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Chip-Firing And A Devil's Staircase

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The devil's staircase – a continuous function on the unit interval $[0,1]$ which is not constant, yet is locally constant on an open dense set – is the sort of exotic creature a combinatorialist might never expect to encounter in “real life.” We show how a devil's staircase arises from the combinatorial problem of parallel chip-firing on the complete graph. This staircase helps explain a previously observed “mode locking” phenomenon, as well as the surprising tendency of parallel chip-firing to find periodic states of small period.

Keywords: Circle map, fixed-energy sandpile, mode locking, non-ergodicity, parallel chip-firing, rotation number, short period attractors

1 Introduction

In this extended abstract, we summarize recent work relating the Poincaré rotation number of a circle map $S^1 \rightarrow S^1$ to the behavior of parallel chip-firing on the complete graph. We use this connection to shed light on two intriguing features of parallel chip-firing, *mode locking* and *short period attractors*. Ever since Bagnoli, Ceconi, Flammini, and Vespignani [1] found evidence of mode locking and short period attractors in numerical experiments in 2003, these two phenomena have called out for a mathematical explanation. The proofs omitted here can be found in [12].

In chip-firing on a finite graph, each vertex v starts with a pile of $\sigma(v) \geq 0$ chips. A vertex is *unstable* if it has at least as many chips as its degree, and can *fire* by sending one chip to each neighbor. In *parallel chip-firing*, at each time step, all unstable vertices fire simultaneously. If it is possible in finitely many firings to reach a stable configuration in which no vertex can fire, then this final configuration does not depend on the order of firings [5]. In this case, the parallel restriction does not affect the final outcome. However, our focus will be on chip configurations that do not stabilize.

Previous work on parallel chip-firing [3, 4, 10, 14] has focused on the periodicity of the dynamics: given a graph G , for which natural numbers q does there exist a parallel chip-firing state on G which first recurs after q time steps? We will have more to say about this question below. In the statistical physics literature, parallel chip-firing often goes by the name “fixed energy sandpile” [1, 6, 7, 15]. The term “sandpile” refers to the Bak-Tang-Wiesenfeld model of self-organized criticality [2], while “fixed energy” refers to the lack of a sink or boundary vertex where chips disappear.

We add loops to the complete graph K_n , so that a vertex with n or more chips is unstable, and fires by sending one chip to each vertex of K_n , including one chip to itself. The *parallel update* rule fires all

unstable vertices simultaneously, yielding a new chip configuration $U\sigma$ given by

$$U\sigma(v) = \begin{cases} \sigma(v) + r(\sigma), & \sigma(v) \leq n - 1 \\ \sigma(v) - n + r(\sigma), & \sigma(v) \geq n. \end{cases} \tag{1}$$

Here

$$r(\sigma) = \#\{v \mid \sigma(v) \geq n\}$$

is the number of unstable vertices. Write $U^0\sigma = \sigma$, and $U^t\sigma = U(U^{t-1}\sigma)$ for $t \geq 1$.

Note that the total number of chips in the system is conserved. In particular, only finitely many different states are reachable from the initial configuration σ , so the sequence $(U^t\sigma)_{t \geq 0}$ is eventually periodic: there exist integers $m \geq 1$ and $t_0 \geq 0$ such that

$$U^{t+m}\sigma = U^t\sigma \quad \forall t \geq t_0. \tag{2}$$

The *activity* of σ is the limit

$$a(\sigma) = \lim_{t \rightarrow \infty} \frac{\alpha_t}{nt}. \tag{3}$$

where

$$\alpha_t = \sum_{s=0}^{t-1} r(U^s\sigma)$$

is the total number of firings performed in the first t updates. By (2), the limit in (3) exists and equals $\frac{1}{mn}(\alpha_{t_0+m} - \alpha_{t_0})$. Since $0 \leq \alpha_t \leq nt$, we have $0 \leq a(\sigma) \leq 1$.

Following [1], we ask: how does the activity change when chips are added to the system? If σ_n is a chip configuration on K_n , write $\sigma_n + k$ for the configuration obtained from σ_n by adding k chips at each vertex. The function

$$\tilde{s}_n(k) = a(\sigma_n + k)$$

is called the *activity phase diagram* of σ_n . Surprisingly, for many choices of σ_n , the function \tilde{s}_n looks like a staircase, with long intervals of constancy punctuated by sudden jumps (Figure 1). This phenomenon is known as *mode locking*: if the system is in a preferred mode, corresponding to a wide stair in the staircase, then even a relatively large perturbation in the form of adding extra chips will not change the activity. For a general discussion of mode locking in dynamical systems, including examples from astronomy and physics, see [11].

To quantify the idea of mode locking in our setting, suppose we are given an infinite family of chip configurations $\sigma_2, \sigma_3, \dots$ with σ_n defined on K_n . Suppose σ_n is stable, i.e.,

$$0 \leq \sigma_n(v) \leq n - 1 \tag{4}$$

for all $v \in [n]$. Moreover, suppose that there is a continuous function $F : [0, 1] \rightarrow [0, 1]$, such that for all $0 \leq x \leq 1$

$$\frac{1}{n} \#\{v \in [n] \mid \sigma_n(v) < nx\} \rightarrow F(x) \tag{5}$$

as $n \rightarrow \infty$. Then according to Theorem 3.1, the activity phase diagrams \tilde{s}_n , suitably rescaled, converge pointwise to a continuous, nondecreasing function $s : [0, 1] \rightarrow [0, 1]$.

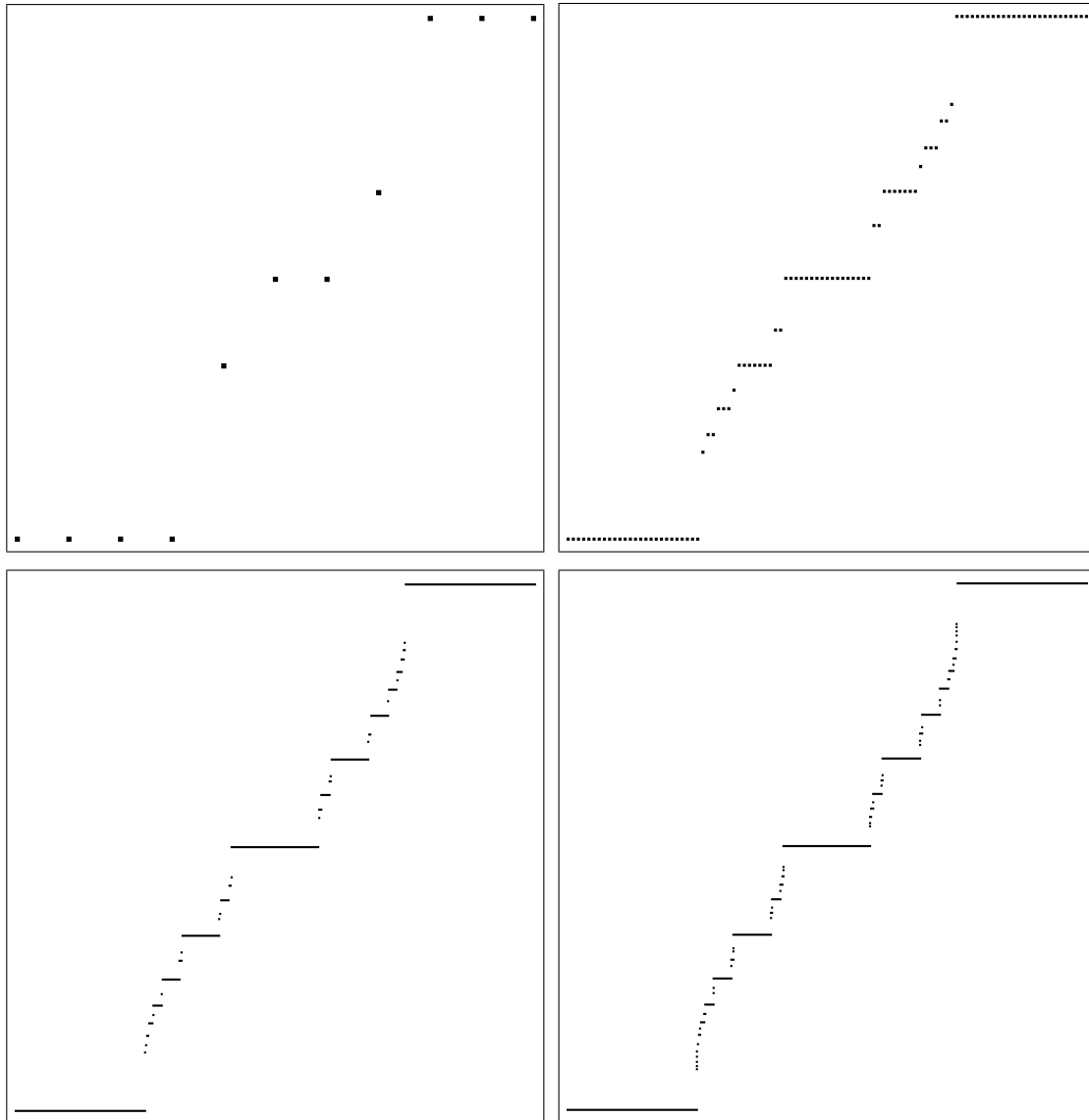


Fig. 1: The activity phase diagrams $a(\sigma_n + k)$, for $n = 10$ (top left), 100 (top right), 1000 (bottom left), and 10000, where σ_n is given by (6). On the horizontal axis, k runs from 0 to n . On the vertical axis, $a(\sigma_n + k)$ runs from 0 to 1.

(Proposition 4.4). Finally, in Theorem 4.11, we find a small “window” in which all states have eventual period two.

Many questions remain concerning parallel chip-firing on graphs other than K_n . If the underlying graph is a tree [4] or a cycle [7], then instead of a devil's staircase of infinitely many preferred modes, there are just three: activity 0, $\frac{1}{2}$ and 1. On the other hand, the numerical experiments of [1] for parallel chip-firing on the $n \times n$ torus suggest a devil's staircase in the large n limit. Our arguments rely quite strongly on the structure of the complete graph, whereas the mode locking phenomenon seems to be widespread. It would be very interesting to relate parallel chip-firing on other graphs to iteration of a circle map (or perhaps a map on a higher-dimensional manifold) in order to explain the ubiquity of mode locking.

2 Construction of the Circle Map

We first identify a certain class of chip configurations on K_n , the *confined states*, with the property that for any configuration σ of activity $a(\sigma) < 1$, we have $U^t \sigma$ confined for all sufficiently large t .

Definition. A chip configuration σ on K_n is *preconfined* if it satisfies

$$(i) \quad \sigma(v) \leq 2n - 1 \text{ for all vertices } v \text{ of } K_n.$$

If, in addition, σ satisfies

$$(ii) \quad \max_v \sigma(v) - \min_v \sigma(v) \leq n - 1$$

then σ is *confined*.

Lemma 2.1. *If σ is preconfined, then $U\sigma$ is confined.*

Lemma 2.2. *If $a(\sigma) < 1$, then $U^t \sigma$ is confined for all sufficiently large t .*

Note that from (1)

$$U\sigma(v) \equiv \sigma(v) + r(\sigma) \pmod{n}.$$

Iterating yields the congruence

$$U^t \sigma(v) \equiv \sigma(v) + \alpha_t \pmod{n} \tag{8}$$

where

$$\alpha_t = \sum_{s=0}^{t-1} r(U^s \sigma)$$

is the total number of firings before time t .

Our next lemma characterizes the vertices that fire at time $t + 1$.

Lemma 2.3. *If $U^t \sigma$ is preconfined, then $U^{t+1} \sigma(v) \geq n$ if and only if*

$$\sigma(v) \equiv -j \pmod{n}$$

for some $\alpha_t < j \leq \alpha_{t+1}$.

Let

$$\phi(j) = \#\{v \in [n] \mid \sigma(v) \equiv -j \pmod{n}\}. \tag{9}$$

By Lemma 2.3, if $U^t\sigma$ is preconfined, then the number of unstable vertices in $U^{t+1}\sigma$ is

$$r_{t+1} = \phi(\alpha_t + 1) + \dots + \phi(\alpha_{t+1}),$$

hence

$$\alpha_{t+2} = \alpha_{t+1} + \sum_{j=\alpha_{t+1}}^{\alpha_{t+1}} \phi(j). \tag{10}$$

This gives a recurrence for α_t relating three consecutive terms α_t , α_{t+1} and α_{t+2} . Our next lemma simplifies this to a recurrence relating just two consecutive terms.

Lemma 2.4. *If σ is preconfined, then for all $t \geq 0$*

$$\alpha_{t+1} = g(\alpha_t),$$

where $g : \mathbb{N} \rightarrow \mathbb{N}$ is given by

$$g(k) = \alpha_1 + \sum_{j=1}^k \phi(j) \tag{11}$$

and ϕ is given by (9).

The function g appearing in Lemma 2.4 satisfies

$$\begin{aligned} g(k+n) &= g(k) + \sum_{j=k+1}^{k+n} \phi(j) \\ &= g(k) + \sum_{j=k+1}^{k+n} \#\{v \mid \sigma(v) \equiv -j \pmod{n}\} \\ &= g(k) + n. \end{aligned} \tag{12}$$

for all $k \in \mathbb{N}$. This suggests that a more natural domain of definition is the unit circle. First extend g to all of \mathbb{Z} by defining

$$g(-k) = g(nk - k) - nk, \quad k \in \mathbb{N}.$$

This is the unique extension with the property that $g - Id$ is periodic mod n . Now for $x \in \mathbb{R}$, let

$$f(x) = \frac{(1 - \{nx\})g(\lfloor nx \rfloor) + \{nx\}g(\lceil nx \rceil)}{n} \tag{13}$$

where $\lfloor y \rfloor$, $\lceil y \rceil$ and $\{y\}$ denote respectively the greatest integer $\leq y$, the least integer $\geq y$, and the fractional part of y .

By (12) we have for all $x \in \mathbb{R}$

$$\begin{aligned} f(x+1) &= \frac{(1 - \{nx\})g(\lfloor nx \rfloor + n) + \{nx\}g(\lceil nx \rceil + n)}{n} \\ &= f(x) + 1. \end{aligned}$$

Hence $f : \mathbb{R} \rightarrow \mathbb{R}$ descends to a circle map $\bar{f} : S^1 \rightarrow S^1$ (viewing S^1 as \mathbb{R}/\mathbb{Z}). Since f is nondecreasing, it has a well-defined *Poincaré rotation number* [8, 13]

$$\rho(f) = \lim_{t \rightarrow \infty} \frac{f^t(x) - x}{t} \tag{14}$$

which does not depend on x . Here f^t denotes the t -fold iterate $f^t(x) = f(f^{t-1}(x))$, with $f^0 = Id$. The rotation number measures the rate at which the sequence of points $x, \bar{f}(x), \bar{f}(\bar{f}(x)), \dots$ winds around the circle.

Theorem 2.5. *If σ is preconfined, then $a(\sigma) = \rho(f)$.*

Note that the map g is defined in terms of α_1 and ϕ , both of which are easily read off from σ . So given a preconfined configuration σ , equations (11) and (13) give an explicit recipe for writing down a circle map f whose rotation number is the activity of σ .

One naturally wonders how to generalize this construction to chip-firing configurations on graphs other than K_n . A key step may involve identifying invariants of the dynamics. On K_n , these invariants take a very simple form: by (8), for any two vertices $v, w \in [n]$, the difference

$$U^t \sigma(v) - U^t \sigma(w) \pmod n$$

does not depend on t . Analogous invariants for parallel chip-firing on the $n \times n$ torus are classified in [6].

3 Devil's Staircase

Let $\sigma_2, \sigma_3, \dots$ be a sequence of chip configurations, with σ_n defined on K_n , satisfying the conditions (4) and (5). Extend F to all of \mathbb{R} by setting

$$F(x + m) = F(x) + m, \quad m \in \mathbb{Z}. \tag{15}$$

Note that (4) and (5) force $F(0) = 0$ and $F(1) = 1$, so this extension is continuous.

The *rescaled activity phase diagram* of σ_n is the function $s_n : [0, 1] \rightarrow [0, 1]$ defined by

$$s_n(y) = a(\sigma_n + \lfloor ny \rfloor).$$

As $n \rightarrow \infty$, the s_n have a pointwise limit identified in our next result. Here and in what follows, $\rho(\cdot)$ denotes the rotation number (14).

Theorem 3.1. *If (4) and (5) hold, then for each $y \in [0, 1]$ we have*

$$s_n(y) \rightarrow s(y) := \rho(R_y \circ \Phi)$$

as $n \rightarrow \infty$, where $\Phi(x) = -F(-x)$, and $R_y(x) = x + y$.

Write $\Phi_y = R_y \circ \Phi$, and let $\bar{\Phi}_y : S^1 \rightarrow S^1$ be the corresponding circle map. We will call a function $s : [0, 1] \rightarrow [0, 1]$ a *devil's staircase* if it is continuous, nondecreasing, nonconstant, and locally constant on an open dense set. Next we show that if

$$(\bar{\Phi}_y)^q \neq Id \quad \text{for all } y \in S^1 \text{ and all } q \in \mathbb{N}, \tag{16}$$

then the limiting function $s(y)$ in Theorem 3.1 is a devil's staircase. Examples of these staircases for different choices of F are shown in Figure 2.

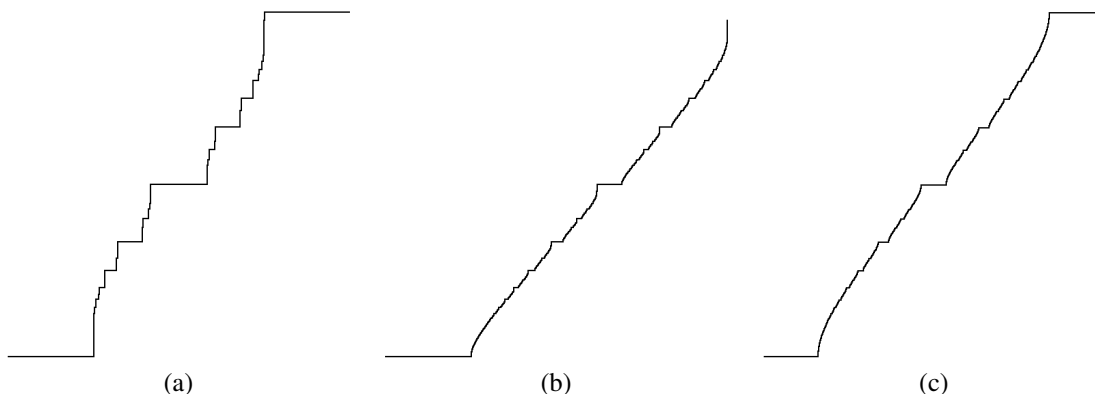


Fig. 2: The devil's staircase $s(y)$, when (a) $F(x)$ is given by (7); (b) $F(x) = \sqrt{x}$ for $x \in [0, 1]$; and (c) $F(x) = x + \frac{1}{2\pi} \sin 2\pi x$. On the horizontal axis y runs from 0 to 1, and on the vertical axis $s(y)$ runs from 0 to 1.

Proposition 3.2. *The function $s(y) = \rho(\Phi_y)$ is continuous and nondecreasing in y . If $z \in [0, 1]$ is irrational, then $s^{-1}(z)$ is a point. Moreover, if (16) holds, then for every rational number $p/q \in [0, 1]$ the fiber $s^{-1}(p/q)$ is an interval of positive length.*

Our next result shows that in the interiors of the stairs, we in fact have $s_n(y) = s(y)$ for sufficiently large n .

Proposition 3.3. *Suppose that (4), (5) and (16) hold. If $s^{-1}(p/q) = [a, b]$, then for any $\epsilon > 0$*

$$[a + \epsilon, b - \epsilon] \subset s_n^{-1}(p/q)$$

for all sufficiently large n .

The results in this section follow from Theorem 2.5 along with a few well-known properties of the rotation number $\rho(f)$. To give a flavor of the proofs, we list here the properties we use. For more background on the rotation number, see [8, 13].

Call a map $f : \mathbb{R} \rightarrow \mathbb{R}$ a *monotone degree one lift* if f is continuous, nondecreasing and satisfies

$$f(x + 1) = f(x) + 1 \tag{17}$$

for all $x \in \mathbb{R}$. Let f, f_n, g be monotone degree one lifts, and denote by $\bar{f}, \bar{f}_n, \bar{g}$ the corresponding circle maps $S^1 \rightarrow S^1$. Write $f \leq g$ if $f(x) \leq g(x)$ for all $x \in \mathbb{R}$, and $f < g$ if $f(x) < g(x)$ for all $x \in \mathbb{R}$.

- **Monotonicity.** If $f \leq g$, then $\rho(f) \leq \rho(g)$.
- **Continuity.** If $\sup |f_n - f| \rightarrow 0$, then $\rho(f_n) \rightarrow \rho(f)$.
- **Conjugation Invariance.** If g is strictly increasing, then $\rho(g \circ f \circ g^{-1}) = \rho(f)$.
- **Instability of an irrational rotation number.** If $\rho(f) \notin \mathbb{Q}$, and $f_1 < f < f_2$, then

$$\rho(f_1) < \rho(f) < \rho(f_2).$$

- **Stability of a rational rotation number.** If $\rho(f) = p/q \in \mathbb{Q}$, and $\bar{f}^q \neq Id : S^1 \rightarrow S^1$, then for sufficiently small $\epsilon > 0$, either

$$\rho(g) = p/q \text{ whenever } f \leq g \leq f + \epsilon,$$

or

$$\rho(g) = p/q \text{ whenever } f - \epsilon \leq g \leq f.$$

4 Short Period Attractors

For a chip configuration σ on K_n and a vertex $v \in [n]$, let

$$u_t(\sigma, v) = \#\{0 \leq s < t \mid U^s \sigma(v) \geq n\}$$

be the number of times v fires during the first t updates. During these updates, the vertex v emits a total of $n u_t(\sigma, v)$ chips and receives a total of $\alpha_t = \sum_w u_t(\sigma, w)$ chips, so that

$$U^t \sigma(v) - \sigma(v) = \alpha_t - n u_t(\sigma, v). \tag{18}$$

An easy consequence is the following.

Lemma 4.1. *A chip configuration σ on K_n satisfies $U^t \sigma = \sigma$ if and only if*

$$u_t(\sigma, v) = u_t(\sigma, w) \tag{19}$$

for all vertices v and w .

According to our next lemma, if σ is confined, then $u_t(\sigma, v)$ and $u_t(\sigma, w)$ differ by at most one.

Lemma 4.2. *If σ is confined, and $\sigma(v) \leq \sigma(w)$, then for all $t \geq 0$*

$$u_t(\sigma, v) \leq u_t(\sigma, w) \leq u_t(\sigma, v) + 1.$$

Lemma 4.3. *If σ is confined, then $U^t \sigma = \sigma$ if and only if $n \mid \alpha_t$.*

Let σ be a confined state on K_n . By the pigeonhole principle, there exist times $0 \leq s < t \leq n$ with

$$\alpha_s \equiv \alpha_t \pmod{n}.$$

By Lemma 4.3 it follows that $U^s \sigma = U^t \sigma$, so σ has eventual period at most n .

Our next result improves this bound a bit. Write $m(\sigma)$ for the eventual period of σ , and

$$\nu(\sigma) = \#\{\sigma(v) \mid v \in [n]\}$$

for the number of distinct heights in σ .

Proposition 4.4. *For any chip configuration σ on K_n ,*

$$m(\sigma) \leq \nu(\sigma).$$

Bitar [3] conjectured that any parallel chip-firing configuration on a connected graph of n vertices has eventual period at most n . A counterexample was found by Kiwi et al. [10]. It would be interesting to investigate for what classes of graphs Bitar's conjecture does hold; for example, no counterexample seems to be known on a regular graph.

Next we relate the period to the activity.

Lemma 4.5. *If $a(\sigma) = p/q$ and $(p, q) = 1$, then $m(\sigma) = q$.*

The proof uses the fact that the rotation number of a circle map determines the periods of its periodic points: if $f : \mathbb{R} \rightarrow \mathbb{R}$ is a monotone degree one lift (17) with $\rho(f) = p/q$ in lowest terms, then all periodic points of $\bar{f} : S^1 \rightarrow S^1$ have period q ; see Proposition 4.3.8 and Exercise 4.3.5 of [8].

Given $1 \leq p < q \leq n$ with $(p, q) = 1$ and $p/q \leq 1/2$, one can check that the chip configuration σ on K_n given by

$$\sigma(v) = \begin{cases} v + p - 1, & v \leq q - 1 - p \\ v + n + p - q - 1, & q - p \leq v \leq q - 1 \\ n + p - 1, & v \geq q. \end{cases}$$

has activity $a(\sigma) = p/q$. For a similar construction on more general graphs in the case $p = 1$, see [14, Prop. 6.8]. In particular, $m(\sigma) = q$ by Lemma 4.5. So for every integer $q = 1, \dots, n$ there exists a chip configuration on K_n of period q .

Despite the existence of states of period as large as n , states of smaller period are in some sense more prevalent. One way to capture this is the following.

Theorem 4.6. *If $\sigma_2, \sigma_3, \dots$ is a sequence of chip configurations satisfying (4), (5) and (16), then for each $q \in \mathbb{N}$ there is a constant $c = c_q > 0$ such that for all sufficiently large n , at least cn of the states $\{\sigma_n + k\}_{k=0}^n$ have eventual period q .*

The proof follows easily from Proposition 3.3, which shows that a constant fraction cn of the states $\sigma_n + k$ have activity $1/q$. By Lemma 4.5 these states have eventual period q . The devil's staircase $s(y)$ determines the best possible constant c_q , namely, the total length of all stairs whose height has denominator q . If $s^{-1}(p/q) = [a_p, b_p]$, then any constant

$$c_q < \sum_{p:(p,q)=1} (b_p - a_p)$$

satisfies the conclusion of the theorem.

The rest of this section outlines the proof of Theorem 4.11, which finds a *period 2 window*: any chip configuration on K_n with total number of chips strictly between $n^2 - n$ and n^2 has eventual period 2. The following lemma is a special case of [14, Prop. 6.2].

Lemma 4.7. *If σ and τ are chip configurations on K_n with $\sigma(v) + \tau(v) = 2n - 1$ for all v , then $a(\sigma) + a(\tau) = 1$.*

Given a chip configuration σ on K_n , for $j = 1, \dots, n$ we define *conjugate* configurations

$$c^j \sigma(v) = \begin{cases} \sigma(v) + j - n, & v \leq j \\ \sigma(v) + j, & v > j. \end{cases}$$

Lemma 4.8. *Let σ be a chip configuration on K_n , and fix $j \in [n]$. For all $t \geq 0$, we have for $v \leq j$*

$$u_t(\sigma, v) - 1 \leq u_t(c^j \sigma, v) \leq u_t(\sigma, v),$$

while for $v > j$

$$u_t(\sigma, v) \leq u_t(c^j \sigma, v) \leq u_t(\sigma, v) + 1.$$

Corollary 4.9. *For any chip configuration σ on K_n and any $j \in [n]$,*

$$a(c^j \sigma) = a(\sigma).$$

It turns out that the circle maps corresponding to σ and $c^j \sigma$ are conjugate to one another by a rotation. This gives an alternative proof of the corollary, in the case when both σ and $c^j \sigma$ are preconfined.

Lemma 4.10. *Let σ be a chip configuration on K_n . If $u_2(\sigma, v) \geq 1$ for all v , then $u_{2t}(\sigma, v) \geq t$ for all v and all $t \geq 1$.*

Write

$$|\sigma| = \sum_{v=1}^n \sigma(v)$$

for the total number of chips in the system.

Theorem 4.11. *Every chip configuration σ on K_n with $n^2 - n < |\sigma| < n^2$ has eventual period 2.*

The outline of the proof runs as follows. Writing

$$\ell(\sigma) = \min\{\sigma(1), \dots, \sigma(n)\}$$

and

$$r(\sigma) = \#\{v \in [n] : \sigma(v) \geq n\},$$

a straightforward calculation shows that if $\sigma(1) \geq \sigma(2) \geq \dots \geq \sigma(n)$ and $n^2 - n < |\sigma| < n^2$, then

$$\sum_{j=1}^n (\ell(c^j \sigma) + r(c^j \sigma)) > n^2 - n.$$

Since each term in the sum on the left is a nonnegative integer, we must have

$$\ell(c^j \sigma) + r(c^j \sigma) \geq n.$$

for some $j \in [n]$. Thus every vertex v fires at least once during the first two updates of $c^j \sigma$. By Corollary 4.9 and Lemma 4.10, this implies

$$a(\sigma) = a(c^j \sigma) \geq \frac{1}{2}.$$

The chip configuration $\tau(v) := 2n - 1 - \sigma(v)$ also satisfies $n^2 - n < |\tau| < n^2$, so $a(\tau) \geq \frac{1}{2}$ as well. By Lemma 4.7 we have $a(\sigma) + a(\tau) = 1$, so $a(\sigma) = a(\tau) = \frac{1}{2}$. Finally, from Lemma 4.5 we conclude that $m(\sigma) = 2$.

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