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# Harmonics for deformed Steenrod operators (Extended Abstract)

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**Abstract.** We explore in this paper the spaces of common zeros of several deformations of Steenrod operators. Proofs are omitted in view of pages limitation for the extended abstract.

**Résumé.** Nous explorons dans cet article l'espace des zéros communs de plusieurs déformations d'opérateurs de Steenrod. Faute de place, les preuves sont omises.

Keywords: Harmonic polynomials, Steenrod Operators

### 1 Introduction

In recent years many authors have studied variations on a striking classical result of invariant theory holding for any finite group W of real  $n \times n$  matrices generated by reflections. Roughly stated, this result asserts that there is a natural decomposition

$$\mathbb{R}[\mathbf{x}] \simeq \mathbb{R}[\mathbf{x}]^W \otimes \mathbb{R}[\mathbf{x}]_W \tag{1}$$

of the ring of polynomials  $\mathbb{R}[\mathbf{x}]$ , in *n* variables  $\mathbf{x} = x_1, x_2, \ldots, x_n$ , as a tensor product of the ring  $\mathbb{R}[\mathbf{x}]^W$ of *W*-invariant polynomials, and the "*W*-coinvariant-space"  $\mathbb{R}[\mathbf{x}]_W$ . This last is simply the space obtained as the quotient of the ring  $\mathbb{R}[\mathbf{x}]$  by the ideal generated by constant-term-free *W*-invariant polynomials. It is well known that  $\mathbb{R}[\mathbf{x}]_W$  is isomorphic as a *W*-module to the space  $\mathcal{H}_W$  of *W*-harmonic polynomials, i.e.: the set of polynomials  $f(\mathbf{x})$  that satisfy all partial differential equations of the form  $p(\partial_{\mathbf{x}})f(\mathbf{x}) = 0$ , where  $p(\partial_{\mathbf{x}})$  any constant-term-free *W*-invariant polynomial in the partial derivatives  $\partial_i$ .

The purpose of this work is to study twisted versions of this setup. Typically, we replace symmetric operators  $\partial_1^k + \ldots + \partial_n^k$ , by operators of the form

$$D_k := \sum_{i=1}^n a_{i,k} x_i \partial_i^{k+1} + b_{i,k} \partial_i^k, \tag{2}$$

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with some parameters  $a_{i,k}$  and  $b_{i,k}$ . We then consider the solution set  $\mathcal{H}_{\mathbf{x}}$  of the system of partial differential equations  $D_k f(\mathbf{x}) = 0$ , for  $k \ge 1$ . Observe that the operators  $D_k$  are homogeneous. We say that they are of degree -k since they lower degree of polynomials by k. It follows that  $\mathcal{H}_{\mathbf{x}}$  is graded by degree. In particular, it makes sense to consider the *Hilbert series* 

$$H_n(t) := \sum_{d \ge 0} t^d \dim(\pi_d(\mathcal{H}_{\mathbf{x}})), \tag{3}$$

with  $\pi_d$  denoting the projection onto the homogeneous component of degree d. Clearly the right-hand side of (3) depends on the choice of the parameters  $a_{i,k}$  and  $b_{i,k}$ . Recall that the Hilbert series of the space  $\mathcal{H}_{\mathfrak{S}_n}$ , of  $\mathfrak{S}_n$ -harmonic polynomials (which corresponds to setting  $a_{i,k} = 0$  and  $b_{i,k} = 1$ ) is the classical t-analog of n!. As we will see later, this is a "generic" value for  $H_n(t)$ .

Before going on with our discussion, let us consider an interesting dual point of view. Following a terminology of Wood [5], we shall say that a polynomial is a *hit-polynomial* if it can be expressed in the form

$$f(\mathbf{x}) = \sum_{k} D_k^* g_k(\mathbf{x}),\tag{4}$$

for some polynomials  $g_k(\mathbf{x})$ , with  $D_k^*$  standing for the dual operator of  $D_k$  with respect to the following scalar product on the ring of polynomials.

For two polynomials f and g in  $\mathbb{R}[\mathbf{x}]$ , one sets

$$\langle f, g \rangle := f(\partial_{\mathbf{x}})g(\mathbf{x})|_{\mathbf{x}=0}.$$
 (5)

In other words, this corresponds to the constant term of the polynomial resulting from the application of the differential operator  $f(\partial_{\mathbf{x}})$  to  $g(\mathbf{x})$ . A straightforward computation reveals that, for two monomials  $\mathbf{x}^{\mathbf{a}}$  and  $\mathbf{x}^{\mathbf{b}}$ , we have  $\langle \mathbf{x}^{\mathbf{a}}, \mathbf{x}^{\mathbf{b}} \rangle = \mathbf{a}!$ , if  $\mathbf{a} = \mathbf{b}$ , and 0 otherwise. Here, as is now almost usual,  $\mathbf{a}!$  stands for  $a_1!a_2!\cdots a_n!$ . This observation makes it clear that (5) indeed defines a scalar product on  $\mathbb{R}[\mathbf{x}]$ . Moreover, the dual of the operator  $\partial_i^k$  is easily checked to be multiplication by  $x_i^k$ . It follows that

$$D_k^* = \sum_{i=1}^n a_{i,k} \, x_i^{k+1} \partial_i + b_{i,k} \, x_i^k.$$

From general basic linear algebra principles, it follows that the space of  $\mathcal{H}_{\mathbf{x}}$ , of general harmonic polynomials, is orthogonal to the space of hit-polynomials. Moreover, since the subspace of hit-polynomials is homogeneous, the corresponding quotient C of  $\mathbb{R}[\mathbf{x}]$ , by this subspace, is isomorphic to  $\mathcal{H}_{\mathbf{x}}$  as a graded space.

For, the special case corresponding to setting  $a_{i,k} = q$ , and  $b_{i,k} = 1$ , for all *i* and *k*, we denote  $\mathcal{H}_{\mathbf{x};q}$  resulting space which has been considered by Hivert and Thiéry (in [3]). Using the notation

$$D_{k;q} := \sum_{i=1}^{n} q \, x_i \partial_i^{k+1} + \partial_i^k,$$

Using a simply Lie-bracket calculation, Hivert and Thiéry have observed that  $\mathcal{H}_{\mathbf{x};q}$  is simply characterized as the common solutions of the two equations  $D_{1;q}f(\mathbf{x}) = 0$ , and  $D_{2;q}f(\mathbf{x}) = 0$ . Recall that the ring of polynomials  $\mathbb{R}[\mathbf{x}]$  can be considered as a  $\mathfrak{S}_n$ -module for the action that corresponds to permutation of

the variables. This action restricts to a natural  $\mathfrak{S}_n$ -action on the space  $\mathcal{H}_{\mathbf{x};q}$ , since the operators  $D_{k;q}$  are symmetric. It is classical that  $\mathcal{H}_{\mathfrak{S}_n} = \mathcal{H}_{\mathbf{x};0}$  is isomorphic, as a  $\mathfrak{S}_n$ -module, to the regular representation of  $\mathfrak{S}_n$ . Hivert-Thiéry go on to state that

**Conjecture 1 (Hivert-Thiéry)** As  $\mathfrak{S}_n$ -modules, the spaces  $\mathcal{H}_{\mathbf{x};q}$  is isomorphic to  $\mathcal{H}_{\mathfrak{S}_n}$ , when q > 0. In particular, this implies that the Hilbert series of  $\mathcal{H}_{\mathbf{x};q}$  is  $[n]!_t$ .

It follows from (1) that the graded Frobenius characteristic  $F_n(t)$  of  $\mathcal{H}_{\mathbf{x};q}$  (and  $\mathcal{H}_{\mathfrak{S}_n}$ ) is

$$F_n(t) = [n]!_t (1-t)^n \sum_{\lambda \vdash n} \prod_{k=1}^n \frac{1}{d_k!} \left(\frac{p_k}{k (1-t^k)}\right)^{d_k},$$
(6)

where  $d_k = d_k(\lambda)$  is the number of size k parts of  $\lambda$ .

In this work we generalize and extend the scope of the above conjecture to include the more general operators of (2). Along the way we prove several related results.

# 2 Tilde-Harmonics and Hat-Harmonics

We first consider another interesting special case of (2). Namely, we suppose that all  $b_{i,k}$ 's vanish, and all  $a_{i,k}$ 's are equal to 1. Thus, we consider the space of common zeros of the operators  $\widetilde{D}_k := \sum_{i=1}^n x_i \partial_i^{k+1}$ , which is called the space of *tilde-harmonics*, and denoted  $\widetilde{\mathcal{H}}_{\mathbf{x}}$ . We easily check that

$$[D_k, D_j] = (k - j)D_{k+j},$$
(7)

hence  $\widetilde{\mathcal{H}}_{\mathbf{x}}$  is simply the set of common zeros of the two equations  $\widetilde{D}_1 f(\mathbf{x}) = 0$ , and  $\widetilde{D}_2 f(\mathbf{x}) = 0$ . The space  $\widetilde{\mathcal{H}}_{\mathbf{x}}$  affords a natural action of the symmetric group, and the associated graded Frobenius characteristic is denoted  $\widetilde{F}_n(t)$ . Computer experimentations suggest that the Hilbert series of  $\widetilde{\mathcal{H}}_{\mathbf{x}}$  seems to be

$$\widetilde{\mathcal{H}}_{n}(t) = \sum_{k=0}^{n} \binom{n}{k} t^{k} [k]_{t}!.$$
(8)

Modulo a natural conjecture, this follows from a very explicit description of  $\hat{\mathcal{H}}_{\mathbf{x}}$  outlined below. To state it we need one more family of operators and yet another version of harmonic polynomials. For each  $k \ge 1$ , consider the operator  $\hat{D}_k = \sum_{i=1}^n x_i \partial_i^{k+1} + (k+1) \partial_i^k$ , and introduce the space

$$\widehat{\mathcal{H}}_{\mathbf{x}} := \big\{ f(\mathbf{x}) \in \mathbb{R}[\mathbf{x}] \mid \widehat{D}_k f(\mathbf{x}) = 0, \quad \forall k \ge 1 \big\},\$$

whose elements are said to be "hat-harmonics". We will soon relate the two notions of tilde and hat harmonics. Experimentation suggest that  $\hat{\mathcal{H}}_{\mathbf{x}}$  has dimension n!, and that even more precisely we have the following.

**Conjecture 2** As a graded  $\mathfrak{S}_n$ -module,  $\widehat{\mathcal{H}}_{\mathbf{x}}$  is isomorphic to the space of  $\mathfrak{S}_n$ -harmonics.

Now, for any given k-subset y of the n variables x, let us consider the space  $\hat{H}_{y}$ , and write

$$e_{\mathbf{y}} := \prod_{x \in \mathbf{y}} x,$$

for the elementary symmetric polynomial of degree k in the variables y. As usual, we define the *support* of a monomial to be the set of variable that appear in it, with non-zero exponent. Clearly,  $y^{a}$  has support y if and only if  $y^{a} = e_{y} y^{b}$ , for some b. Observe that we have the operator identity we can easily check the operator identity

$$\widetilde{D}_k e_{\mathbf{x}} = e_{\mathbf{x}} \, \widehat{D}_k,\tag{9}$$

where  $e_x$  stands for the operator of multiplication by  $e_x$ . We can now state the following remarkable fact.

**Theorem 1** The space of tilde-harmonics has the direct sum decomposition

$$\widetilde{\mathcal{H}}_{\mathbf{x}} = \bigoplus_{\mathbf{y} \subseteq \mathbf{x}} e_{\mathbf{y}} \widehat{\mathcal{H}}_{\mathbf{y}},\tag{10}$$

*if we consider that hat-harmonics for*  $\mathbf{y} = \emptyset$  *are simply the scalars.* 

The same holds for the more general case of operators  $a_k \widetilde{D}_k$  and  $a_k \widehat{D}_k$ , with the  $a_k$ 's equal to 0 or 1. The intent here is to restrict the set of equations considered to those k for which  $a_k$  takes the value 1. The corresponding spaces are denoted  $\widetilde{\mathcal{H}}^{\mathbf{a}}_{\mathbf{x}}$  and  $\widehat{\mathbf{H}}^{\mathbf{a}}_{\mathbf{x}}$ , with similar convention for the corresponding Hilbert series and graded Frobenius characteristics. It follows that, even in this more general context, we have

**Corollary 2** For all choices of  $a_k$ ,

$$\widetilde{\mathcal{H}}_{n}^{\mathbf{a}}(t) = \sum_{k=0}^{n} \binom{n}{k} t^{k} \, \widehat{\mathcal{H}}_{k}^{\mathbf{a}}(t).$$
(11)

In particular, if conjecture 2 holds then (8) holds. There is an even finer corollary of Theorem 1.

**Corollary 3** The graded Frobenius characteristic of  $\widetilde{\mathcal{H}}^{\mathbf{a}}_{\mathbf{x}}$  is given by the symmetric function

$$\widetilde{F}_{n}^{\mathbf{a}}(t) = \sum_{k=0}^{n} t^{k} \widehat{F}_{k}^{\mathbf{a}}(t) h_{n-k}(\mathbf{z})$$
(12)

A conjecture of Wood [5, conjecture 7.3] is thus partially addressed in a very explicit manner. Indeed, in view of Theorem 1, Wood's conjecture is a consequence of Conjecture 2 and the fact that  $\widetilde{C}$  is isomorphic to  $\widetilde{\mathcal{H}}_{\mathbf{x}}$  as a graded  $\mathfrak{S}_n$ -module.

# 3 More on *q*-harmonics

We now link the study of harmonics of the  $\widetilde{D}_k$  to further our understanding of the common zeros of the operators  $D_{k:q}$ , in the case when q is considered as a formal parameter. Our point of departure is the following important fact. Denote by  $\nabla_k := \sum_{i=1}^n \partial_i^k$  the generalized Laplacian, and observe that  $D_{k:q} = q\widetilde{D}_k + \nabla_k$ , then we get the following.

**Theorem 4** Up to a power of q, every q-harmonic polynomial f may be written in the form

$$f = f_0 + qf_1 + q^2f_2 + \dots + q^m f_m$$
(13)

with  $f_i \in \mathbb{R}[\mathbf{x}]$ , and such that for all  $k \geq 1$  we have

(a) 
$$\nabla_k f_0 = 0$$
  
(b)  $\nabla_k f_i = -\widetilde{D}_k f_{i-1}$ , for all  $i = 2, \dots, m-1$ , (14)  
(c)  $\widetilde{D}_k f_m = 0$ .

In particular, it follows that for any  $r \ge 0$ , and any choice of  $k_1, k_2, \ldots, k_r \ge 1$ , the element

$$\nabla_{k_1} \nabla_{k_2} \cdots \nabla_{k_r} f_r \tag{15}$$

is a  $\mathfrak{S}_n$ -harmonic polynomial in the usual sense.

Let us now reformulate the expansion of (13) in the form

$$f = q^r (f_0 + q f_1 + \dots + q^m f_m) \qquad (\text{with each } f_i \in \mathbb{R}[\mathbf{x}], \ f_i \neq 0)$$

We call  $f_0$  the *first term* of f and denote it "FT(f)". Analogously we say that  $f_m$  is the *last term* of f and denote it "LT(f)". The integer m will be called the *length* of f. We also set

$$\mathcal{H}_{\mathbf{x}}^{F} := \mathcal{L}[\mathrm{FT}(f) \mid f \in \mathcal{H}_{\mathbf{x};q}] \quad \text{and} \quad \mathcal{H}_{\mathbf{x}}^{L} := \mathcal{L}[\mathrm{LT}(f) \mid f \in \mathcal{H}_{\mathbf{x};q}]$$
(16)

to respectively stand for the span of first terms of q-harmonics and last terms. Using a theorem of [3] we then get the following remarkable corollary.

**Corollary 5** The three spaces  $\mathcal{H}_{\mathbf{x}}^{F}$ ,  $\mathcal{H}_{\mathbf{x}}^{L}$  and  $\mathcal{H}_{\mathbf{x};q}$  are isomorphic as graded  $\mathfrak{S}_{n}$ -modules and therefore they are all isomorphic to a submodule of the Harmonics of  $\mathfrak{S}_{n}$ .

Since the dimension of  $\mathcal{H}_{\mathbf{x};q}$  is thus bounded above, the single equality dim  $\mathcal{H}_{\mathbf{x};q} = n!$  would imply that  $\mathcal{H}_{\mathbf{x};q}$  affords the regular representation of  $\mathfrak{S}_n$ . In particular this would yield that  $\mathcal{H}_{\mathbf{x}}^F$  is none other than the space of harmonics of  $\mathfrak{S}_n$ . Since  $\mathcal{H}_{\mathbf{x};q}$  is isomorphic to  $\mathcal{H}_{\mathbf{x}}^F$ , as a graded  $\mathfrak{S}_n$ -module, it would follow that  $\mathcal{H}_{\mathbf{x};q}$  itself is isomorphic to the space of harmonics of  $\mathfrak{S}_n$  (as a graded  $\mathfrak{S}_n$ -module). Thus the Hivert-Thiéry conjecture would result.

# 4 The Kernel of $D_k$

To compute the general space  $\mathcal{H}_{\mathbf{x}}$  of harmonic polynomials, we need to find common solutions of the differential equations  $D_k f(\mathbf{x}) = 0$ , for k > 0. For each k, the kernel of the operator  $D_k$  may be given a precise explicit description whenever  $a_{i,k} d + b_{i,k} \neq 0$ , for all  $d \in \mathbb{N}$ . We lighten the notation by writing simply  $a_i$  instead of  $a_{i,k}$ .

The case k = 1 illustrates all aspects of the method. We construct a set

$$\{\mathbf{y}^{\mathbf{r}} + \Psi_1(\mathbf{y}^{\mathbf{r}})\}_{\mathbf{r} \in \mathbb{N}^{n-1}}$$
(17)

which is a basis of the solution set of  $D_1 f(\mathbf{x}) = 0$ . Here,  $\Psi_1$  is a linear operator described below. Simply writing x for  $x_n$ , and y for the set of variables  $x_1, \ldots, x_{n-1}$ , we expand  $f \in \mathbb{R}[\mathbf{x}]$  as polynomials in x:

$$f = \sum_{d} f_{d} \frac{x^{d}}{d!}, \quad \text{with } f_{d} \in \mathbb{R}[\mathbf{y}].$$
(18)

The effect of  $D_1$  can then be described in the format

$$D_1\left(\sum_d f_d \frac{x^d}{d!}\right) = \sum_d \left[D_1(f_d) + (d\,a_n + b_n)f_{d+1}\right] \frac{x^d}{d!}.$$
(19)

Setting  $a := a_n$  and  $b := b_n$ , we now assume that  $a d + b \neq 0$ , for all  $d \in \mathbb{N}$ . Then, the right-hand side of (19) vanishes if and only we choose f to be such that

$$f_{d+1} = \frac{-1}{a\,d+b}\,D_1(f_d),\tag{20}$$

for all  $d \ge 0$ . Unfolding this recurrence for the  $f_d$ 's, we find that every element of the kernel of  $D_1$  can be written as  $f_0 + \Psi_1(f_0)$ , if we define the linear operator  $\Psi_1$  as

$$\Psi_1(g) := \sum_{m \ge 1} (-1)^m \frac{D_1^m(g)}{[a;b]_m} \frac{x^m}{m!}, \quad \text{for } g \in \mathbb{R}[\mathbf{y}].$$
(21)

Here we use the notation  $[a; b]_m := b(a+b)(2a+b)\cdots((m-1)a+b)$ . This leads to the following theorem.

**Theorem 6** The collection of polynomials  $\mathbf{y}^{\mathbf{r}} + \Psi_1(\mathbf{y}^{\mathbf{r}})$  is a basis for the kernel of  $D_1$ . In fact, given any polynomial f in the kernel of  $D_1$ , its expansion in terms of this basis is simply obtained as

$$f = \sum_{\mathbf{r}} a_{\mathbf{r}} (\mathbf{y}^{\mathbf{r}} + \Psi_1(\mathbf{y}^{\mathbf{r}}))$$
(22)

with  $(f \mod x) = \sum_{\mathbf{r}} a_{\mathbf{r}} \mathbf{y}^{\mathbf{r}}$ .

It follows readily that, whenever  $a d + b \neq 0$  for all  $d \in \mathbb{N}$ , the Hilbert series of the dimension of the kernel of  $D_1$  is  $(1-t)^{1-n}$ . In view of Theorem 1, this implies that the Hilbert series of the kernel of  $\widetilde{D}_1$  is

$$1 + \sum_{k=1}^{n} \binom{n}{k} t^{k} \frac{1}{(1-t)^{k-1}}.$$
(23)

In fact, we can get an explicit description of this kernel using (9).

We can generalize formula (21) to get a description of the kernel of  $D_k$  as follows. Observe as before that

$$D_k\left(\sum_{d} f_d \frac{x^d}{d!}\right) = \sum_{d} \left[D_k(f_d) + (a\,d+b)f_{d+k}\right] \frac{x^d}{d!}.$$
(24)

For this expression to be zero, we must have

$$f_{d+k} = \frac{-1}{a\,d+b}\,D_k(f_d),$$

with the same conditions as before on a and b. This recurrence has a unique solution given initial values for  $f_d$ ,  $0 \le d \le k - 1$ . Clearly these can be fixed at leisure. Substituting the solution of the recurrence in f, we get an element of the kernel of  $D_k$  if and only if f is of the form

$$f = (f \mod x^k) + \Psi_k(f \mod x^k),$$

with  $\Psi_k$  the linear operator defined as

$$\Psi_k\left(\sum_{r=0}^{k-1} f_r \frac{x^r}{r!}\right) := \sum_{m \ge 1} \sum_{r=0}^{k-1} (-1)^m \frac{D_k^m(f_r)}{[a\,k; a\,r+b]_m} \frac{x^{k\,m+r}}{(k\,m+r)!}.$$
(25)

In particular, it follows that the Hilbert series of the kernel of  $D_k$  is  $(1 + t + ... t^{k-1}) (1 - t)^{1-n}$ .

# 5 Some explicit harmonic polynomials

Common zeros of all  $D_k$ 's are exactly what we are looking for. Some of these are easy to find when the  $D_k$ 's are symmetric. Let  $\lambda$  be any partition of n, and consider a tableau  $\tau$  of shape  $\lambda$ , this is to say a bijection

$$\tau: \lambda \longrightarrow \{1, 2, \dots, n\},\$$

with  $\lambda$  identified with the set of cells of its Ferrers diagram. Recall that, for  $\lambda = \lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_k > 0$ , the *cells* of  $\lambda$  are the *n* pairs (i, j) in  $\mathbb{N}^2$ , such that

$$1 \le i \le \lambda_j, \qquad 1 \le j \le k.$$

The value  $\tau(i, j)$  is called an *entry* of  $\tau$ , and it is said to lie in *column* i of  $\tau$ . The *Garnir polynomial* of a  $\lambda$ -shape tableau  $\tau$ , is defined to be

$$\Delta_{\tau}(\mathbf{x}) := \prod_{i, j < k} (x_{\tau(i,j)} - x_{\tau(i,k)}).$$

In other terms, the factors that appear in  $\Delta_{\tau}(\mathbf{x})$  are differences of entries of  $\tau$  that lie in the same column.

Now, define  $\mathcal{V}_{\lambda}$  to be the linear span of the polynomials  $\Delta_{\tau}$ , for  $\tau$  varying in the set of tableaux of shape  $\lambda$ . In formula,

$$\mathcal{V}_{\lambda} := \mathbb{R}\{\Delta_{\tau} \mid \tau \text{ tableau of shape } \lambda\}.$$

In other words,  $\mathcal{V}_{\lambda}$  is the linear span of the  $\Delta_{\tau}$ . It is well known that this homogeneous (invariant) subspace is an irreducible representation of of  $\mathfrak{S}_n$  of dimension equal to the number of standard Young tableaux. Moreover, in the ring  $\mathbb{R}[\mathbf{x}]$ , there exists no isomorphic copy of this irreducible representation lying in some homogeneous component of degree lower then that in which lies  $\mathcal{V}_{\lambda}$ . It is easy to check that the degrees of all of the  $\Delta_{\tau}$ 's, for a tableau of shape  $\lambda$ , are all equal to

$$\sum_{i=1}^{\ell(\lambda)} (i-1)\,\lambda_i,$$

which is usually denoted  $n(\lambda)$  in the literature (see [4]). This is the smallest possible value for the cocharge of a standard tableau of shape  $\lambda$ . This fact has the following easy implication.

**Proposition 7** For any tableau  $\tau$  of shape  $\lambda$ , the Garnir polynomial  $\Delta_{\tau}(\mathbf{x})$  is a zero of  $D_k$ , for  $k \ge 1$ , whenever  $D_k$  is symmetric.

A direct consequence of this is that there is at least one copy of each irreducible representation of  $\mathfrak{S}_n$ in  $\mathcal{H}_x$ , when the  $D_k$ 's are all symmetric. Moreover, under the same conditions, we have

$$\sum_{\lambda \vdash n} f_{\lambda} t^{n(\lambda)} \ll H_n(t)$$

with " $\ll$ " denoting coefficient wise inequality, and  $H_n(t)$  as in (3).

# 6 A new regular sequence and a universal dimension bound

The goal of this section is to establish a bound for the dimension of  $\mathcal{H}_{\mathbf{x};q}$  which is valid for all values of q. To carry this out we need some auxiliary results from commutative algebra. Let  $\mathbb{F}$  be an algebraically closed field and let  $\theta_1(\mathbf{x}), \theta_2(\mathbf{x}), \ldots, \theta_n(\mathbf{x})$  be homogeneous polynomials of  $\mathbb{F}[\mathbf{x}]$  of respective degrees  $d_1, d_2, \ldots, d_n$ . The following result is basic.

**Proposition 8** The polynomials  $\theta_1(\mathbf{x}), \theta_2(\mathbf{x}), \ldots, \theta_n(\mathbf{x})$  form a regular sequence in  $\mathbb{F}[\mathbf{x}]$  if and only if the system of equations

$$\theta_1(\mathbf{x}) = 0, \ \theta_2(\mathbf{x}) = 0, \ \dots, \ \theta_n(\mathbf{x}) = 0$$

has, for  $\mathbf{x} \in \mathbb{F}^n$ , the unique solution

$$x_1 = 0, x_2 = 0, \ldots, x_n = 0$$

We next make use of this proposition to study the sequence of polynomials

$$\varphi_m(\mathbf{x}) := \sum_{i=1}^n a_i x_i^m,$$

for  $m \ge 0$ . More precisely we seek to obtain conditions on the coefficient sequence

$$\mathbf{a} = (a_1, a_2, \dots, a_n) \in \mathbb{F}^n \tag{26}$$

which assure that, for a given  $k \ge 1$ , that the polynomials

$$\varphi_k(\mathbf{x}), \ \varphi_{k+1}(\mathbf{x}), \ \dots, \ \varphi_{k+n-1}(\mathbf{x})$$

form a regular sequence in  $\mathbb{F}[\mathbf{x}]$ .

We first observe that the polynomials  $\varphi_m(\mathbf{x})$ , for m > n, may be expressed in term of the  $\varphi_k(\mathbf{x})$ 's, for  $1 \le k \le n$ . Indeed, recall that the ordinary elementary symmetric functions  $e_r(\mathbf{x})$  may be presented in the form of the identity

$$(t-x_1)(t-x_2)\cdots(t-x_n) = \sum_{r=0}^n (-1)^r e_r(\mathbf{x}) t^{n-r}.$$

Setting  $t = x_i$ , we obtain

$$\sum_{r=0}^{n} (-1)^{r} e_{r}(\mathbf{x}) \, x_{i}^{n-r} = 0.$$

Multiplying both sides by  $a_i x_i^{m-n}$  and isolating  $a_i x_i^m$ , we get

$$a_i x_i^m = -\sum_{r=1}^n (-1)^r e_r(\mathbf{x}) a_i x_i^{m-r}.$$

Thus, summing up on *i*, the following recurrence results

$$\varphi_m(\mathbf{x}) = \sum_{r=1}^n (-1)^{r+1} e_r(\mathbf{x}) \,\varphi_{m-r}(\mathbf{x}). \tag{27}$$

Unfolding this recurrence, we conclude that  $\varphi_m$  lies in the ideal  $(\varphi_1, \varphi_2, \ldots, \varphi_n)_{\mathbb{F}[\mathbf{x}]}$ , for all  $m \ge 1$ .

Remark 1 It is interesting to observe that identity (27) yields that

$$\varphi_1(\mathbf{x}), \, \varphi_2(\mathbf{x}), \, \dots, \, \varphi_n(\mathbf{x})$$
 (28)

is never a regular sequence when  $a_1 + a_2 + \cdots + a_n = 0$ . Indeed, setting m = n in (27), we get

$$\varphi_m(\mathbf{x}) = \sum_{r=1}^{n-1} \varphi_{m-r}(\mathbf{x})(-1)^{r+1} e_r(\mathbf{x}) + (-1)^{n+1} e_n(\mathbf{x}) (a_1 + a_2 + \dots + a_n)$$

and thus the vanishing of  $a_1 + a_2 + \cdots + a_n$  forces  $\varphi_n(\mathbf{x})$  to vanish modulo the ideal

$$(\varphi_1,\varphi_2,\ldots,\varphi_{n-1})_{\mathbb{F}[\mathbf{x}]}.$$

Let us now denote

$$\Phi_n^k := (\varphi_k, \varphi_{k+1}, \ldots, \varphi_{k+n-1})_{\mathbb{F}[\mathbf{x}]},$$

the ideal in  $\mathbb{F}[\mathbf{x}]$  generated by the *n* polynomials  $\varphi_{\ell}(\mathbf{x})$ , with  $k \leq \ell \leq k + n - 1$ . We also write  $\Phi_n$  for  $\Phi_n^1$ . Proposition 8 and (27) combine to yield the following remarkable result.

**Theorem 9** For any  $k \ge 1$  the sequence

$$\varphi_k(\mathbf{x}), \ \varphi_{k+1}(\mathbf{x}), \ \dots, \ \varphi_{k+n-1}(\mathbf{x}),$$
(29)

is regular if and only if the sequence

$$\varphi_1(\mathbf{x}), \ \varphi_2(\mathbf{x}), \ \ldots, \ \varphi_n(\mathbf{x}),$$
(30)

is regular.

This given, here and after we need only be concerned with finding conditions on  $a_1, a_2, \ldots, a_n$  that assure the regularity of sequence  $\varphi_1, \varphi_2, \ldots, \varphi_n$ . The following result offers a useful criterion.

**Theorem 10** In the ring  $\mathbb{F}[\mathbf{x}]$ , the polynomials

$$\varphi_1, \varphi_2, \ldots, \varphi_n$$

form a regular sequence if and only if we have

$$x_i^{\binom{n}{2}+1} \in \Phi_n. \tag{31}$$

When this happens we have the Hilbert series equalities

$$F_{\mathbb{F}[x]/\Phi_n^k}(t) = [k]_t [k+1]_t \cdots [k+n-1]_t$$
(32)

and, in particular,

$$\dim \mathbb{F}[x]/\Phi_n^k = (k)(k+1)\cdots(k+n-1).$$

Going along the lines of Remark 1, we are now ready to assert the following characterization of the  $a_i$ 's for which we have regularity.

**Theorem 11** For k > 1, the sequences

$$\varphi_k, \varphi_{k+1}, \dots, \varphi_{k+n-1} \tag{33}$$

is regular if and only if we have

$$a_{i_1} + a_{i_2} + \dots + a_{i_k} \neq 0, \tag{34}$$

for all  $1 \le i_1 < i_2 < \cdots < i_k \le n$ .

We intend to derive the consequences of this assumption in the theory of q-harmonics. First, we simply reformulated every statement modulo the substitution of variables

$$(a_1, a_2, \dots, a_n) \quad \mapsto \quad \mathbf{x} = (x_1, x_2, \dots, x_n),$$
$$\mathbf{x} \quad \mapsto \quad \xi = (\xi_1, \xi_2, \dots, \xi_n),$$

and we now have

$$\Phi_n^k = (\varphi_k(\xi), \varphi_{k+1}(\xi), \dots, \varphi_{k+n-1}(\xi))_{\mathbb{F}_{\mathbf{x}}[\xi]}.$$

This given, from Theorem (10) we can derive the following facts about the ring

$$\mathbb{F}_{\mathbf{x}}[\xi_1,\xi_2,\ldots,\xi_n],$$

where now,  $\mathbb{F}_{\mathbf{x}}$  denotes the field of rational functions in  $\mathbf{x}$  with coefficients in  $\mathbb{F}.$ 

#### Theorem 12 Let

 $u_1(\xi), u_2(\xi), \ldots, u_{(n+1)!}(\xi)$ 

be a monomial basis for the quotient

$$\mathbb{F}_{\mathbf{x}}[\xi]/\Phi_n^k,$$

and let  $deg(u_i) = d_i$ . Then every polynomial  $f(\xi) \in \mathbb{F}_{\mathbf{x}}[\xi]$ , which is homogeneous of degree d, has a unique expansion of the form

$$f(\xi) = \sum_{i=1}^{(n+1)!} u_i(\xi) \sum_{\sum_k r_k(k+1) = d - d_i} a_{i;\mathbf{r}}(\mathbf{x}) \varphi_1^{r_1}(\xi), \varphi_2^{r_2}(\xi) \cdots \varphi_n^{r_n}(\xi),$$
(35)

where the coefficients  $a_{i;\mathbf{r}}(\mathbf{x})$  are rational functions of  $\mathbf{x}$ , for  $\mathbf{r} \in \mathbb{N}^n$ . In particular if  $d > \binom{n+1}{2}$  then

$$f(\xi) \equiv 0 \mod \Phi_n^k. \tag{36}$$

Let us now denote by  $\mathcal{D}(\mathbf{x})$  the algebra of differential operators with coefficients in  $\mathbb{F}_{\mathbf{x}}$ . Moreover, let  $\mathcal{D}_d(\mathbf{x})$  denote the subspace of  $\mathcal{D}(\mathbf{x})$  consisting of operators of order d. More precisely we have  $D \in \mathcal{D}_d(\mathbf{x})$  if and only if D may be expanded in the form

$$D = \sum_{|\mathbf{r}| \le d} a_{\mathbf{r}}(\mathbf{x}) \,\partial_{\mathbf{x}}^{\mathbf{r}} \tag{37}$$

with coefficients  $a_{\mathbf{r}}(\mathbf{x}) \in \mathbb{F}_{\mathbf{x}}$  such that  $a_{\mathbf{r}}(\mathbf{x}) \neq 0$  at least once when  $|\mathbf{r}| = d$ . We are here extending our vectorial notation to operators, so that

$$\partial_{\mathbf{x}}^{\mathbf{r}} = \partial_1^{r_1} \partial_2^{r_2} \cdots \partial_n^{r_n}$$

is an operator of order  $|\mathbf{r}| = r_1 + r_2 + \ldots + r_n$ . The degree condition in (37) imply that the polynomial

$$\sigma(D) := \sum_{|\mathbf{r}|=d} a_{\mathbf{r}}(\mathbf{x}) \, \xi^{\mathbf{r}}.$$

does not identically vanish. We will refer to  $\sigma(D)$  as the "symbol" of D.

This given, as a corollary of Theorem (10), we obtain the following basic result for Steenrod operators

**Theorem 13** Every operator  $D \in \mathcal{D}_d(\mathbf{x})$  has an expansion of the form

$$D = \sum_{i=1}^{(n+1)!} \sum_{\sum_{\ell} r_k(k+1) \le d-d_i} a_{i;\mathbf{r}}(\mathbf{x}) u_i(\partial_{\mathbf{x}}) D_{1;q}^{r_1} D_{2;q}^{r_2} \cdots D_{n;q}^{r_n}$$

where  $d_i = \deg(u_i)$  and  $a_{i;\mathbf{r}}(\mathbf{x}) \in \mathbb{F}_{\mathbf{x}}$ . Note that this holds true for any rational value of q.

We may now establish the main goal of this section.

**Theorem 14** For any value of q the dimension of the space of q-Harmonic polynomials in  $\mathbf{x}$  does not exceed (n + 1)!

# 7 Last Considerations

Further computer experiments suggest that we have

**Conjecture 3** The set  $\mathcal{D}_n^{\mathbf{a}}$  of common polynomial zeros of the operators

$$\sum_{i=1}^n a_i \,\partial_{x_i}^k \partial_{y_i}^j,$$

for all  $k, j \in \mathbb{N}$  such that k + j > 0, is of a bigraded space of dimension  $(n + 1)^{n-1}$ , whenever we have  $\mathbf{a} = (a_1, \dots, a_n)$  such that

$$\sum_{k \in K} a_k \neq 0, \tag{38}$$

for all nonempty subsets K of  $\{1, \ldots, n\}$ .

Another interesting experimental observation concerning the space of common zeros of  $D_1$  and  $D_2$  with general operators

$$D_1 := \sum_{i=1}^n a_i x_i \partial_i^2 + b_i \partial_i,$$
  
$$D_2 := \sum_{i=1}^n c_i x_i \partial_i^3 + d_i \partial_i^2,$$

is that there seem to be conditions, similar to (38), for which this space is always n!-dimensional.

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