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A preorder-free construction of the Kazhdan-Lusztig representations of Hecke algebras $H_n(q)$ of symmetric groups

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Abstract. We use a quantum analog of the polynomial ring $\mathbb{Z}[x_{1,1}, \ldots, x_{n,n}]$ to modify the Kazhdan-Lusztig construction of irreducible $H_n(q)$-modules. This modified construction produces exactly the same matrices as the original construction in [*Invent. Math.* **53** (1979)], but does not employ the Kazhdan-Lusztig preorders. Our main result is dependent on new vanishing results for immanants in the quantum polynomial ring.

Résumé. Nous utilisons un analogue quantique de l'anneau $\mathbb{Z}[x_{1,1}, \ldots, x_{n,n}]$ pour modifier la construction Kazhdan-Lusztig des modules-$H_n(q)$ irreductibles. Cette construction modifiée produit exactement les mêmes matrices que la construction originale dans [*Invent. Math.* **53** (1979)], mais sans employer les préordres de Kazhdan-Lusztig. Notre résultat principal dépend de nouveaux résultats de disparaition pour des immanants dans l'anneau polynôme de quantique.

Resumen. Utilizamos un analog cuántico del anillo $\mathbb{Z}[x_{1,1}, \ldots, x_{n,n}]$ para modificar la construcción de Kazhdan-Lusztig de módulos-$H_n(q)$ irreducibles. Esta construcción modificada produce exactamente las mismas matrices que la construcción original en [*Invent. Math.* **53** (1979)], pero sin emplear los preórdenes de Kazhdan-Lusztig. Nuestro resultado principal es depende en los nuevos resultados de desaparición para los imanantes en el anillo polinómico del cuántico.

Keywords: Kazhdan-Lusztig, immanants, irreducible representations, Hecke algebra

1 Introduction

In 1979, Kazhdan and Lusztig introduced [8] a family of modules for Coxeter groups and related Hecke algebras. These modules, which happen to be irreducible for Coxeter groups of type-A and have many fascinating properties, also aid in the understanding of modules for quantum groups and other algebras. Important ingredients in the construction of the Kazhdan-Lusztig modules are the computation of certain polynomials in $\mathbb{Z}[q]$ known as *Kazhdan-Lusztig polynomials*, and the description of preorders on Coxeter group elements known as the *Kazhdan-Lusztig preorders*. These two tasks, which present something of an obstacle to one wishing to construct the modules, have become fascinating research topics in their own right. Even in the simplest case of a Coxeter group, the symmetric group S_n, the Kazhdan-Lusztig polynomials and preorders are somewhat poorly understood, see [2, Chp. 6], [13].

As an alternative to the "traditional" Kazhdan-Lusztig construction of type-A modules in terms of subspaces of the type-A Hecke algebra $H_n(q)$, one may construct modules in terms of subspaces of a non-commutative "quantum polynomial ring". Theoretically, this alternative offers no special advantage over the

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original construction. On the other hand, a simple modification of this alternative completely eliminates the need for the Kazhdan-Lusztig preorders in a new construction of $H_n(q)$-modules.

In Sections 2-3, we review essential definitions for the symmetric group, Hecke algebra, and Kazhdan-Lusztig modules. In Section 4 we review definitions related to a quantum analog of the polynomial ring $\mathbb{Z}[x_{1,1}, \ldots, x_{n,n}]$ and a particular $n!$-dimensional subspace called the *quantum immanant space*. In Section 4, we use the basis of Kazhdan-Lusztig immanants studied in [10] to transfer the traditional Kazhdan-Lusztig representations of $H_n(q)$ to the immanant space.

Results of Clausen [4] will then motivate us to modify the above representations in Section 5 and to apply vanishing properties of Kazhdan-Lusztig immanants similar to those obtained in [11]. This leads to our main result that the resulting representations, which do not rely upon the Kazhdan-Lusztig preorders, have matrices equal to those corresonding to the original Kazhdan-Lusztig representations in [8].

2 Tableaux and the symmetric group

We call a weakly decreasing sequence $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ of positive integers with $\sum_{i=1}^{\ell} \lambda_i = r$ an *integer partition* of r, and we denote this by $\lambda \vdash r$ or $|\lambda| = r$. A partial ordering on integer partitions of r called *dominance order* is given by $\lambda \succeq \mu$ if and only if

$$\lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i, \text{ for all } i = 1, \ldots, \ell. \tag{1}$$

From an integer partition λ we can construct a *Ferrers diagram* which has λ_i left justified dots in row i. When we replace the dots in a diagram with $1, \ldots, r$ we have a *Young tableau* where the *shape* of the tableau is λ. An *injective tableau* is merely one in which the replacing is performed injectively, i.e. the $1, \ldots, r$ appear exactly once in the tableau. We call a tableau *column-(semi)strict* if its entries are (weakly) increasing downward in columns. A tableau is *row-(semi)strict* if entries (weakly) increase from left to right in rows. We call a tableau *semistandard* if it is column-strict and row-semistrict, and *standard* if it is semistandard and injective. We define transposition of partitions $\lambda \mapsto \lambda^\top$ (also known as conjugation) and tableaux $T \mapsto T^\top$ in a manner analogous to matrix transposition. We define a *bitableau* to be a pair of tableaux of the same shape, and say that it posesses a certain tableau property if both of its tableaux posess this property.

For each partition λ we define the *superstandard tableau* of shape λ to be the tableau $T(\lambda)$ having entries in reading order. For example,

$$T((4,2,1)) = \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 5 & 6 & & \\ 7 & & & \end{array}. \tag{2}$$

The standard presentation of S_n is given by generators $s_1, \ldots, s_{n-1}$ and relations

$$\begin{aligned} s_i^2 &= 1, && \text{for } i = 1, \ldots, n-1, \\ s_i s_j s_i &= s_j s_i s_j, && \text{if } |i-j| = 1, \\ s_i s_j &= s_j s_i, && \text{if } |i-j| \geq 2. \end{aligned} \tag{3}$$

Let S_n act on rearrangements of the letters $[n] = \{1, \ldots, n\}$ by

$$s_i \circ v_1 \cdots v_n \underset{\text{def}}{=} v_1 \cdots v_{i-1} v_{i+1} v_i v_{i+2} \cdots v_n. \tag{4}$$

For each permutation $w = s_{i_1} \cdots s_{i_\ell} \in S_n$ we define the *one-line notation* of w to be the word

$$w_1 \cdots w_n \underset{\text{def}}{=} s_{i_1} \circ (\cdots (s_{i_\ell} \circ (1 \cdots n)) \cdots). \tag{5}$$

For each $w \in S_n$ we define two tableaux, $P(w), Q(w)$ which are obtained from the *Robinson-Schensted correspondence* using column insertion to the one-line notation of w. (See, e.g., [12, Sec. 3.1].) It is well known that these tableaux satisfy $P(w^{-1}) = Q(w)$. Since $\mathrm{sh}(P(w)) = \mathrm{sh}(Q(w))$ we can define the *shape of a permutation* as $\mathrm{sh}(w) = \mathrm{sh}(P(w))$.

Given a permutation $w \in S_n$ expressed in terms of generators $w = s_{i_1} \cdots s_{i_\ell}$ we say this expression is *reduced* if w cannot be expressed as a shorter product of generators of S_n. We call the *length* of a permutation $w \in S_n$ $\ell(w) = \ell$, in the previous equation. We define the *Bruhat order* on S_n by $v \leq w$ if some (equivalently every) reduced expression for w contains a reduced expression for v as a subword (The reader is referred to [2] for more on this topic). Throughout this paper we will use w_0 to denote the unique maximal element in the Bruhat order. Multiplying a permutation on the right by w_0 also changes the bitableau of the Robinson-Schensted correspondence for that permutation. Specifically, this change can be described in terms of transposition. (See [2, Appendix].)

Lemma 2.1 *If $v \in S_n$, then $Q(v) = (Q(vw_0))^\top$.*

3 Kazhdan-Lusztig representations

Given an indeterminate q we define the *Hecke algebra*, $H_n(q)$, to be the $\mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$-algebra with multiplicative identity $\widetilde{T}_e$ generated by $\{\widetilde{T}_{s_i}\}_{i=1}^{n-1}$ with relations

$$\widetilde{T}_{s_i}^2 = (q^{\frac{1}{2}} - q^{\frac{-1}{2}})\widetilde{T}_{s_i} + \widetilde{T}_e, \qquad \text{for } i = 1, \ldots, n-1, \tag{6}$$

$$\widetilde{T}_{s_i}\widetilde{T}_{s_j}\widetilde{T}_{s_i} = \widetilde{T}_{s_j}\widetilde{T}_{s_i}\widetilde{T}_{s_j}, \qquad \text{if } |i-j| = 1, \tag{7}$$

$$\widetilde{T}_{s_i}\widetilde{T}_{s_j} = \widetilde{T}_{s_j}\widetilde{T}_{s_i}, \qquad \text{if } |i-j| \geq 2. \tag{8}$$

We then can define $\widetilde{T}_w$ for any $w \in S_n$ by $\widetilde{T}_w = \widetilde{T}_{s_{i_1}} \cdots \widetilde{T}_{s_{i_l}}$ where $w = s_{i_1} \cdots s_{i_l}$ is any reduced expression. Inverses of generators are given by

$$\widetilde{T}_{s_i}^{-1} = \widetilde{T}_{s_i} - (q^{\frac{1}{2}} - q^{\frac{-1}{2}})\widetilde{T}_e = \widetilde{T}_{s_i} - q^{\frac{-1}{2}}(q-1)\widetilde{T}_e. \tag{9}$$

When $q = 1$ we see that this presentation is simply that of the symmetric group algebra $\mathbb{Z}[S_n]$.

An important involution of the Hecke algebra is the so called *bar involution*. The involution is defined as

$$\sum_w a_w \widetilde{T}_w \mapsto \overline{\sum_w a_w \widetilde{T}_w} = \sum_w \overline{a_w}\,\overline{\widetilde{T}_w} \tag{10}$$

where

$$\overline{q} = q^{-1}, \qquad \overline{\widetilde{T}_w} = \left(\widetilde{T}_{w^{-1}}\right)^{-1}. \tag{11}$$

The *Kazhdan-Lusztig basis*, $\{C'_w(q) \mid w \in S_n\}$, is the unique basis of $H_n(q)$ such that the basis elements are invariant under the bar involution, $\overline{C'}_w(q) = C'_w(q)$ for all $w \in S_n$, and that $C'_w(q)$ in terms of the $\{\widetilde{T}_v \mid v \in S_n\}$ is given by

$$C'_w(q) = \sum_{v \leq w} q_{v,w}^{-1} P_{v,w}(q)\, \widetilde{T}_v, \tag{12}$$

where $P_{v,w}(q)$ are polynomials in q of degree at most $\frac{\ell(w)-\ell(v)-1}{2}$ and where we define the convenient notation $\epsilon_{v,w} = (-1)^{\ell(w)-\ell(v)}$, $q_{v,w} = (q^{\frac{1}{2}})^{\ell(w)-\ell(v)}$. These polynomials are known as the *Kazhdan-Lusztig polynomials* and in fact belong to $\mathbb{N}[q]$.

Kazhdan and Lusztig also introduced another basis $\{C_w(q) \mid w \in S_n\}$ with similar properties which is traditionally known as the Kazhdan-Lusztig basis, but for our purposes the $\{C'_w(q) \mid w \in S_n\}$ basis is more convenient. $C_w(q)$ and $C'_w(q)$ are related by $C'_w(q) = \psi(C_w(q))$, where ψ is the semilinear map defined by

$$\psi : q^{\frac{1}{2}} \mapsto q^{\frac{-1}{2}} \text{ and } \widetilde{T}_w \mapsto \epsilon_{e,w}\widetilde{T}_w. \tag{13}$$

Thus $C_w(q)$ is also bar invariant and its expression in terms of $\{\widetilde{T}_v \mid v \in S_n\}$ is

$$C_w(q) = \sum_{v \leq w} \epsilon_{v,w} q_{v,w} \overline{P_{v,w}(q)} \widetilde{T}_v. \tag{14}$$

As a preliminary to the proof of the existence and uniqueness of their bases Kazhdan and Lusztig also defined the following function

$$\mu(u,v) \underset{\text{def}}{=} \begin{cases} \text{coefficient of } q^{(\ell(v)-\ell(u)-1)/2} \text{ in } P_{u,v}(q) & \text{if } u < v, \\ 0 & \text{otherwise.} \end{cases} \tag{15}$$

Note that $\mu(u,v) = 0$ if $\ell(v) - \ell(u)$ is even since $P_{u,v}(q)$ has only integer powers of q. Also, it is well known that $P_{u,v}(q) = P_{w_0uw_0,w_0vw_0}(q)$, and therefore that $\mu(u,v) = \mu(w_0uw_0, w_0vw_0)$. Kazhdan and Lusztig showed further [8, Cor. 3.2] $\mu(u,v) = \mu(w_0v, w_0u)$, even though $P_{u,v}(q)$ and $P_{w_0v,w_0u}(q)$ are not equal in general.

In the existence proof of the Kazhdan-Lusztig basis in [8, Pf. of Thm. 1.1] an expression for the action of $\widetilde{T}_s$, s a basic transposition, on the basis element $C'_w(q)$ is given by

$$C'_w(q)\widetilde{T}_s = \begin{cases} -q^{\frac{-1}{2}} C'_w(q) + C'_{ws}(q) + \sum\limits_{\substack{v<w \\ vs<v}} \mu(v,w) C'_v(q) & \text{if } ws > w, \\ q^{\frac{1}{2}} C'_w(q) & \text{if } ws < w. \end{cases} \tag{16}$$

Along with these bases Kazhdan and Lusztig defined a preorder on S_n in order to construct representations of $H_n(q)$. This preorder, called the *right preorder*, is denoted by $\leq_R$ and is defined as the transitive closure of $\lessdot_R$ where $u \lessdot_R v$ if $C'_u(q)$ has nonzero coefficient in the expression of $C'_v(q)\widetilde{T}_w$ for some $w \in S_n$. It follows from a result in [1] that $w \leq_R v$ implies $\mathrm{sh}(v) \preceq \mathrm{sh}(w)$.

We follow the description in [7, Appendix] of the Kazhdan-Lusztig construction of an irreducible $H_n(q)$-module indexed by partition $\lambda \vdash n$. Here and henceforth the span will be over the Laurent polynomial ring $\mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$. Choosing tableau T of shape λ, we allow $H_n(q)$ to act by right multiplication on

$$K^\lambda \underset{\text{def}}{=} \mathrm{span}\{C'_w(q) \mid Q(w) = T\}, \tag{17}$$

regarded as the quotient

$$\mathrm{span}\{C'_v(q) \mid v \leq_R w\} / \mathrm{span}\{C'_v(q) \mid v \leq_R w, v \not\geq_R w\}. \tag{18}$$

The quotient is necessary because K^λ is not in general closed under the action of $H_n(q)$. In particular, for $\lambda \neq (1^n)$ we have the containments $K^\lambda \subset H_n(q)K^\lambda \subseteq K^\lambda \oplus \mathrm{span}\{C'_v(q) \mid v \leq_R w, v \not\geq_R w\}$.

4 The quantum polynomial ring and Kazhdan-Lusztig immanants

Let $x = (x_{i,j})$ be an $n \times n$-matrix of variables. The polynomial ring $\mathbb{Z}[x]$ has a natural grading $\mathbb{Z}[x] = \oplus_{r\geq 0}\mathcal{A}_r$, where $\mathcal{A}_r$ is the span of all monomials of total degree r. Further decomposing each space $\mathcal{A}_r$, we define a multigrading

$$\mathbb{Z}[x] = \bigoplus_{r\geq 0}\mathcal{A}_r = \bigoplus_{r\geq 0}\bigoplus_{L,M}\mathcal{A}_{L,M}, \tag{19}$$

where $L = \{\ell(1) \leq \ldots \leq \ell(r)\}$ and $M = \{m(1) \leq \ldots \leq m(r)\}$ are r-element multisets of $[n]$, written as weakly increasing sequences, and where $\mathcal{A}_{L,M}$ is the span of monomials whose row and column indices are given by L and M, respectively. We define the *generalized submatrix* of x with respect to (L, M) by

$$x_{L,M} = \begin{bmatrix} x_{\ell(1),m(1)} & \cdots & x_{\ell(1),m(r)} \\ x_{\ell(2),m(1)} & \cdots & x_{\ell(2),m(r)} \\ \vdots & & \vdots \\ x_{\ell(r),m(1)} & \cdots & x_{\ell(r),m(r)} \end{bmatrix}. \tag{20}$$

We refer to the space

$$\mathcal{A}_{[n],[n]} = \operatorname{span}\{x_{1,w_1}\cdots x_{n,w_n} \mid w \in S_n\}, \tag{21}$$

as the *immanant space*, and define the notation $x^{u,v} = x_{u_1,v_1}\cdots x_{u_n,v_n}$ for permutations $u, v \in S_n$. Immanants are a generalization of the determinant and permanent of a matrix introduced in [9].

A natural S_n-action on $\mathbb{Z}[x]$ is given by

$$g(x) \circ s_i \underset{\text{def}}{=} g(xs_i), \tag{22}$$

where $g \in \mathbb{Z}[x]$ and xs_i is interpreted as the product of x and the permutation matrix of s_i.

We now define a generalization of the polynomial ring $\mathbb{Z}[x]$ called the *quantum polynomial ring*, $\mathcal{A}(n;q)$. The ring $\mathcal{A}(n;q)$ is a noncommutative $\mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$-algebra on n^2 generators $x = (x_{1,1}\ldots, x_{n,n})$ with relations (assuming $i < j$ and $k < \ell$),

$$\begin{aligned} x_{i,\ell}x_{i,k} &= q^{\frac{1}{2}}x_{i,k}x_{i,\ell}, \\ x_{j,k}x_{i,k} &= q^{\frac{1}{2}}x_{i,k}x_{j,k}, \\ x_{j,k}x_{i,\ell} &= x_{i,\ell}x_{j,k}, \\ x_{j,\ell}x_{i,k} &= x_{i,k}x_{j,\ell} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})x_{i,\ell}x_{j,k}. \end{aligned} \tag{23}$$

A natural basis for the quantum polynomial ring consists of the set of monomials in lexicographic order. Analogous to the multigrading of $\mathbb{Z}[x]$ is the multigrading

$$\mathcal{A}(n;q) = \bigoplus_{r\geq 0}\mathcal{A}_r(n;q) = \bigoplus_{r\geq 0}\bigoplus_{L,M}\mathcal{A}_{L,M}(n;q), \tag{24}$$

where $\mathcal{A}_r(n;q)$ is the span of all monomials of total degree r, and where $\mathcal{A}_{L,M}(n;q)$ is the span of monomials whose row and column indices are given by r-element multisets L and M of $[n]$. We again call the space $\mathcal{A}_{[n],[n]}(n;q) = \operatorname{span}\{x^{e,w} \mid w \in S_n\}$ the *immanant space* of $\mathcal{A}(n;q)$ or the *quantum immanant space*.

Define a right action of the Hecke algebra on $\mathcal{A}_{[n],[n]}(n;q)$ by

$$x^{e,v} \circ \widetilde{T}_{s_i} = \begin{cases} x^{e,vs_i} & \text{if } vs_i > v, \\ x^{e,vs_i} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})x^{e,v} & \text{if } vs_i < v. \end{cases} \tag{25}$$

Related to the bar involution on $H_n(q)$ is another *bar involution* on $\mathcal{A}_{[n],[n]}(n;q)$ defined by

$$\sum_w a_w x^{e,w} \mapsto \overline{\sum_w a_w x^{e,w}} = \sum_w \overline{a_w}\ \overline{x^{e,w}} \tag{26}$$

where

$$\overline{q} = q^{-1}, \qquad \overline{x^{e,w}} = x^{w_0,w_0 w} = x_{n,w_n} \cdots x_{1,w_1}. \tag{27}$$

Lemma 4.1 *The bar involutions of (10) and (26) are compatible with the action of $H_n(q)$ on $\mathcal{A}_{[n],[n]}(n;q)$. That is,*

$$\overline{x^{e,v} \circ \widetilde{T}_{s_i}} = \overline{x^{e,v}} \circ \overline{\widetilde{T}_{s_i}} \tag{28}$$

for all $v \in S_n$.

Proof: Omitted. □

It is known that there is a unique, bar-invariant basis of $\mathcal{A}_{[n],[n]}(n;q)$ closely related to the Kazhdan-Lusztig basis of the Hecke algebra. We call the elements of this basis the *Kazhdan-Lusztig immanants* $\{\mathrm{Imm}_v(x;q) \mid v \in S_n\}$. First appearing in Du [5],[6], this basis has the following theorem-definition. (See, e.g., [3, Thm. 5.3])

Theorem 4.2 *For any $v \in S_n$, there is a unique element $\mathrm{Imm}_v(x;q) \in \mathcal{A}_{[n],[n]}(n;q)$ such that*

$$\overline{\mathrm{Imm}_v(x;q)} = \mathrm{Imm}_v(x;q) \tag{29}$$

$$\mathrm{Imm}_v(x;q) = \sum_{w \geq v} \epsilon_{v,w} q_{v,w}^{-1} Q_{v,w}(q) x^{e,w}, \tag{30}$$

where $Q_{v,w}(q)$ are polynomials in q of degree $\leq \frac{\ell(w)-\ell(v)-1}{2}$ if $v < w$ and $Q_{v,v}(q) = 1$.

The polynomials $Q_{u,v}(q)$ above are actually the *inverse Kazhdan-Lusztig polynomials*, found in [8, Sec. 3]. They are related to the Kazhdan-Lusztig polynomials by

$$Q_{u,v}(q) = P_{w_0 v, w_0 u}(q) = P_{v w_0, u w_0}(q). \tag{31}$$

We can now describe a right action of $H_n(q)$ on the immanant space by its action on the Kazhdan-Lusztig immanants.

Corollary 4.3 *The right action of the Hecke algebra on $\mathcal{A}_{[n],[n]}(n;q)$ is described by*

$$\mathrm{Imm}_v(x;q) \circ \widetilde{T}_{s_i} = \begin{cases} q^{\frac{1}{2}} \mathrm{Imm}_v(x;q) + \mathrm{Imm}_{vs_i}(x;q) + \sum\limits_{\substack{w>v \\ ws_i>w}} \mu(v,w) \mathrm{Imm}_w(x;q) & \text{if } vs_i < v, \\ -q^{-\frac{1}{2}} \mathrm{Imm}_v(x;q) & \text{if } vs_i > v. \end{cases} \tag{32}$$

Proof: Omitted. □

A deeper connection between the Kazhdan-Lusztig immanants and the Kazhdan-Lusztig basis is evident in the $\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$-bilinear form on $\mathcal{A}_{[n],[n]}(n;q) \times H_n(q)$ defined by by $\langle x^{e,v}, \widetilde{T}_w \rangle = \delta_{v,w}$. Specifically, we

have $\langle \mathrm{Imm}_v(x;q), C'_w(q)\rangle = \delta_{v,w}$, so the Kazhdan-Lusztig basis is dual to the basis of Kazhdan-Lusztig immanants.

In the following lemma we relate the definition of the right preorder in the Hecke algebra with these Kazhdan-Lusztig immanants. The results in the proof will also be essential in describing the relationship of the $H_n(q)$-representations associated with the Kazhdan-Lusztig basis and immanants.

Lemma 4.4 *Let $v, v' \in S_n$. Then $v \lessdot_R v'$ if $\mathrm{Imm}_{v'}(x;q)$ appears with nonzero coefficient in the Kazhdan-Lusztig immanant expansion of $\mathrm{Imm}_v(x;q) \circ \widetilde{T}_u$ for some $u \in S_n$.*

Proof: Omitted. □

With Lemma 4.4 we can now express the preorder in terms of the Kazhdan-Lusztig immanants. We can now construct $H_n(q)$-modules indexed by $\lambda \vdash n$, as in [7, Appendix], with the Kazhdan-Lusztig immanants. We choose a tableau T of shape λ and allow $H_n(q)$ to act by right multiplication on

$$V^\lambda \underset{\mathrm{def}}{=} \mathrm{span}\{\mathrm{Imm}_w(x;q) \,|\, Q(w) = T\}, \tag{33}$$

regarded as the quotient

$$\mathrm{span}\{\mathrm{Imm}_v(x;q) \,|\, v \geq_R w\}/\mathrm{span}\{\mathrm{Imm}_v(x;q) \,|\, v \geq_R w, v \not\leq_R w\}. \tag{34}$$

The quotient is necessary because like K^λ, V^λ is not in general closed under the action of $H_n(q)$. In particular, whenever $\lambda \neq (1^n)$ we have the containments

$$V^\lambda \subset H_n(q)V^\lambda \subseteq V^\lambda \oplus \mathrm{span}\{\mathrm{Imm}_v(x;q) \,|\, v \geq_R w, v \not\leq_R w\}. \tag{35}$$

5 Generalized submatrices and vanishing properties of immanants

In [11] Rhoades and Skandera stated conditions on immanants $\mathrm{Imm}_w(x)$ in $\mathbb{Z}[x]$ and on $n \times n$-matrices A which imply that $\mathrm{Imm}_w(A) = 0$. Here we present new, analogous vanishing results for immanants in $\mathcal{A}_{[n],[n]}(n;q)$. Specifically we will state conditions on quantum immanants $\mathrm{Imm}_w(x;q)$ in $\mathcal{A}(n;q)$ and on generalized submatrices $x_{L,M}$ of the quantum matrix x, which imply that $\mathrm{Imm}_w(x_{L,M};q) = 0$. Using these results we can eliminate the quotient needed in the construction (34) of the $H_n(q)$-modules. This provides a quantum analog of the authors' results in [3].

To express the vanishing results we need to define the *row repetition partition* of an $n \times n$-matrix A by

$$\mu_{[j]}(A) \underset{\mathrm{def}}{=} (\mu_1, \ldots, \mu_k), \tag{36}$$

where k is the number of distinct rows in the $n \times j$-submatrix $A_{[n],[j]}$, and $\mu_1, \ldots, \mu_k$ are the multiplicities with which distinct rows appear, written in weakly decreasing order. Also we define the permutation $w_{[j]} \in S_j$ from $w \in S_n$ by arranging $1, \ldots, j$ in the same relative order of the first j terms in the one line notation of w.

Lemma 5.1 *Fix a permutation $w \in S_n$ and indices $1 \leq j \leq n$. If $\mathrm{sh}(w_{[j]}) \not\geq \mu_{[j]}(x_{L,[n]})$, then*

$$\mathrm{Imm}_w(x_{L,[n]};q) = 0. \tag{37}$$

An immediate consequence of this vanishing result follows after defining a partial order found in [11]. A partial order on standard tableaux is the *iterated dominance of tableaux*. Given two standard tableau T, U both having n boxes, we define $U \lhd_I T$ if for $j = 1, \ldots, n$ we have

$$\mathrm{sh}(U_{[j]}) \prec \mathrm{sh}(T_{[j]}), \tag{38}$$

where $U_{[j]}$ is the subtableau of U consisting of all entries less than or equal to j.

Corollary 5.2 *Fix a partition $\lambda \vdash n$ and define the multiset $L = 1^{\lambda_1} \cdots n^{\lambda_n}$, where n^k is shorthand for n appearing k times. For each permutation w satisfying $\mathrm{sh}(w) \not\succeq \lambda$ or satisfying $\mathrm{sh}(w) = \lambda$ and $Q(w) \neq T(\lambda)$, we have that $\mathrm{Imm}_w(x_{L,[n]}) = 0$.*

Proof: If w satisfies $\mathrm{sh}(w) \not\succeq \lambda$ then the case with $j = n$ of Lemma 5.1 implies that $\mathrm{Imm}_w(x_{L,[n]}; q) = 0$. Suppose that $\mathrm{sh}(w) = \lambda$ and $Q(w) \neq T(\lambda)$. Since the tableau $T(\lambda)$ is greatest in iterated dominance among all tableaux of shape λ, we have that $Q(w) \lhd_I T(\lambda)$ and there exists an index j such that

$$\mathrm{sh}(Q(w)_{[j]}) \prec \mathrm{sh}(T(\lambda)_{[j]}) = \mu_{[j]}(x_{L,[n]}). \tag{39}$$

Then by the fact that $\mathrm{sh}(w_{[j]}) = \mathrm{sh}(Q(w)_{[j]})$ we see that $\mathrm{sh}(w_{[j]}) \prec \mu_{[j]}(x_{L,[n]})$, which by Lemma 5.1 implies that $\mathrm{Imm}_w(x_{L,[n]}; q) = 0$. □

We can define a right action of $H_n(q)$ on $\mathcal{A}_{L,[n]}(n; q)$ by the formula

$$(x_{L,[n]})^{e,w} \circ \widetilde{T}_s = \begin{cases} (x_{L,[n]})^{e,ws}, & ws > w \\ (x_{L,[n]})^{e,ws} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})(x_{L,[n]})^{e,w}, & ws < w. \end{cases} \tag{40}$$

We can then extend this action for the Kazhdan-Lusztig immanants evaluated at generalized submatrices.

Corollary 5.3 *Fix $u \in S_n$ and an n-element multiset L of $[n]$. For a basic transposition s, the right action of $H_n(q)$ on the element $\mathrm{Imm}_u(x_{L,[n]}; q)$ of $\mathcal{A}_{L,[n]}(n; q)$ is given by*

$$\mathrm{Imm}_u(x_{L,[n]}; q) \circ \widetilde{T}_s = \begin{cases} q^{\frac{1}{2}} \mathrm{Imm}_u(x_{L,[n]}; q) + \mathrm{Imm}_{us}(x_{L,[n]}; q) + \sum\limits_{\substack{w > u \\ ws > w}} \mu(u, w) \mathrm{Imm}_w(x_{L,[n]}; q), & us < u \\ -q^{\frac{-1}{2}} \mathrm{Imm}_u(x_{L,[n]}; q), & us > u. \end{cases} \tag{41}$$

Proof: For $u \in S_n$ the Kazhdan-Lusztig immanant indexed by u evaluated at the matrix $x_{L,[n]}$ is given by

$$\mathrm{Imm}_u(x_{L,[n]}; q) = \sum_{w \geq u} \epsilon_{u,w} q_{u,w}^{-1} Q_{u,w}(q) (x_{L,[n]})^{e,w}. \tag{42}$$

Now we have an action of $H_n(q)$ on the immanants by (40),

$$\begin{aligned}
\mathrm{Imm}_u(x_{L,[n]};q)\circ\widetilde{T}_s &= \sum_{w\geq u}\epsilon_{u,w}q_{u,w}^{-1}Q_{u,w}(q)\,(x_{L,[n]})^{e,w}\circ\widetilde{T}_s \\
&= \sum_{\substack{w\geq u\\ ws>w}}\epsilon_{u,w}q_{u,w}^{-1}Q_{u,w}(q)(x_{L,[n]})^{e,ws} \\
&+ \sum_{\substack{w\geq u\\ ws<w}}\epsilon_{u,w}q_{u,w}^{-1}Q_{u,w}(q)\left((x_{L,[n]})^{e,ws}+(q^{\frac{1}{2}}-q^{\frac{-1}{2}})(x_{L,[n]})^{e,w}\right) \\
&= -q^{\frac{-1}{2}}\sum_{\substack{w\geq u\\ ws>w}}\epsilon_{u,w}q_{u,w}^{-1}Q_{u,ws}(q)(x_{L,[n]})^{e,w} \\
&+ \sum_{\substack{w\geq u\\ ws<w}}\epsilon_{u,w}q_{u,w}^{-1}\left((q^{\frac{1}{2}}-q^{\frac{-1}{2}})Q_{u,w}(q)-q^{\frac{1}{2}}Q_{u,ws}(q)\right)(x_{L,[n]})^{e,w}.
\end{aligned} \tag{43}$$

If $us>u$ we know that $Q_{u,w}(q)=Q_{u,ws}(q)$ for any permutation w. Thus we have from (43) the action of $\widetilde{T}_s$ is

$$\begin{aligned}
\mathrm{Imm}_u(x_{L,[n]};q)\circ\widetilde{T}_s &= -q^{\frac{-1}{2}}\sum_{\substack{w\geq u\\ ws>w}}\epsilon_{u,w}q_{u,w}^{-1}Q_{u,w}(q)(x_{L,[n]})^{e,w} \\
&+ \sum_{\substack{w\geq u\\ ws<w}}\epsilon_{u,w}q_{u,w}^{-1}\left((q^{\frac{1}{2}}-q^{\frac{-1}{2}})Q_{u,w}(q)-q^{\frac{1}{2}}Q_{u,w}(q)\right)(x_{L,[n]})^{e,w} \\
&= -q^{\frac{-1}{2}}\mathrm{Imm}_u(x_{L,[n]};q),
\end{aligned} \tag{44}$$

as we expected.

If $us<u$ we know that $Q_{us,w}(q)=Q_{us,ws}(q)$ for any permutation w. By careful application of the recursive formula for the inverse Kazhdan-Lusztig polynomials we can also observe the following relationships. If $ws>w$ then we see that

$$Q_{u,ws}(q)=Q_{us,w}(q)-qQ_{u,w}(q)+\sum_{\substack{u<v\leq w\\ v<vs}}q_{v,w}\mu(u,v)Q_{v,w}(q). \tag{45}$$

If $ws<w$ then we see that

$$Q_{u,w}(q)+qQ_{u,ws}(q) = Q_{us,ws}(q)+\sum_{\substack{u<v\leq ws\\ v<vs}}q_{v,ws}\mu(u,v)Q_{v,ws}(q) \tag{46}$$

$$= Q_{us,w}(q)+\sum_{\substack{u<v\leq ws\\ v<vs}}q_{v,ws}\mu(u,v)Q_{v,ws}(q). \tag{47}$$

Thus we have from (43) the action of $\widetilde{T}_s$ is

$$\begin{aligned}
&\mathrm{Imm}_u(x_{L,[n]};q)\circ\widetilde{T}_s \\
&\quad= -q^{\frac{-1}{2}}\sum_{\substack{w\geq u\\ ws>w}}\epsilon_{u,w}q_{u,w}^{-1}\left(Q_{us,w}(q)-qQ_{u,w}(q)+\sum_{\substack{u<v\leq w\\ v<vs}}q_{v,w}\mu(u,v)Q_{v,w}(q)\right)(x_{L,[n]})^{e,w} \\
&\quad+\sum_{\substack{w\geq u\\ ws<w}}\epsilon_{u,w}q_{u,w}^{-1}\left(q^{\frac{1}{2}}Q_{u,w}(q)-q^{\frac{-1}{2}}\left(Q_{us,w}(q)+\sum_{\substack{u<v\leq ws\\ v<vs}}q_{v,ws}\mu(u,v)Q_{v,ws}(q)\right)\right)(x_{L,[n]})^{e,w} \\
&\quad=\sum_{\substack{w\geq u\\ ws>w}}\epsilon_{u,w}q_{u,w}^{-1}\left(q^{\frac{1}{2}}Q_{u,w}(q)-q^{\frac{-1}{2}}Q_{us,w}(q)-q^{\frac{-1}{2}}\sum_{\substack{u<v\leq w\\ v<vs}}q_{v,w}\mu(u,v)Q_{v,w}(q)\right)(x_{L,[n]})^{e,w} \\
&\quad+\sum_{\substack{w\geq u\\ ws<w}}\epsilon_{u,w}q_{u,w}^{-1}\left(q^{\frac{1}{2}}Q_{u,w}(q)-q^{\frac{-1}{2}}Q_{us,w}(q)-q^{\frac{-1}{2}}\sum_{\substack{u<v\leq ws\\ v<vs}}q_{v,ws}\mu(u,v)Q_{v,ws}(q)\right)(x_{L,[n]})^{e,w} \\
&\quad= q^{\frac{1}{2}}\mathrm{Imm}_u(x_{L,[n]};q)+\mathrm{Imm}_{us}(x_{L,[n]};q)+\sum_{\substack{v>u\\ v<vs}}\mu(u,v)\mathrm{Imm}_v(x_{L,[n]};q), \qquad (48)
\end{aligned}$$

as we expected. □

We can now see that the right $H_n(q)$-action defined in Corollary 5.3 actually describes an $H_n(q)$-module if we evaluate the immanants at generalized submatrices.

Theorem 5.4 *Let $\lambda\vdash n$ and set $L=1^{\lambda_1}\cdots n^{\lambda_n}$. Define*

$$W^\lambda \underset{\mathrm{def}}{=} \mathrm{span}\{\mathrm{Imm}_w(x_{L,[n]};q)\,|\,Q(w)=T(\lambda)\}, \qquad (49)$$

where $T(\lambda)$ is the superstandard tableau of shape λ. Then W^λ is an $H_n(q)$-module.

Proof: By (35) we know that it suffices to show that $\mathrm{Imm}_v(x_{L,[n]};q)=0$ for $v>_R w$ where $Q(w)=T(\lambda)$. Since $v>_R w$ then we know that $\mathrm{sh}(w)\succ\mathrm{sh}(v)$. The row multiplicity partition of $x_{L,[n]}$ is $\mu(x_{L,[n]})=\lambda$. So $\mathrm{sh}(v)\prec\mathrm{sh}(w)=\mu(x_{L,[n]})$. Thus $\mathrm{sh}(v)\not\preceq\mu(x_{L,[n]})$. Therefore, by Lemma 5.1, $\mathrm{Imm}_v(x_{L,[n]};q)=0$ for all $v>_R w$. □

The condition for inclusion in the basis of this module is $Q(w)^\top=T(\lambda)$ unlike the condition, $Q(w)=T$ where $\mathrm{sh}(T)=\lambda$, used in the definition of V^λ above. The need for the change in conditions is due to the result Corollary 5.2.

We would now like to show that these modules, W^λ, are isomorphic to the modules constructed by the action $H_n(q)$ on the Kazhdan-Lusztig basis. We shall then show that the action of $\widetilde{T}_{s_i}$ on either basis yields equal matrices, up to ordering of the basis elements. Let $\rho_1: H_n(q)\to\mathrm{End}(K^\lambda)$ and $\rho_2: H_n(q)\to\mathrm{End}(W^\lambda)$ be the representations of $H_n(q)$ defined by the right actions described in (16) and Corollary 4.3, respectively.

Theorem 5.5 *Let $X_1(h)$, $X_2(h)$ be the matrices of $\rho_1(h)$, $\rho_2(h)$ with respect to the Kazhdan-Lusztig basis and the Kazhdan-Lusztig immanant basis. Then $X_1(h)=X_2(h)$.*

Proof: First, we construct K^λ as in (17) with $T = T(\lambda)$. Let $B = \{v \in S_n \mid Q(v) = T(\lambda)\}$. From Lemma 2.1 we see that if $C_w(q)$ is a basis element of K^λ, i.e. $w \in B$, then $Q(ww_0)^\top = Q(w) = T(\lambda)$. Thus if $w \in Bw_0$, then $\mathrm{Imm}_w(x_{L,[n]}; q)$ is a basis element of W^λ, as in (49). Define coefficients $a^{s_i}_{v,w}$ for each generators s_i of S_n and $v, w \in B$ so that

$$C'_v(q)\widetilde{T}_{s_i} = \sum_{w \in B} a^{s_i}_{v,w} C'_w(q). \tag{50}$$

Then from the proof of Lemma 4.4 and Corollary 5.3 we see that for all $v \in B$

$$\mathrm{Imm}_{vw_0}(x_{L,[n]}; q) \circ \widetilde{T}_{s_i} = \sum_{w \in B} a^{s_i}_{v,w} \mathrm{Imm}_{ww_0}(x_{L,[n]}; q). \tag{51}$$

Thus $X_1(\widetilde{T}_{s_i}) = X_2(\widetilde{T}_{s_i})$. Since any element of $v \in S_n$ is a product of generators we have that $X_1(\widetilde{T}_v) = X_2(\widetilde{T}_v)$ and thus for any element $h \in H_n(q)$ we have that $X_1(h) = X_2(h)$. □

Corollary 5.6 *The modules $\mathbb{C}(q^{\frac{1}{2}}) \otimes W^\lambda$ indexed by partitions $\lambda \vdash n$ are the irreducible $\mathbb{C}(q^{\frac{1}{2}}) \otimes H_n(q)$-modules.*

This result follows immediately from the fact that the modules K^λ are the irreducible $H_n(q)$-modules.

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