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Record statistics in integer compositions

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A composition $\sigma = a_1 a_2 \dots a_m$ of n is an ordered collection of positive integers whose sum is n . An element a_i in σ is a strong (weak) *record* if $a_i > a_j$ ($a_i \geq a_j$) for all $j = 1, 2, \dots, i - 1$. Furthermore, the position of this record is i . We derive generating functions for the total number of strong (weak) records in all compositions of n , as well as for the sum of the positions of the records in all compositions of n , where the parts a_i belong to a fixed subset A of the natural numbers. In particular when $A = \mathbb{N}$, we find the asymptotic mean values for the number, and for the sum of positions, of records in compositions of n .

Keywords: Composition, Record, Left-to-right maxima, Generating function, Mellin transforms, Asymptotic estimates

Introduction

Let $\pi = a_1 a_2 \dots a_n$ be any permutation of length n , an element a_i in π is a *record* if $a_i > a_j$ for all $j = 1, 2, \dots, i - 1$. Furthermore, the position of this record is i . The number of records was first studied by Rényi [13], compare also [7]. A survey of results on this topic can be found in [2]. In the literature records are also referred to as a *left-to-right maxima* or *outstanding elements*. In particular the study of records has applications to observations of extreme weather problems, test of randomness, determination of minimal failure, and stresses of electronic components. The recent paper by Kortchemski [8] defines a new statistic *srec*, where $srec(\pi)$ is the sum over the positions of all records in π . For instance, the permutation $\pi = 451632$ has 3 records 4, 5, 6 and $srec(\pi) = 1 + 2 + 4 = 7$.

A word over an alphabet A , a set of positive integers, is defined as any ordered sequence of possibly repeated elements of A . Recently, Prodinger [12] studied the statistic *srec* for words over the alphabet $\mathbb{N} = \{1, 2, 3, \dots\}$, equipped with geometric probabilities p, pq, pq^2, \dots , with $p + q = 1$. In the case of words there two versions: A *strong record* in a word $a = a_1 \dots a_n$ is an element a_i such that $a_i > a_j$ for all $j = 1, 2, \dots, i - 1$ (that is, must be strictly larger than elements to the left) and *weak record* is an element $a_i \geq a_j$ for all $j = 1, 2, \dots, i - 1$ (must be only larger or equal to elements to the left). Furthermore, the position i is called the position of the strong record (weak record). We denote the sum of

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the positions of all strong (respectively, weak) records in a word a by $ssrec$ (respectively, $wsrec$). In [12], Prodinger found the expected value of the sum of the positions of strong records, in random geometrically distributed words of length n . Previously, Prodinger [10] also studied the number of strong and weak records, in samples of geometrically distributed random variables. He also studied further properties of such records in papers such as [11] and references therein.

A composition $\sigma = \sigma_1 \sigma_2 \dots \sigma_m$ of n is an ordered collection of positive integers whose sum is n . Thus a composition σ of n with parts in A is a restricted word over the alphabet A . We denote the set of all compositions of n with m parts in A by $C_A(n, m)$. It is well known that the number of compositions of $n \geq 1$ with m parts in \mathbb{N} is given by $\binom{n-1}{m-1}$ and that the total number of compositions of n is 2^{n-1} .

In this paper we find generating functions for these parameters, number of strong records, number of weak records, sum of positions of strong records, and sum positions of weak records in a random composition of n with parts in $A = [d] := \{1, 2, \dots, d\}$ or $A = \mathbb{N}$. We also study the mean values of these parameters as $n \rightarrow \infty$ in the case $A = \mathbb{N}$ by means of rational function asymptotics and Mellin transforms. Details of some of the lengthier proofs will be left to the full version of the paper. We remark that in [5], an asymptotic correspondence is established between compositions of n and samples of geometric variable of parameter $p = 1/2$ and length $n/2$. By exploiting this correspondence, and using the already established results of Prodinger for samples of geometric random variables, alternative derivations of our asymptotic results can be obtained.

1 Number strong records and weak records

Let $NSR_A(z, y, q)$ and $NWR_A(z, y, q)$ be the generating function for the number of compositions of n with m parts in A according to the number of strong and weak records, respectively, that is,

$$\begin{aligned} NSR_A(z, y, q) &= \sum_{n, m \geq 0} \sum_{\sigma \in C_A(n, m)} z^n y^m q^{nsr(\sigma)}, \\ NWR_A(z, y, q) &= \sum_{n, m \geq 0} \sum_{\sigma \in C_A(n, m)} z^n y^m q^{nwr(\sigma)}, \end{aligned}$$

where $nsr(\sigma)$ and $nwr(\sigma)$ is the number of strong and weak records in composition σ , respectively. In this section we find an explicit formulas for those generating functions.

Theorem 1.1 *The generating function $NSR_{[d]}(z, y, q)$ is given by*

$$NSR_{[d]}(z, y, q) = \prod_{j=1}^d \left(1 + \frac{z^j y q}{1 - y \sum_{i=1}^j z^i} \right).$$

Proof: We denote the number of occurrences of the part d in the composition $\sigma \in C_{[d]}(n, m)$ by $\ell(\sigma)$. Now let us write equation for the generating function $NSR_{[d]}(z, y, q)$. The contribution of the case $\ell(\sigma) = 0$ is given by $NSR_{[d-1]}(z, y, q)$. Assume $\ell(\sigma) > 0$, then σ can be decomposed as $\sigma' d \sigma''$, where σ' is a composition with parts in $[d-1]$ and σ'' is a composition with parts in $[d]$. Thus, the contribution of the case $\ell(\sigma) > 0$ equals $z^d y q NSR_{[d-1]}(z, y, q) NSR_{[d]}(z, y, 1)$. Therefore,

$$NSR_{[d]}(z, y, q) = NSR_{[d-1]}(z, y, q) + z^d y q NSR_{[d-1]}(z, y, q) NSR_{[d]}(z, y, 1).$$

For $q = 1$ and by induction we have that

$$NSR_{[d]}(z, y, 1) = \frac{1}{1 - y \sum_{j=1}^d z^j}.$$

Hence,

$$NSR_{[d]}(z, y, q) = \prod_{j=1}^d \left(1 + \frac{z^j y q}{1 - y \sum_{i=1}^j z^i} \right),$$

as claimed. \square Theorem 1.1 with $q = 1$ gives that the generating function for the number of compositions of n with m parts in $[d]$ is given by $NSR_{[d]}(z, y, 1) = \frac{1}{1 - y \sum_{i=1}^d z^i}$, see for example [4].

Also from Theorem 1.1 we get that

$$\begin{aligned} \frac{\partial}{\partial q} NSR_{[d]}(z, y, 1) &= \prod_{j=1}^d \left(1 + \frac{z^j y}{1 - y \sum_{i=1}^j z^i} \right) \left(\sum_{j=1}^d \frac{z^j y}{1 - y \sum_{i=1}^{j-1} z^i} \right) \\ &= \frac{1}{1 - y \sum_{i=1}^d z^i} \left(\sum_{j=1}^d \frac{z^j y}{1 - y \sum_{i=1}^{j-1} z^i} \right). \end{aligned}$$

Hence, the generating function for the number strong records in all compositions of n with parts in \mathbb{N} is given by

$$f(z) := \frac{1}{1 - \sum_{i \geq 1} z^i} \sum_{j \geq 1} \frac{z^j}{1 - \sum_{i=1}^{j-1} z^i} = \frac{1-z}{1-2z} \sum_{j \geq 1} \frac{z^j}{1 - \sum_{i=1}^{j-1} z^i}.$$

Theorem 1.2 *The average number E_n^s of strong left-to-right maxima in the context of compositions of n has the asymptotic expansion*

$$E_n^s = \frac{1}{2} \left[\log_2 n - \frac{1}{2} + \frac{\gamma}{L} - \delta(\log_2 n) \right] + o(1).$$

Here and in the rest of the paper, $L = \log 2$; γ is Euler's constant and $\delta(x)$ is a periodic function of period 1 and mean 0 and small amplitude, which is given by the Fourier series

$$\delta(x) = \frac{1}{L} \sum_{k \neq 0} \Gamma(-\chi_k) e^{2k\pi i x}.$$

The complex numbers χ_k are given by $\chi_k = 2k\pi i / L$.

Proof: Firstly by summing the finite geometric series and using partial fraction decomposition,

$$f(z) = \frac{z - z^2}{1 - 2z} + (1 - z)^2 \sum_{k \geq 2} \left[\frac{1}{1 - 2z} - \frac{1}{1 - 2z + z^k} \right].$$

Hence the average number E_n^s of strong left-to-right maxima in compositions of n satisfies

$$E_n^s = \frac{1}{2^{n-1}} [z^n] f(z) = \frac{1}{2} + \frac{1}{2^{n-1}} [z^n] (1-z)^2 \sum_{k=2}^n \left[\frac{1}{1-2z} - \frac{1}{1-2z+z^k} \right].$$

Let ρ_k be the smallest positive root of the denominator polynomial $1-2z+z^k$ that lies between $1/2$ and 1 . An application of the principle of the argument or Rouché's Theorem shows such a root to exist with all other roots of modulus greater than $3/4$. By dominant pole analysis,

$$q_{n,k} := [z^n] \frac{(1-z)^2}{1-2z+z^k} = c_k \rho_k^{-n} + O\left(\left(\frac{4}{3}\right)^n\right) \quad \text{with} \quad c_k = \frac{(1-\rho_k)^2}{\rho_k(2-k\rho_k^{k-1})},$$

for large n but fixed k . The denominator polynomial $1-2z+z^k$ behaves like a perturbation of $1-2z$ near $z = 1/2$. By "bootstrapping" we find that

$$\rho_k = \frac{1}{2} + 2^{-k-1} + O(k2^{-2k})$$

and hence $c_k = \frac{1}{4} + O(k2^{-k})$. The use of this approximation can be justified for a wide range of values of k and n (see for example [3] or [6]).

Let us now restrict our attention to those k for which $n^{-3} \leq 2^{-k} \leq \frac{\log n}{n}$. For such k we can show that

$$q_{n,k} = 2^{n-2} \left(\exp\left(-\frac{n}{2^k}\right) + O\left(\frac{\log^3 n}{n}\right) \right). \quad (1)$$

Turning next to smaller values of $k \geq 2$, that is, k such that $2^{-k} > \frac{\log n}{n}$, we find that now the coefficients $q_{n,k}$ are relatively small, since for such k , $q_{n,k} = O\left(\frac{2^n}{n}\right)$ as $n \rightarrow \infty$. Finally we must consider larger values of $k \leq n$ that is, k for which $n^{-3} > 2^{-k}$, or equivalently, $k \geq 3 \log_2 n$. In this range we find that

$$q_{n,k} = 2^{n-2} \left(\exp\left(-\frac{n}{2^k}\right) + O\left(\frac{1}{n^2}\right) \right). \quad (2)$$

Then combining the estimates for $q_{n,k}$ over the range $2 \leq k \leq n$ above,

$$\begin{aligned} E_n^s - \frac{1}{2} &= \frac{1}{2} \sum_{k=2}^n \left(1 - \frac{q_{n,k}}{2^{n-2}} \right) \\ &\sim \frac{1}{2} \sum_{k \geq 0} \left(1 - \exp\left(-\frac{n}{2^k}\right) \right) - 1, \end{aligned}$$

as the additional tail sum $\sum_{k > n} (1 - \exp(-\frac{n}{2^k}))$ is exponentially small. It remains to estimate

$$h(n) := \sum_{k \geq 0} \left(1 - \exp\left(-\frac{n}{2^k}\right) \right),$$

as $n \rightarrow \infty$. For this we use Mellin transforms and find (see [1, Appendix B.7, equation (48)])

$$h(n) = \log_2 n + \frac{1}{2} + \frac{\gamma}{L} - \delta(\log_2 n) + O(1/n). \quad (3)$$

The asymptotic estimate for E_n^s follows. \square

Remarks Asymptotically we find that the expected number of strict left-to-right maxima is half the expected size of the *largest part* in a random composition of n (see [9]). Also, as mentioned in the introduction, the asymptotic correspondence established in [5] would allow one to use the results of Prodinger [10] in the case $p = 1/2$, to give an alternative proof of Theorem 1.2.

A similar approach to that of Theorem 1.3 leads to

Theorem 1.3 *The generating function $NWR_{[d]}(z, y, q)$ is given by*

$$NWR_{[d]}(z, y, q) = \prod_{j=1}^d \frac{1}{1 - \frac{z^j y q}{1 - y \sum_{i=1}^{j-1} z^i}}.$$

The generating function for the total number of weak records in compositions over \mathbb{N} is then

$$\begin{aligned} g(z) &:= \left. \frac{\partial NWR_{[\mathbb{N}]}(z, 1, q)}{\partial q} \right|_{q=1} = \frac{1-z}{1-2z} \sum_{k \geq 1} \frac{z^k}{1 - \sum_{i=1}^k z^i} \\ &= \frac{(1-z)^2}{z} \sum_{k \geq 2} \left[\frac{1}{1-2z} - \frac{1}{1-2z+z^k} \right]. \end{aligned}$$

Theorem 1.4 *The average number E_n^w of weak left-to-right maxima in the context of compositions of n has the asymptotic expansion*

$$E_n^w = \log_2 n - \frac{3}{2} + \frac{\gamma}{L} - \delta(\log_2 n) + o(1).$$

Proof: The average number E_n^w of weak left-to-right maxima in compositions of n satisfies

$$E_n^w = \frac{1}{2^{n-1}} [z^n] g(z) = \frac{1}{2^{n-1}} [z^{n+1}] (1-z)^2 \sum_{k=2}^n \left[\frac{1}{1-2z} - \frac{1}{1-2z+z^k} \right].$$

Then using the $q_{n,k}$ notation in the proof of Theorem 1.2

$$E_n^w = \sum_{k=2}^{n+1} \left(1 - \frac{q_{n+1,k}}{2^{n-1}} \right) = 2E_{n+1}^s - 1.$$

The asymptotic estimate then follows from that of Theorem 1.2. \square

2 The statistics $ssrec$ and $wsrec$ on the set of compositions

Let $NSR_A(z, y, q)$ and $NWR_A(z, y, q)$ be the generating function for the number of compositions of n with m parts in A according to the statistic $ssrec$ and $wsrec$, respectively, that is,

$$\begin{aligned} PSR_A(z, y, q) &= \sum_{n,m \geq 0} \sum_{\sigma \in C_A(n,m)} z^n y^m q^{ssrec(\sigma)}, \\ PWR_A(z, y, q) &= \sum_{n,m \geq 0} \sum_{\sigma \in C_A(n,m)} z^n y^m q^{wsrec(\sigma)}. \end{aligned}$$

Theorem 2.1 The generating function $PSR_{[d]}(z, y, q)$ is given by

$$1 + \sum_{k=1}^d q^k \left(\sum_{d \geq j_1 > j_2 > \dots > j_k \geq 1} \prod_{i=1}^k \frac{z^{j_i} y q^{i-1}}{1 - y q^{i-1} \sum_{\ell=1}^{j_i} z^\ell} \right).$$

Proof: We denote the number of occurrences of the part d in the composition $\sigma \in C_{[d]}(n, m)$ by $\ell(\sigma)$. Decomposing according to $\ell(\sigma) = 0$ and $\ell(\sigma) > 0$ leads to

$$PSR_{[d]}(z, y, q) = PSR_{[d-1]}(z, y, q) + z^d y q PSR_{[d-1]}(z, qy, q) PSR_{[d]}(z, y, 1). \quad (4)$$

For $q = 1$, $PSR_{[d]}(z, y, 1) = \frac{1}{1 - y \sum_{j=1}^d z^j}$. Hence,

$$\begin{aligned} PSR_{[d]}(z, y, q) &= PSR_{[d-1]}(z, y, q) + \frac{z^d y q}{1 - y \sum_{i=1}^d z^i} PSR_{[d-1]}(z, qy, q) \\ &= PSR_{[d-2]}(z, y, q) + \sum_{j=d-1}^d \frac{z^j y q}{1 - y \sum_{i=1}^j z^i} PSR_{[j-1]}(z, qy, q) \\ &\vdots \\ &= 1 + \sum_{j=1}^d \frac{z^j y q}{1 - y \sum_{i=1}^j z^i} PSR_{[j-1]}(z, qy, q). \end{aligned}$$

Iterating the above recurrence relation d times we get the desired result. \square

From this we derive

Corollary 2.2 The generating function $v_d(z) = \frac{\partial}{\partial q} PSR_{[d]}(z, 1, q) \big|_{q=1}$ is given by

$$\frac{z}{1 - \sum_{j=1}^d z^j} \sum_{j=0}^{d-1} \frac{z^j}{\left(1 - \sum_{i=1}^j z^i\right)^2}.$$

The above corollary gives that the generating function for the number of compositions of n according to the total of the statistic $ssrec$ is given by

$$v(z) := \frac{z(1-z)}{1-2z} \sum_{j \geq 0} \frac{z^j}{\left(1 - \sum_{i=1}^j z^i\right)^2}.$$

The rather lengthy proof of the asymptotic behaviour of the coefficients of $v(z)$ will be left for the journal version of the paper. We obtain

Theorem 2.3 The average sum of the positions of the strong records e_n^s in compositions of n has the asymptotic expansion

$$e_n^s = \frac{n}{4 \log 2} (1 + \delta_2 (\log_2 n)) + o(n),$$

where $\delta_2(x)$ is a periodic function of period 1, mean zero and small amplitude, which is given by the Fourier series

$$\delta_2(x) = \sum_{k \neq 0} \chi_k \Gamma(-1 - \chi_k) e^{2k\pi i x}.$$

With reference again to [5], Theorem 2.3 is seen to correspond to the $p = 1/2$ case of the results of Prodinger [12].

The corresponding results for $PWR_{[d]}(z, y, q)$ are as follows.

Theorem 2.4 *The generating function $PWR_{[d]}(z, y, q)$ satisfies the following recurrence relation*

$$PWR_{[d]}(z, y, q) = PWR_{[d-1]}(z, y, q) + \frac{z^d y q}{1 - y \sum_{i=1}^{d-1} z^i} PWR_{[d]}(z, qy, q).$$

Corollary 2.5 *The generating function $w_d(z) = \frac{\partial}{\partial q} PWR_{[d]}(z, 1, q) \big|_{q=1}$ is given by*

$$w_d(z) = \frac{1}{1 - \sum_{j=1}^d z^j} \sum_{j \geq 1} \frac{z^j}{\left(1 - \sum_{i=1}^j z^i\right)^2}.$$

The above corollary gives that the generating function for the number of compositions of n according to the total of the statistic *surec* is given by

$$w(z) := \frac{1-z}{1-2z} \sum_{j \geq 1} \frac{z^j}{\left(1 - \sum_{i=1}^j z^i\right)^2}.$$

Theorem 2.6 *The average sum of the positions of the weak records e_n^w in compositions of n has the asymptotic expansion*

$$e_n^w = \frac{n}{2 \log 2} (1 + \delta_2(\log_2 n)) + o(n),$$

where $\delta_2(x)$ is the same periodic function that occurred in Theorem 2.3.

Proof: The generating functions $v(z)$ and $w(z)$ are related as follows,

$$v(z) = \frac{z(1-z)}{1-2z} + zw(z).$$

From this we see that

$$[z^{n+1}]v(z) = 2^{n-1} + [z^n]w(z).$$

So that $e_n^w = 2e_{n+1}^s - 1$. The result then follows from Theorem 2.3. \square

Now, our aim is to present a combinatorial explanation for the fact that the number (sum) of the positions of weak records in all compositions of n plus 2^{n-1} equals the number (sum) of the positions of strong records in all compositions of $n+1$, for $n \geq 1$. In order to do that we need the following notations.

Let $sw_{n,r}$ (respectively, $ss_{n,r}$) be the sum of r -th power of the positions of weak (respectively, strong) records in all the compositions of n , namely,

$$\begin{aligned} sw_{n,r} &= \sum_{\sigma \in C_n} \sum_{\sigma_i \text{ is a weak record of } \sigma} i^r, \\ ss_{n,r} &= \sum_{\sigma \in C_n} \sum_{\sigma_i \text{ is a strong record of } \sigma} i^r, \\ sw'_{n,r} &= \sum_{\sigma \in C_n(A)} \sum_{\sigma_i \text{ is a weak record of } \sigma, i > 1} i^r, \\ ss'_{n,r} &= \sum_{\sigma \in C_n(A)} \sum_{\sigma_i \text{ is a strong record of } \sigma, i > 1} i^r, \end{aligned}$$

where $C_n = \cup_{m=1}^n C_{n,m}$ is the set of all compositions of n . From the definitions, each first letter is a weak (strong) record. Therefore,

$$sw_{n,r} = |C_n| + sw'_{n,r} \text{ and } ss_{n,r} = |C_n| + ss'_{n,r}, \quad (5)$$

where $|C_n| = 2^{n-1}$ is the number of compositions of n .

Theorem 2.7 For all $n \geq 1$,

$$ss_{n+1,r} = sw_{n,r} + 2^{n-1}.$$

Proof: It is not hard to see that $\sigma_1 \cdots \sigma_m$ is a composition of n and $\sigma_i, i > 1$, is a weak record if and only if $\sigma_1 \cdots \sigma_{i-1}(\sigma_i + 1)\sigma_{i+1} \cdots \sigma_m$ is a composition of n and $\sigma_i + 1, i > 1$, is a strong record. Therefore, the multiset of all positions $i, i > 1$, of the weak records in all compositions of n is the same multiset as all positions $i, i > 1$, of the strong records in all compositions on $n + 1$. In other words, $ss'_{n+1,r} = sw'_{n,r}$ for all n and r .

Hence, by (5) we have

$$ss_{n+1,r} = 2^n + ss'_{n+1,r} = 2^n + sw'_{n,r} = 2^{n-1} + 2^{n-1} + sw'_{n,r} = 2^{n-1} + sw_{n,r},$$

as requested. \square

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