  $|\mathcal{C}| = \lceil \lambda n^2/k \rceil$ .*

**Proof:** Let  $n = k + r$ . From the assumption  $k < n < 2k$ , we have  $0 < r < k$ . Let  $\lambda r^2 = tk + s$  such that  $s$  and  $t$  are integers with  $0 \leq s < k$ . Note that  $t = \lfloor \lambda r^2/k \rfloor$ . Hence  $\lfloor \lambda n^2/k \rfloor = \lfloor \lambda(k+r)^2/k \rfloor = \lambda(k+2r) + t$  and

$$\left\lceil \frac{\lambda n^2}{k} \right\rceil = \left\lceil \frac{\lambda(k+r)^2}{k} \right\rceil = \begin{cases} \lambda(k+2r) + t & \text{if } s = 0 \\ \lambda(k+2r) + t + 1 & \text{if } s > 0. \end{cases}$$

Since  $\lambda(n-k)^2 \geq k$ ,  $t \geq 1$ . Let  $p_0 = \lceil t/2 \rceil$  and  $p_1 = \lfloor t/2 \rfloor$ . We have  $p_0 = 1$  and  $p_1 = 0$  for  $t = 1$ , and  $p_0 \geq p_1 \geq 1$  for  $t \geq 2$ . In the sequel, we will show that  $\lambda K_{n,n}$  has a packing  $\mathcal{P}$  consisting of  $t$  copies of  $k$ -cycles and  $\lambda(k+2r)$  copies of  $k$ -stars with leave  $P_{s+1}$  (except in the case  $s = 0$ , in which the leave is the empty graph), and a covering  $\mathcal{C}$  with  $|\mathcal{C}| = \lceil \lambda n^2/k \rceil$ .

Let  $A_0 = \{a_0, a_1, \dots, a_{k/2-1}\}$ ,  $A_1 = \{a_{k/2}, a_{k/2+1}, \dots, a_{k-1}\}$ ,  $A_2 = A - (A_0 \cup A_1)$ ,  $B_0 = \{b_0, b_1, \dots, b_{k-1}\}$  and  $B_1 = B - B_0$ . In addition, letting  $A'_1 = \{a_{k/2}, a_{k/2+1}, \dots, a_{\lceil (k+s)/2 \rceil - 1}\}$  for  $s > 0$  and  $G_i = \lambda K_{n,n}[A_i \cup B_1]$  for  $i = 0, 1$ . Clearly,  $G_0$  and  $G_1$  are isomorphic to  $\lambda K_{k/2,k}$ . By Lemma 3.9, there exist  $p_i$  edge-disjoint  $k$ -cycles in  $G_i$  for  $i \in \{0, 1\}$ , and there exist  $p_1 + 1$  edge-disjoint  $k$ -cycles in  $G_1$  for  $\lambda \geq 2$  or  $r \leq k - 2$ . Let  $\delta = 0$  for  $p_1 = 0$  and  $\delta = 1$  for  $p_1 \geq 1$ . Suppose that  $Q_{i,0}, Q_{i,1}, \dots, Q_{i,p_i-1}$  are edge-disjoint  $k$ -cycles in  $G_i$  for  $0 \leq i \leq \delta$ . Moreover, for  $\lambda \geq 2$  or  $r \leq k - 2$ , let  $Q$  be a  $k$ -cycle in  $G_1$  which is edge-disjoint with  $Q_{1,j}$  for  $0 \leq j \leq p_1 - 1$ . Without loss of generality, we assume that

$$Q = (b_{j_1}, a_{k/2}, b_{j_2}, a_{k/2+1}, \dots, b_{j_{k/2}}, a_{k-1}).$$

Note, for  $\lambda = 1$  and  $r = k - 1$ , that  $\lambda r^2 = (k - 1)^2 = k(k - 2) + 1$ , which implies  $t = k - 2$  and  $s = 1$ . For  $s > 0$ , define an  $(s + 1)$ -path  $P$  as follows:

$$P = \begin{cases} a_{k/2} b_\ell & \text{if } \lambda = 1, r = k - 1, \\ b_{j_1} a_{k/2} b_{j_2} a_{k/2+1} \dots b_{j_{s/2}} a_{(k+s)/2-1} b_{j_{s/2+1}} & \text{if } \lambda \geq 2 \text{ or } r \leq k - 2, s \text{ is even,} \\ b_{j_1} a_{k/2} b_{j_2} a_{k/2+1} \dots b_{j_{(s+1)/2}} a_{(k+s+1)/2-1} & \text{if } \lambda \geq 2 \text{ or } r \leq k - 2, s \text{ is odd,} \end{cases}$$

where  $a_{k/2} b_\ell$  is any edge (incident with  $a_{k/2}$ ) not in  $Q_{1,0}, Q_{1,1}, \dots, Q_{1,p_1-1}$ . Let

$$H = \lambda K_{n,n} - E\left(\bigcup_{i=0}^{\delta} \bigcup_{h=0}^{p_i-1} Q_{i,h}\right) \cup P.$$



Note that  $V(H) = V(\lambda K_{n,n})$ ,  $|E(H)| = \lambda n^2 - (tk + s) = \lambda n^2 - \lambda r^2 = \lambda k(k + 2r)$ , and  $\mu(uv) \leq \lambda$  for all  $u, v \in V(H)$ . Moreover, for  $H' = H[A \cup B_0]$ , we have

$$\deg_{H'} v = \begin{cases} \lambda k - 2\lceil t/2 \rceil & \text{if } v \in A_0, \\ \lambda k - 2(\lfloor t/2 \rfloor + 1) & \text{if } s > 0 \text{ and } v \in A'_1 - \{a_{\lceil (k+s)/2 \rceil - 1}\}, \\ \lambda k - 2\lfloor t/2 \rfloor - \rho & \text{if } s > 0 \text{ and } v = a_{\lceil (k+s)/2 \rceil - 1}, \\ \lambda k - 2\lfloor t/2 \rfloor & \text{if } s > 0 \text{ and } v \in A_1 - A'_1, \text{ or } s = 0 \text{ and } v \in A_1, \\ \lambda k & \text{if } v \in A_2, \end{cases}$$

where  $\rho = 1$  if  $s$  is odd, and  $\rho = 2$  if  $s$  is even. Define a function  $c : V(H) \rightarrow \mathbb{N}$  as follows:

$$c(v) = \begin{cases} 0 & \text{if } v \in B_0, \\ \lambda & \text{otherwise.} \end{cases}$$

Now we show that there exists an  $S_k$ -decomposition  $\mathcal{D}$  of  $H$  with central function  $c$  by Proposition 2.1.

First,  $k \sum_{v \in V(H)} c(v) = k\lambda(k + 2r) = |E(H)|$ . This proves (i). Next, if  $u, v \in B_0$ , then  $c(u) + c(v) = 0 = \mu(uv)$ ; otherwise,  $c(u) + c(v) \geq \lambda \geq \mu(uv)$ . This proves (ii). Finally, for  $S \subseteq V(H)$ ,  $i \in \{0, 1, 2\}$ , and  $j \in \{0, 1\}$ , let  $S \cap A_i = X_i$  and  $S \cap B_j = Y_j$ . Moreover, letting  $S \cap A'_1 = X'_1$ ,  $X = X_0 \cup X_1 \cup X_2$ , and  $Y = Y_0 \cup Y_1$ . Define a set  $T$  of ordered pairs of vertices as follows:

$$T = \{(u, v) | u \in X, v \in B_1 - Y_1 \text{ or } u \in X_2, v \in B_0 - Y_0 \text{ or } u \in Y_1, v \in A - X\}.$$

Note that

$$k \sum_{w \in S} c(w) = k\lambda(|X| + |Y_1|), \quad (4)$$

$$\varepsilon(S) = \lambda(|X||Y_1| + |X_2||Y_0|) + \sum_{u \in X_0 \cup X_1, v \in Y_0} \mu(uv), \quad (5)$$

and for  $u \in S$  and  $v \in V(H) - S$

$$\min\{c(u), \mu(uv)\} = \begin{cases} \lambda & \text{if } (u, v) \in T, \\ \mu(uv) & \text{if } u \in X_0 \cup X_1, v \in B_0 - Y_0, \\ 0 & \text{otherwise.} \end{cases} \quad (6)$$

For  $S \subseteq V(H)$ , let

$$g(S) = \varepsilon(S) + \sum_{u \in S, v \in V(H) - S} \min\{c(u), \mu(uv)\} - k \sum_{w \in S} c(w).$$

Note that

$$\begin{aligned} & \sum_{u \in X_0 \cup X_1, v \in Y_0} \mu(uv) + \sum_{u \in X_0 \cup X_1, v \in B_0 - Y_0} \mu(uv) \\ = & \sum_{u \in X_0 \cup X_1, v \in B_0} \mu(uv) \\ = & \begin{cases} |X_0|(\lambda k - 2\lceil t/2 \rceil) + |X_1|(\lambda k - 2\lfloor t/2 \rfloor) & \text{if } s = 0, \\ |X_0|(\lambda k - 2\lceil t/2 \rceil) + |X_1|(\lambda k - 2\lfloor t/2 \rfloor) - 2|X'_1| & \text{if } s > 0, a_{\lceil (k+s)/2 \rceil - 1} \notin X'_1, \\ |X_0|(\lambda k - 2\lceil t/2 \rceil) + |X_1|(\lambda k - 2\lfloor t/2 \rfloor) - 2|X'_1| + 2 - \rho & \text{if } s > 0, a_{\lceil (k+s)/2 \rceil - 1} \in X'_1. \end{cases} \end{aligned}$$

By (4)–(6) and  $|X_0| + |X_1| + |X_2| = |X|$ , we have

$$\begin{aligned} g(S) &= \lambda(|X||Y_1| + |X_2||Y_0|) + \sum_{u \in X_0 \cup X_1, v \in Y_0} \mu(uv) \\ &\quad + \lambda(|X|(r - |Y_1|) + |X_2|(k - |Y_0|) + |Y_1|(k + r - |X|)) \\ &\quad + \sum_{u \in X_0 \cup X_1, v \in B_0 - Y_0} \mu(uv) - k\lambda(|X| + |Y_1|) \\ &= \lambda(r|X| + |Y_1|(r - |X|)) + m, \end{aligned}$$

where

$$m = \begin{cases} -2(|X_0|\lceil t/2 \rceil + |X_1|\lfloor t/2 \rfloor) & \text{if } s = 0, \\ -2(|X_0|\lceil t/2 \rceil + |X_1|\lfloor t/2 \rfloor) - 2|X'_1| & \text{if } s > 0, a_{\lceil (k+s)/2 \rceil - 1} \notin X'_1, \\ -2(|X_0|\lceil t/2 \rceil + |X_1|\lfloor t/2 \rfloor) - 2|X'_1| + 2 - \rho & \text{if } s > 0, a_{\lceil (k+s)/2 \rceil - 1} \in X'_1. \end{cases}$$

If  $a_{\lceil (k+s)/2 \rceil - 1} \notin X'_1$ , then  $|X'_1| \leq |A'_1| - 1 = \lceil s/2 \rceil - 1$ . Hence  $-2|X'_1| \geq -2(\lceil s/2 \rceil - 1) \geq -s$ . If  $a_{\lceil (k+s)/2 \rceil - 1} \in X_1$ , then  $|X'_1| \leq |A'_1| = \lceil s/2 \rceil$ . In addition,  $\rho = 1$  for odd  $s$  and  $\rho = 2$  for even  $s$ . Therefore,  $-2|X'_1| + 2 - \rho \geq -2\lceil s/2 \rceil + 2 - \rho = -s$ . Together with the fact  $\max\{|X_0|, |X_1|\} \leq k/2$ , we have

$$m \geq -2(k/2\lceil t/2 \rceil + k/2\lfloor t/2 \rfloor) - s = -(kt + s) = -\lambda r^2.$$

Thus for  $|X| \geq r$ , we have

$$g(S) \geq \lambda(r|X| - |Y_1|(|X| - r)) - \lambda r^2 = \lambda(|X| - r)(r - |Y_1|) \geq 0.$$

So it remains to consider the case  $|X| < r$ . Recall that  $t = k - 2$  and  $s = 1$  for  $(\lambda, r) = (1, k - 1)$ . Thus  $\lceil t/2 \rceil = \lfloor t/2 \rfloor = (\lambda r - 1)/2$ . In addition,  $|X'_1| = 0$  for  $a_{\lceil (k+s)/2 \rceil - 1} \notin X'_1$ , and  $\rho = 1$  as well as  $|X'_1| = 1$  (which implies  $|X_1| \geq 1$ ) for  $a_{\lceil (k+s)/2 \rceil - 1} \in X'_1$ . Hence for  $a_{\lceil (k+s)/2 \rceil - 1} \notin X'_1$ ,

$$m = -2(|X_0| + |X_1|)(\lambda r - 1)/2 \geq -\lambda r(|X_0| + |X_1|),$$

and for  $a_{\lceil (k+s)/2 \rceil - 1} \in X'_1$ ,

$$\begin{aligned} m &= -2(|X_0| + |X_1|)(\lambda r - 1)/2 - 1 \\ &= -\lambda r(|X_0| + |X_1|) + |X_0| + |X_1| - 1 \\ &\geq -\lambda r(|X_0| + |X_1|). \end{aligned}$$

On the other hand, for  $\lambda \geq 2$  or  $r \leq k - 2$ , we have  $\lfloor t/2 \rfloor + 1 \leq \lambda r/2$  by Lemma 3.9, this implies

$$\begin{aligned} m &\geq -2(|X_0|\lceil t/2 \rceil + |X'_1|(\lfloor t/2 \rfloor + 1) + (|X_1| - |X'_1|)\lfloor t/2 \rfloor) \\ &\geq -2(|X_0| + |X_1|)(\lambda r/2) \\ &= -\lambda r(|X_0| + |X_1|). \end{aligned}$$

Therefore, for  $|X| < r$ , we have

$$g(S) \geq \lambda(r|X| + |Y_1|(r - |X|)) - \lambda r(|X_0| + |X_1|) = \lambda(r|X_2| + |Y_1|(r - |X|)) \geq 0.$$

This settles (iii).

Let  $\mathcal{P} = \mathcal{D} \cup_{i=0}^{\delta} \{Q_{i,0}, Q_{i,1}, \dots, Q_{i,p_i-1}\}$ . Clearly,  $\mathcal{P}$  is the required packing. Let

$$\mathcal{C} = \begin{cases} \mathcal{P} & \text{if } s = 0, \\ \mathcal{P} \cup \{Q\} & \text{if } s \geq 1. \end{cases}$$

It is easy to check that  $\mathcal{C}$  is the covering as required.  $\square$

Now, we are ready for the main result of this section.

**Theorem 3.11** *If  $\lambda$  and  $n$  are positive integers and  $k$  is a positive even integer with  $4 \leq k \leq n$ , then  $p(\lambda K_{n,n}; C_k, S_k) = \lfloor \lambda n^2/k \rfloor$  and  $c(\lambda K_{n,n}; C_k, S_k) = \lceil \lambda n^2/k \rceil$ .*

**Proof:** Obviously,

$$p(\lambda K_{n,n}; C_k, S_k) \leq \left\lfloor \frac{\lambda n^2}{k} \right\rfloor \leq \left\lceil \frac{\lambda n^2}{k} \right\rceil \leq c(\lambda K_{n,n}; C_k, S_k),$$

Let  $n = qk + r$  where  $q$  and  $r$  are integers with  $0 \leq r < k$ . For  $q = 1$ , the result follows from Lemmas 3.5, 3.6, and 3.10. If  $q \geq 2$ , then  $\lambda K_{n,n} = \lambda K_{k+r, k+r} \cup \lambda K_{k+r, (q-1)k} \cup \lambda K_{(q-1)k, n}$ . Note that  $\lambda K_{k+r, k+r}$  has a  $(C_k, S_k)$ -packing  $\mathcal{P}$  with  $|\mathcal{P}| = \lfloor \lambda(k+r)^2/k \rfloor$  and a  $(C_k, S_k)$ -covering  $\mathcal{C}$  with  $|\mathcal{C}| = \lceil \lambda(k+r)^2/k \rceil$ . Trivially,  $\lambda K_{k+r, (q-1)k}$  and  $\lambda K_{(q-1)k, n}$  have  $S_k$ -decompositions  $\mathcal{D}$  and  $\mathcal{D}'$  with  $|\mathcal{D}| = \lambda(k+r)(q-1)$  and  $|\mathcal{D}'| = \lambda(q-1)n$ , respectively. Since  $\lambda(k+r)^2/k + \lambda(k+r)(q-1) + \lambda(q-1)n = \lambda(qk+r)^2/k = \lambda n^2/k$ ,  $\mathcal{P} \cup \mathcal{D} \cup \mathcal{D}'$  is a  $(C_k, S_k)$ -packing of  $\lambda K_{n,n}$  with cardinality  $\lfloor \lambda n^2/k \rfloor$  and  $\mathcal{C} \cup \mathcal{D} \cup \mathcal{D}'$  is a  $(C_k, S_k)$ -covering of  $\lambda K_{n,n}$  with cardinality  $\lceil \lambda n^2/k \rceil$ . This completes the proof.  $\square$

Clearly, if  $\lambda K_{n,n}$  admits a  $(C_k, S_k)$ -decomposition, then  $4 \leq k \leq n$  and  $k$  is even and  $\lambda n^2$  is divisible by  $k$ . When  $k$  divides  $\lambda n^2$ , a  $(C_k, S_k)$ -packing  $\mathcal{P}$  with  $|\mathcal{P}| = \lfloor \lambda n^2/k \rfloor$  is a  $(C_k, S_k)$ -decomposition. Therefore, with the aid of Theorem 3.11, we have the following.

**Corollary 3.12** *For positive integers  $\lambda$ ,  $k$  and  $n$ , the balanced complete bipartite multigraph  $\lambda K_{n,n}$  is  $(C_k, S_k)$ -decomposable if and only if  $4 \leq k \leq n$ ,  $k$  is even, and  $\lambda n^2$  is divisible by  $k$ .*

## 4 Packing and covering with 4-cycles and 4-stars

In this section a complete solution to the maximum packing and minimum covering problem of  $\lambda K_{n,n}$  with  $C_4$  and  $S_4$  is given. Before that, we need more notations. For multigraphs  $G$  and  $H$ ,  $G \uplus H$  denotes the disjoint union of  $G$  and  $H$ ,  $G \odot H$  denotes the union of  $G$  and  $H$  with a common vertex. For a set  $\mathcal{R}$  and a positive integer  $t$ ,  $t\mathcal{R}$  denotes the multiset in which each element in  $\mathcal{R}$  appears  $t$  times. In addition,  $M_t$  denotes the graph induced by  $t$  nonadjacent edges. We begin with the discussion for the possible minimum leaves and paddings of  $\lambda K_{n,n}$  with  $C_4$  and  $S_4$ .

Note that  $|E(\lambda K_{n,n})| = \lambda n^2$ . If  $\lambda \equiv 0 \pmod{4}$  or  $n \equiv 0 \pmod{2}$ , then  $|E(\lambda K_{n,n})| \equiv 0 \pmod{4}$ . By Corollary 3.12, both of the possible minimum leave and the possible minimum padding are the empty graph. If  $\lambda \equiv 1 \pmod{4}$  and  $n \equiv 1 \pmod{2}$ , then  $|E(\lambda K_{n,n})| \equiv 1 \pmod{4}$ . This implies that the possible minimum leave is only  $P_2$ , and the possible minimum paddings are  $S_3$ ,  $P_4$ ,  $P_3 \uplus P_2$ ,  $M_3$ ,  $2P_2 \uplus P_2$ ,  $2P_2 \odot P_2$ , and  $3P_2$ . If  $\lambda \equiv 2 \pmod{4}$  and  $n \equiv 1 \pmod{2}$ , then  $|E(\lambda K_{n,n})| \equiv 2 \pmod{4}$ . This implies that the possible minimum leaves are  $P_3$ ,  $M_2$ , and  $2P_2$ , so are the possible minimum paddings. If  $\lambda \equiv 3 \pmod{4}$  and  $n \equiv 1 \pmod{2}$ , then  $|E(\lambda K_{n,n})| \equiv 3 \pmod{4}$ . This implies that the possible minimum leaves are  $S_3$ ,  $P_4$ ,  $P_3 \uplus P_2$ ,  $M_3$ ,  $2P_2 \uplus P_2$ ,  $2P_2 \odot P_2$ , and  $3P_2$ , and the possible minimum padding is only  $P_2$ .

**Lemma 4.1**  $K_{5,5}$  has no  $(C_4, S_4)$ -covering with padding  $3P_2$ .

**Proof:** It suffices to show that  $K_{5,5} + 3\{a_0b_0\}$  is not  $(C_4, S_4)$ -decomposable. Suppose, to the contrary of the conclusion, that there exists a  $(C_4, S_4)$ -decomposition  $\mathcal{D}$  of  $K_{5,5} + 3\{a_0b_0\}$ . Since there are at most two star with center  $a_0$  (or  $b_0$ ) and each edge joining  $a_0$  and  $b_0$  lies in exactly one subgraph in  $\mathcal{D}$ , there are exactly three possibilities for the edges joining  $a_0$  and  $b_0$  to lie in the decomposition: in four 4-cycles, in three 4-cycles and a 4-star, or in two 4-cycles and two 4-stars. Let  $G_1$  be the graph obtained from  $K_{5,5} + 3\{a_0b_0\}$  by deleting the edges of four 4-cycles, and let  $G_2$  be the graph obtained from  $K_{5,5} + 3\{a_0b_0\}$  by deleting the edges of three 4-cycles or deleting the edges of two 4-cycles. Note that  $\deg_{G_1} x = 3$  for  $x \notin \{a_0, b_0\}$ , which implies that there is no 4-star in  $G_1$ . Since  $\deg_{G_2} x \leq 3$  for  $x \in \{a_0, b_0\}$ , there is no 4-star with center at  $a_0$  or  $b_0$  in  $G_2$ . This leads to a contradiction and completes the proof.  $\square$

We summarize the results discussed above in Table 1.

**Tab. 1:** The possible minimum leaves and paddings of  $\lambda K_{n,n}$  with  $C_4$  and  $S_4$

$\lambda \pmod{4}$ $n \pmod{2}$	$\lambda \equiv 0$ or $n \equiv 0$	$\lambda \equiv 1$ and $n \equiv 1$	$\lambda \equiv 2$ and $n \equiv 1$	$\lambda \equiv 3$ and $n \equiv 1$
Leave	$\emptyset$	$P_2$	$P_3, M_2, 2P_2$	$S_3, P_4, P_3 \uplus P_2,$ $M_3, 2P_2 \uplus P_2,$ $2P_2 \odot P_2, 3P_2$
Padding	$\emptyset$	$S_3, P_4, P_3 \uplus P_2,$ $M_3, 2P_2 \uplus P_2,$ $2P_2 \odot P_2, 3P_2$ ( $3P_2$ for $\lambda \neq 1$ )	$P_3, M_2, 2P_2$	$P_2$

**Lemma 4.2** Let  $r \in \{1, 2, 3, 5\}$ .

(a) There exists a  $(C_4, S_4)$ -packing of  $rK_{5,5}$  with leave  $L$  where

$$\begin{cases} L = P_2 & \text{if } r = 1 \text{ or } r = 5, \\ L \in \{P_3, M_2, 2P_2\} & \text{if } r = 2, \\ L \in \{S_3, P_4, P_3 \uplus P_2, M_3, 2P_2 \uplus P_2, 2P_2 \odot P_2, 3P_2\} & \text{if } r = 3. \end{cases}$$

(b) There exists a  $(C_4, S_4)$ -covering of  $rK_{5,5}$  with padding  $R$  where

$$\begin{cases} R \in \{S_3, P_4, P_3 \uplus P_2, M_3, 2P_2 \uplus P_2, 2P_2 \odot P_2\} & \text{if } r = 1, \\ R \in \{P_3, M_2, 2P_2\} & \text{if } r = 2, \\ R = P_2 & \text{if } r = 3, \\ R \in \{S_3, P_4, P_3 \uplus P_2, M_3, 2P_2 \uplus P_2, 2P_2 \odot P_2, 3P_2\} & \text{if } r = 5. \end{cases}$$

**Proof:** The proof is divided into four parts according to the value of  $r$ .

**Case 1.**  $r = 1$ .

Let  $A_1 = \{a_1, a_2, a_3, a_4\}$  and  $B_1 = \{b_1, b_2, b_3, b_4\}$ , and let  $H = K_{5,5}[A_1 \cup B_1]$ . Trivially,  $H$  is isomorphic to  $K_{4,4}$ . By Corollary 3.12, there exists a  $(C_4, S_4)$ -decomposition  $\mathcal{D}$  of  $K_{4,4}$ . Let  $\mathcal{P} =$

$\mathcal{P} \cup \{(a_0; b_1, b_2, b_3, b_4), (b_0; a_1, a_2, a_3, a_4)\}$ . Clearly,  $\mathcal{P}$  is a  $(C_4, S_4)$ -packing of  $K_{5,5}$  with leave  $P_2 : \{a_0b_0\}$ .

Now we give the required coverings of  $K_{5,5}$ . Note that  $\mathcal{P} \cup \{(a_0; b_0, b_1, b_2, b_3)\}$  is a  $(C_4, S_4)$ -covering of  $K_{5,5}$  with padding  $S_3 : \{(a_0; b_1, b_2, b_3)\}$ , and  $\mathcal{P} \cup \{(a_0, b_1, a_1, b_0)\}$  is a  $(C_4, S_4)$ -covering of  $K_{5,5}$  with padding  $P_4 : \{a_0b_1a_1b_0\}$ . Without loss of generality, we assume that  $\mathcal{P}$  contains a 4-star  $(a_4; b_1, b_2, b_3, b_4)$ . Thus  $\mathcal{P} - \{(a_0; b_1, b_2, b_3, b_4), (a_4; b_1, b_2, b_3, b_4)\} \cup \{(a_0, b_3, a_4, b_4), (a_0; b_0, b_1, b_2, b_3), (a_4; b_0, b_1, b_2, b_4)\}$  is a  $(C_4, S_4)$ -covering of  $K_{5,5}$  with padding  $P_3 \uplus P_2 : \{b_0a_4b_4, a_0b_3\}$ . In addition,  $\{(a_3, b_3, a_4, b_4), (a_0; b_0, b_2, b_3, b_4), (a_1; b_0, b_1, b_3, b_4), (a_2; b_1, b_2, b_3, b_4), (b_0; a_0, a_2, a_3, a_4), (b_1; a_0, a_1, a_3, a_4), (b_2; a_1, a_2, a_3, a_4)\}$  is a  $(C_4, S_4)$ -covering of  $K_{5,5}$  with padding  $M_3 : \{a_0b_0, a_1b_1, a_2b_2\}$ ,  $\mathcal{P} - \{(a_4; b_1, b_2, b_3, b_4)\} \cup \{(a_0, b_0, a_4, b_4), (a_4; b_0, b_1, b_2, b_3)\}$  is a  $(C_4, S_4)$ -covering of  $K_{5,5}$  with padding  $2P_2 \uplus P_2 : 2\{b_0a_4\} \cup \{a_0b_4\}$ , and  $\mathcal{P} - \{(a_0; b_1, b_2, b_3, b_4), (a_4; b_1, b_2, b_3, b_4)\} \cup \{(a_0, b_0, a_4, b_4), (a_0; b_0, b_1, b_2, b_3), (a_4; b_0, b_1, b_2, b_3)\}$  is a  $(C_4, S_4)$ -covering of  $K_{5,5}$  with padding  $2P_2 \odot P_2 : 2\{b_0a_4\} \cup \{b_0a_0\}$ .

**Case 2.**  $r = 2$ .

First, we use  $\mathcal{P}$  to construct the required packings of  $2K_{5,5}$ . Exchanging  $b_0$  with  $b_1$  in  $\mathcal{P}$ , we obtain a packing  $\mathcal{P}'$  of  $K_{5,5}$  with leave  $a_0b_1$ . Let  $\mathcal{P}_1 = \mathcal{P} \cup \mathcal{P}'$ . One can see that  $\mathcal{P}_1$  is a packing of  $2K_{5,5}$  with leave  $P_3 : \{b_0a_0b_1\}$ . Next, rename the vertices  $a_0, a_1, b_0, b_1$  in  $\mathcal{P}$  to  $a_1, a_0, b_1, b_0$ , respectively, we obtain a packing  $\mathcal{P}''$  of  $K_{5,5}$  with leave  $a_1b_1$ . Let  $\mathcal{P}_2 = \mathcal{P} \cup \mathcal{P}''$ . It is easy to see that  $\mathcal{P}_2$  is a packing of  $2K_{5,5}$  with leave  $M_2 : \{a_0b_0, a_1b_1\}$ . Finally,  $2\mathcal{P}$  is clearly a packing of  $2K_{5,5}$  with leave  $2P_2 : 2\{a_0b_0\}$ .

Now we use packings to construct the required coverings of  $2K_{5,5}$ . Note that  $\mathcal{P}_1 \cup \{(a_0; b_0, b_1, b_2, b_3)\}$  is a  $(C_4, S_4)$ -covering of  $2K_{5,5}$  with padding  $P_3 : \{b_2a_0b_3\}$ , and  $\mathcal{P}_2 \cup \{(a_0, b_0, a_1, b_1)\}$  is a  $(C_4, S_4)$ -covering of  $2K_{5,5}$  with padding  $M_2 : \{a_0b_1, a_1b_0\}$ . Moreover,  $2\mathcal{P} - \{(a_0; b_1, b_2, b_3, b_4), (a_4; b_1, b_2, b_3, b_4)\} \cup \{(a_0, b_0, a_4, b_4), (a_0; b_0, b_1, b_2, b_3), (a_4; b_0, b_1, b_2, b_3)\}$  is a  $(C_4, S_4)$ -covering of  $2K_{5,5}$  with padding  $2P_2 : 2\{b_0a_4\}$ .

**Case 3.**  $r = 3$ .

First, we use packings of  $K_{5,5}$  and  $2K_{5,5}$  to construct the required packings of  $3K_{5,5}$ . Exchanging  $b_0$  with  $b_2$  in  $\mathcal{P}$ , we obtain a packing  $\mathcal{R}$  of  $K_{5,5}$  with leave  $a_0b_2$ . Hence  $\mathcal{P}_1 \cup \mathcal{R}$  is a packing of  $3K_{5,5}$  with leave  $S_3 : \{(a_0; b_0, b_1, b_2)\}$ . Next, rename the vertices  $a_0, a_2, b_0, b_2$  in  $\mathcal{P}$  to  $a_2, a_0, b_2, b_0$ , respectively, we obtain a packing  $\mathcal{R}'$  of  $K_{5,5}$  with leave  $a_2b_2$ . Thus  $\mathcal{P}_2 \cup \mathcal{R}'$  is a packing of  $3K_{5,5}$  with leave  $M_3 : \{a_0b_0, a_1b_1, a_2b_2\}$ . Note that  $\mathcal{P}_1 \cup \mathcal{P}''$  is a packing of  $3K_{5,5}$  with leave  $P_4 : \{b_0a_0b_1a_1\}$ . In addition,  $\mathcal{P}_1 \cup \mathcal{R}'$  is a packing of  $3K_{5,5}$  with leave  $P_3 \uplus P_2 : \{b_0a_0b_1\} \cup \{a_2b_2\}$ ,  $2\mathcal{P} \cup \mathcal{R}'$  is a packing of  $3K_{5,5}$  with leave  $2P_2 \uplus P_2 : 2\{a_0b_0\} \cup \{a_2b_2\}$ ,  $2\mathcal{P} \cup \mathcal{R}$  is a packing of  $3K_{5,5}$  with leave  $2P_2 \odot P_2 : 2\{a_0b_0\} \cup \{a_0b_2\}$ , and  $3\mathcal{P}$  is clearly a packing of  $3K_{5,5}$  with leave  $3P_2 : 3\{a_0b_0\}$ .

Finally, since  $3(5-4)^2 = 3 < 4$ , there exists a  $(C_4, S_4)$ -covering of  $3K_{5,5}$  with leave  $P_2$  by Lemma 3.6.

**Case 4.**  $r = 5$ .

By Corollary 3.12,  $(C_4, S_4) \mid 4K_{5,5}$ . Since  $5K_{5,5} = K_{5,5} \cup 4K_{5,5}$ , it suffices to show that there exists a  $(C_4, S_4)$ -covering of  $5K_{5,5}$  with padding  $3P_2$ . Note that  $5K_{5,5} = 2K_{5,5} \cup 3K_{5,5}$ . Since  $2K_{5,5}$  has a  $(C_4, S_4)$ -covering with padding  $2P_2 : 2\{b_0a_4\}$  and  $3K_{5,5}$  has a  $(C_4, S_4)$ -covering with padding  $P_2$  (say  $\{b_0a_4\}$ ), we have the required covering.  $\square$

**Lemma 4.3** *Let  $r$  be a positive integer and let  $m$  be a positive odd integer with  $m \geq 5$ . If  $rK_{m,m}$  has a  $(C_4, S_4)$ -packing (resp.  $(C_4, S_4)$ -covering) with leave  $L$  (resp. padding  $R$ ), then  $rK_{m+2, m+2}$  also has a  $(C_4, S_4)$ -packing (resp.  $(C_4, S_4)$ -covering) with leave  $L$  (resp. padding  $R$ ).*

**Proof:** Let  $m = 2t + 1$  where  $t$  is a positive integer with  $t \geq 2$ . Let  $A_1 = \{a_0, a_1, \dots, a_{2t}\}$  and

$B_1 = \{b_0, b_1, \dots, b_{2t}\}$ . Letting  $G_1 = K_{m+2, m+2}[A_1 \cup B_1]$  and  $G_2 = K_{m+2, m+2} - E(G_1)$ . Clearly,  $G_1$  is isomorphic to  $K_{m, m}$ . Note that  $\{(a_{2t+1}, b_{2i}, a_{2t+2}, b_{2i+1}), (a_{2i}, b_{2t+1}, a_{2i+1}, b_{2t+2}) : i = 0, 1, \dots, t-2\} \cup \{(a_{2t+1}; b_{2t-2}, b_{2t-1}, b_{2t}, b_{2t+1}), (a_{2t+2}; b_{2t-2}, b_{2t-1}, b_{2t}, b_{2t+2}), (b_{2t+1}; a_{2t-2}, a_{2t-1}, a_{2t}, a_{2t+2}), (b_{2t+2}; a_{2t-2}, a_{2t-1}, a_{2t}, a_{2t+1})\}$  is a  $(C_4, S_4)$ -decomposition of  $G_2$ . Since  $rK_{m+2, m+2} = rG_1 \cup rG_2$ ,  $rK_{m+2, m+2}$  has the required packings and coverings.  $\square$

Now, we are ready for the main result of this section.

**Theorem 4.4** *Let  $\lambda$  and  $n$  be positive integers with  $n \geq 4$ .*

(A)  $\lambda K_{n, n}$  has a maximum  $(C_4, S_4)$ -packing with leave  $L$  if and only if

$$\left\{ \begin{array}{ll} L = \emptyset & \text{if } \lambda n^2 \equiv 0 \pmod{4}, \\ L = P_2 & \text{if } \lambda n^2 \equiv 1 \pmod{4}, \\ L \in \{P_3, M_2, 2P_2\} & \text{if } \lambda n^2 \equiv 2 \pmod{4}, \\ L \in \{S_3, P_4, P_3 \uplus P_2, M_3, 2P_2 \uplus P_2, 2P_2 \odot P_2, 3P_2\} & \text{if } \lambda n^2 \equiv 3 \pmod{4}. \end{array} \right.$$

(B)  $\lambda K_{n, n}$  has a minimum  $(C_4, S_4)$ -covering with padding  $R$  if and only if

$$\left\{ \begin{array}{ll} R = \emptyset & \text{if } \lambda n^2 \equiv 0 \pmod{4}, \\ R \in \{S_3, P_4, P_3 \uplus P_2, M_3, 2P_2 \uplus P_2, 2P_2 \odot P_2\} & \text{if } \lambda n^2 \equiv 1 \pmod{4} \text{ and } \lambda = 1, \\ R \in \{S_3, P_4, P_3 \uplus P_2, M_3, 2P_2 \uplus P_2, 2P_2 \odot P_2, 3P_2\} & \text{if } \lambda n^2 \equiv 1 \pmod{4} \text{ and } \lambda \geq 5, \\ R \in \{P_3, M_2, 2P_2\} & \text{if } \lambda n^2 \equiv 2 \pmod{4}, \\ R = P_2 & \text{if } \lambda n^2 \equiv 3 \pmod{4}. \end{array} \right.$$

**Proof:** The necessity follows from the arguments above Table 1. It suffices to show that  $\lambda K_{n, n}$  has required packings and coverings. The result for  $\lambda n^2 \equiv 0 \pmod{4}$  follows from Corollary 3.12 immediately. So it remains to consider the case  $\lambda n^2 \equiv r \pmod{4}$  for  $r \in \{1, 2, 3\}$ . Note that  $\lambda n^2 \equiv r \pmod{4}$  if and only if  $\lambda \equiv r \pmod{4}$  and  $n \equiv 1 \pmod{2}$ . When  $\lambda \in \{1, 2, 3, 5\}$ , the result for  $n = 5$  follows from Lemma 4.2, and the result for  $n > 5$  can be obtained by using Lemma 4.3 recursively. Now consider  $\lambda \equiv r \pmod{4}$  and  $\lambda > 5$ . Note that  $\lambda K_{n, n} = rK_{n, n} \cup (\lambda - r)K_{n, n}$ . Since  $(\lambda - r)K_{n, n}$  is  $(C_4, S_4)$ -decomposable by Corollary 3.12, we have the result.  $\square$

## References

- [1] A. Abueida, S. Clark, and D. Leach, Multidecomposition of the complete graph into graph pairs of order 4 with various leaves. *Ars Combin.*, 93:403–407, 2009.
- [2] A. Abueida and M. Daven, Multidesigns for graph-pairs of order 4 and 5. *Graphs Combin.*, 19:433–447, 2003.
- [3] A. Abueida and M. Daven, Multidecompositons of the complete graph. *Ars Combin.*, 72:17–22, 2004.
- [4] A. Abueida and M. Daven, Multidecompositions of several graph products. *Graphs Combin.*, 29:315–326, 2013.
- [5] A. Abueida, M. Daven, and K. J. Roblee, Multidesigns of the  $\lambda$ -fold complete graph for graph-pairs of order 4 and 5. *Australas J. Combin.*, 32:125–136, 2005.

- [6] A. Abueida and C. Hampson, Multidecomposition of  $K_n - F$  into graph-pairs of order 5 where  $F$  is a Hamilton cycle or an (almost) 1-factor. *Ars Combin.*, 97:399–416, 2010.
- [7] A. Abueida and T. O’Neil, Multidecomposition of  $\lambda K_m$  into small cycles and claws. *Bull. Inst. Combin. Appl.*, 49:32–40, 2007.
- [8] D.G. Hoffman, The real truth about star designs. *Discrete Math.*, 284:177–180, 2004.
- [9] H.-C. Lee, Multidecompositions of complete bipartite graphs into cycles and stars. *Ars Combin.*, 108:355–364, 2013.
- [10] H.-C. Lee and J.-J. Lin, Decomposition of the complete bipartite graph with a 1-factor removed into cycles and stars. *Discrete Math.*, 313:2354–2358, 2013.
- [11] J. Ma, L. Pu, and H. Shen, Cycle decompositions of  $K_{n,n} - I$ . *SIAM J. Discrete Math.*, 20:603–609, 2006.
- [12] H. M. Priyadharsini and A. Muthusamy,  $(G_m, H_m)$ -multifactorization of  $\lambda K_m$ . *J. Combin. Math. Combin. Comput.*, 69:145–150, 2009.
- [13] H. M. Priyadharsini and A. Muthusamy,  $(G_m, H_m)$ -multidecomposition of  $K_{m,m}(\lambda)$ . *Bull. Inst. Combin. Appl.*, 66:42–48, 2012.
- [14] T.-W. Shyu, Decomposition of complete graphs into paths and stars. *Discrete Math.*, 310:2164–2169, 2010.
- [15] T.-W. Shyu, Decompositions of complete graphs into paths and cycles. *Ars Combin.*, 97:257–270, 2010.
- [16] T.-W. Shyu, Decomposition of complete graphs into paths of length three and triangles. *Ars Combin.*, 107:209–224, 2012.
- [17] T.-W. Shyu, Decomposition of complete graphs into cycles and stars. *Graphs Combin.*, 29:301–313, 2013.
- [18] T.-W. Shyu, Decomposition of complete bipartite graphs into paths and stars with same number of edges. *Discrete Math.*, 313:865–871, 2013.
- [19] D. Soiteau, Decomposition of  $K_{m,n}$  ( $K_{m,n}^*$ ) into cycles (circuits) of length  $2k$ . *J. Combin. Theory, Ser. B*, 30:75–81, 1981.