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# *$q, t$ -Fuß-Catalan numbers for complex reflection groups*

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**Abstract.** In type  $A$ , the  $q, t$ -Fuß-Catalan numbers  $\text{Cat}_n^{(m)}(q, t)$  can be defined as a bigraded Hilbert series of a module associated to the symmetric group  $\mathcal{S}_n$ . We generalize this construction to (finite) complex reflection groups and exhibit some nice conjectured algebraic and combinatorial properties of these polynomials in  $q$  and  $t$ . Finally, we present an idea how these polynomials could be related to some graded Hilbert series of modules arising in the context of rational Cherednik algebras. This is work in progress.

**Résumé.** Dans le cas du type  $A$ , les  $q, t$ -nombres de Fuß-Catalan  $\text{Cat}_n^{(m)}(q, t)$  peuvent être définis comme la série de Hilbert bigraduée d'un certain module associé au groupe symétrique  $\mathcal{S}_n$ . Nous généralisons cette construction aux groupes de réflexion complexes (finis) et nous formulons de jolies propriétés (conjecturales) algébriques et combinatoires de ces polynômes en  $q$  et  $t$ . Enfin, nous décrivons une idée sur la manière dont ces polynômes pourraient être liés à certaines séries de Hilbert de modules apparaissant dans le contexte des algèbres de Cherednik rationnelles. Ceci est un travail en cours.

**Keywords:**  $q, t$ -Catalan numbers, reflection group, Shi arrangement, coinvariant ring, rational Cherednik algebras

## 1 Introduction

Within the last 15 years the  $q, t$ -Fuß-Catalan numbers of type  $A$ ,  $\text{Cat}_n^{(m)}(q, t)$ , arose in more and more contexts in mathematics, namely in *symmetric functions theory*, *algebraic* and *enumerative combinatorics*, *representation theory* and *algebraic geometry*. They first appeared in a paper by A. Garsia and M. Haiman in the context of *modified Macdonald polynomials*, [11]. Later, in his work on the  $n!$ - and on the  $(n + 1)^{n-1}$ -conjecture, M. Haiman showed that  $\text{Cat}_n^{(m)}(q, t)$  is equal to the Hilbert series of the alternating component of the *diagonal coinvariant ring*, [15]. J. Haglund, [14], and N. Loehr, [18], found a very interesting (partially conjectured) combinatorial interpretation of  $\text{Cat}_n^{(m)}(q, t)$ . The  $q, t$ -Catalan numbers have many interesting algebraic and combinatorial properties. To mention some: they are symmetric functions in  $q$  and  $t$  with positive integer coefficients and give a  $q, t$ -extension of the famous *Fuß-Catalan numbers*

$$\text{Cat}_n^{(m)} := \frac{1}{mn + 1} \binom{(m + 1)n}{n}.$$

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Specializing  $t = 1$  in  $\text{Cat}_n^{(m)}(q, t)$  gives some  $q$ -Fuß-Catalan numbers introduced by J. Fülrlinger and J. Hofbauer in [9] and specializing  $t = q^{-1}$  gives another  $q$ -extension of the Fuß-Catalan numbers introduced in [19] by P.A. MacMahon.

We generalize the definition of Fuß-Catalan numbers by means of a Hilbert series from the symmetric group to arbitrary finite complex reflection groups. Furthermore, we present conjectured generalizations of the properties mentioned above concerning specializations of  $q$  and  $t$ , and combinatorial interpretations, see Conjectures 3.10, 3.11 and 4.17. Finally, we present an idea which would relate them to some graded Hilbert series constructed by I. Gordon [12] and by Y. Berest, P. Etingof and V. Ginzburg [4] in the context of *rational Cherednik algebras*, see Conjecture 5.3 and the following corollary.

This extended abstract is organized as follows: in Section 2, we define alternating polynomials of type  $A$  and generalize this definition to complex reflection groups. In Section 3, we first define  $q, t$ -Fuß-Catalan numbers of type  $A$  as a bigraded Hilbert series and present some properties. Then we generalize this definition to  $q, t$ -Fuß-Catalan numbers for complex reflection groups,  $\text{Cat}_n^{(m)}(W, q, t)$ , and present two conjectures concerning the specializations  $q = t = 1$  and  $t = q^{-1}$ . In Section 4, we combinatorially define  $q$ -Fuß-Catalan numbers in terms of the *extended Shi arrangement* and conjecture that they appear in the specialization  $t = 1$  of  $\text{Cat}_n^{(m)}(W, q, t)$ . In Section 5, we present another conjecture which would connect  $\text{Cat}_n^{(m)}(W, q, t)$  to a graded Hilbert series in the context of *rational Cherednik algebras*.

## 2 Alternating polynomials

### 2.1 Alternating polynomials associated to the symmetric group

The symmetric group  $\mathcal{S}_n$ , which is the reflection group of type  $A_{n-1}$ , acts on the polynomial ring

$$\mathbb{C}[\mathbf{x}, \mathbf{y}] := \mathbb{C}[x_1, y_1, \dots, x_n, y_n]$$

by permuting the coordinates in  $\mathbf{x}$  and  $\mathbf{y}$  simultaneously amongst themselves. This is the *diagonal action*

$$\sigma(x_i) := x_{\sigma(i)}, \sigma(y_i) := y_{\sigma(i)} \quad \text{for } \sigma \in \mathcal{S}_n.$$

A polynomial  $p \in \mathbb{C}[\mathbf{x}, \mathbf{y}]$  is called *alternating* if

$$\sigma(p) = \text{sgn}(\sigma)p \quad \text{for all } \sigma \in \mathcal{S}_n,$$

where  $\text{sgn}(\sigma)$  is the usual sign of the permutation  $\sigma$ . We denote the space of all alternating polynomials by  $\mathbb{C}[\mathbf{x}, \mathbf{y}]^\epsilon$ . As a vector space,  $\mathbb{C}[\mathbf{x}, \mathbf{y}]^\epsilon$  has a well-known basis: For  $G = \{(\alpha_1, \beta_1), \dots, (\alpha_n, \beta_n)\} \subseteq \mathbb{N} \times \mathbb{N}$  define a bivariate analogue of the Vandermonde determinant  $\Delta_G$  by

$$\Delta_G = \det \begin{pmatrix} x_1^{\alpha_1} y_1^{\beta_1} & \dots & x_1^{\alpha_n} y_1^{\beta_n} \\ \vdots & & \vdots \\ x_n^{\alpha_1} y_n^{\beta_1} & \dots & x_n^{\alpha_n} y_n^{\beta_n} \end{pmatrix}.$$

The set

$$\mathcal{B} := \{\Delta_G : G \subseteq \mathbb{N} \times \mathbb{N}, |G| = n\}$$

forms a vector space basis of  $\mathbb{C}[\mathbf{x}, \mathbf{y}]^\epsilon$  and in particular the ideal generated by  $\mathcal{B}$  is the same as the ideal generated by all alternating polynomials.

## 2.2 Alternating polynomials associated to any complex reflection group

The concept for polynomials to be alternating can be generalized to any (finite) complex reflection group in the following way: let  $V$  be an  $n$ -dimensional complex vector space and let  $W \subseteq \text{GL}(V)$  be a (finite) complex reflection group acting on  $V$ . For definitions and further information on complex reflection groups see e.g. [6].

The *contragredient action* of  $W$  on  $V^* = \text{Hom}(V, \mathbb{C})$  is given by

$$\omega(\rho) := \rho \circ \omega^{-1}.$$

This induces an action of  $W$  on the symmetric algebra  $S(V^*)$  which is equal to  $\mathbb{C}[\mathbf{x}]$ . “Doubling up” this action diagonally defines a *diagonal action* of  $W$  on  $\mathbb{C}[\mathbf{x}, \mathbf{y}]$ .

**Definition 2.1** *Let  $W$  be a complex reflection group acting on a complex vector space of dimension  $n$ . We call a polynomial  $p \in \mathbb{C}[\mathbf{x}, \mathbf{y}]$  alternating if*

$$\det(\omega)\omega(p) = p \quad \text{for all } \omega \in W.$$

For  $W$  being the complex reflection group of type  $A_{n-1}$  - which is the symmetric group  $\mathcal{S}_n$  - this definition reduces to the definition of alternating polynomials given above. We denote the space of all alternating polynomials by  $\mathbb{C}[\mathbf{x}, \mathbf{y}]^{\epsilon, W}$ .

**Remark 2.2** *If  $W$  is a real reflection group or, equivalently, if  $W$  is a (finite) Coxeter group then*

$$\det(\omega) = (-1)^{l(\omega)},$$

where  $l$  is the length function in the Coxeter group  $W$ .

The reason why we call a polynomial  $p$  alternating if  $\det(\omega)\omega(p) = p$  and not if  $\omega(p) = \det(\omega)p$  is the following: Define the *sign idempotent*  $\mathbf{e}_\epsilon$  by

$$\mathbf{e}_\epsilon := \frac{1}{|W|} \sum_{\omega \in W} \det(\omega)\omega$$

and the *sign representation*  $\epsilon$  by  $\omega(z) := \det(\omega)z$  for all  $\omega \in W$  and  $z \in \mathbb{C}$ . Then

$$\mathbb{C}[\mathbf{x}, \mathbf{y}]^{\epsilon, W} = (\mathbb{C}[\mathbf{x}, \mathbf{y}] \otimes \epsilon)^W = \mathbf{e}_\epsilon \mathbb{C}[\mathbf{x}, \mathbf{y}].$$

As for the symmetric group,  $\mathbb{C}[\mathbf{x}, \mathbf{y}]^{\epsilon, W}$  has a vector space basis given by

$$\mathcal{B}_W := \{ \mathbf{e}_\epsilon(m) : m \text{ monomial in } \mathbf{x}, \mathbf{y} \text{ with } \mathbf{e}_\epsilon(m) \neq 0 \}.$$

**Remark 2.3** • For  $W$  of type  $A$ ,  $\mathcal{B}_W$  reduces to  $\mathcal{B}$  defined above,

- for  $W$  of type  $B$ ,  $\mathcal{B}_W$  reduces to  $\{ \Delta_G : G \subseteq \mathbb{N} \times \mathbb{N}, |G| = n, \alpha_i + \beta_i \equiv 1 \pmod{2} \}$ ,
- for  $W$  of type  $D$ ,  $\mathcal{B}_W$  reduces to  $\{ \Delta_G : G \subseteq \mathbb{N} \times \mathbb{N}, |G| = n, \alpha_i + \beta_i \equiv \alpha_j + \beta_j \pmod{2} \}$ .

### 3 $q, t$ -Fuß-Catalan numbers

Before we define  $q, t$ -Fuß-Catalan numbers in general, we review the definition and the properties about the well-studied case  $W = \mathcal{S}_n$  which they seem to generalize. To refer to the parameter  $m \in \mathbb{N}$ , we use the term *Fuß-Catalan* which has commonly been used in the literature for *higher* Catalan numbers of general type, see e.g. [2] and [8]. In the literature concerning only the case  $W = \mathcal{S}_n$ , the name *generalized  $q, t$ -Catalan numbers* was more usual.

#### 3.1 $q, t$ -Fuß-Catalan numbers associated to the symmetric group

Let  $\mathcal{S}_n$  act on  $\mathbb{C}[\mathbf{x}, \mathbf{y}]$  as described in Section 2.1 and let  $I \trianglelefteq \mathbb{C}[\mathbf{x}, \mathbf{y}]$  be the ideal generated by all alternating polynomials. Define the  $\mathcal{S}_n$ -module  $M^{(m)}$  to be the *minimal generating space* of  $I^m$ ,

$$M^{(m)} := I^m / \langle \mathbf{x}, \mathbf{y} \rangle I^m.$$

It carries a natural bigrading by degree in  $\mathbf{x}$  and degree in  $\mathbf{y}$ ,  $M^{(m)} = \bigoplus_{i,j \geq 0} M_{ij}^{(m)}$ .

**Remark 3.1** *The name minimal generating space comes from the fact that - as a vector space -  $M^{(m)}$  is isomorphic to the complex vector space with basis in one-to-one correspondence to any homogeneous minimal generating set of  $I^m$ .*

The following definition is due to M. Haiman:

**Definition 3.2** *The  $q, t$ -Fuß-Catalan numbers of type  $A_{n-1}$  are defined as the bigraded Hilbert series of the  $\mathcal{S}_n$ -module  $M^{(m)}$ ,*

$$\text{Cat}_n^{(m)}(q, t) := \mathcal{H}(M^{(m)}; q, t) = \sum_{i,j \geq 0} \dim(M_{ij}^{(m)}) q^i t^j.$$

**Remark 3.3**  $\text{Cat}_n^{(m)}(q, t)$  was originally defined in [11] as the complicated rational function

$$\sum_{\mu \vdash n} \frac{q^{(m+1)n(\mu')} t^{(m+1)n(\mu)} (1-q)(1-t) \Pi_\mu(q, t) B_\mu(q, t)}{\prod_{c \in D(\mu)} (q^{a(c)} - t^{l(c)+1}) (t^{l(c)} - q^{a(c)+1})}$$

in the context of modified Macdonald polynomials. M. Haiman later showed that this rational function is in fact equal to the bigraded Hilbert series in the definition above, [16].

In [10], A. Garsia and J. Haglund proved a simple combinatorial interpretation of  $\text{Cat}_n(q, t) := \text{Cat}_n^{(1)}(q, t)$  which was conjectured by J. Haglund in [14], where he introduced the *bounce statistic* on *Catalan paths* or, equivalently, on the set  $\mathcal{D}_n$  of all partitions that fit inside the partition  $(n-1, \dots, 2, 1)$ :

$$\text{Cat}_n(q, t) = \sum_{\lambda \in \mathcal{D}_n} q^{\text{area}(\lambda)} t^{\text{bounce}(\lambda)}.$$

Together with N. Loehr, they extended the definitions of *area* and *bounce* to the set  $\mathcal{D}_n^{(m)}$  of  $m$ -Catalan paths, these are partitions that fit inside the partition  $((n-1)m, \dots, 2m, m)$ , and conjectured a combinatorial interpretation of  $\text{Cat}_n^{(m)}(q, t)$  in terms of these statistics, [18]:

$$\text{Cat}_n^{(m)}(q, t) = \sum_{\lambda \in \mathcal{D}_n^{(m)}} q^{\text{area}(\lambda)} t^{\text{bounce}(\lambda)}.$$

**Remark 3.4** The specialization  $t = 1$  was proved by A. Garsia and M. Haiman in [11].

**Corollary 3.5** The specialization  $q = t = 1$  reduces  $\text{Cat}_n^{(m)}(q, t)$  to the  $\text{Cat}_n^{(m)}$ .

Also in [11], it was shown that the specialization  $t = q^{-1}$  yields the following  $q$ -extension of  $\text{Cat}_n^{(m)}$ :

**Theorem 3.6 (Garsia, Haiman)**

$$q^{m\binom{n}{2}} \text{Cat}_n^{(m)}(q, q^{-1}) = \frac{1}{[mn + 1]_q} \begin{bmatrix} (m + 1)n \\ n \end{bmatrix}_q,$$

where  $[k]_q := 1 + q + \dots + q^{k-1}$ ,  $[k]_{q!} := [1]_q [2]_q \dots [k]_q$  and  $\begin{bmatrix} k \\ l \end{bmatrix}_q := [k]_{q!} / [l]_{q!} [k - l]_{q!}$ .

**Remark 3.7** For  $m = 1$ , this reduces to the  $q$ -Catalan numbers defined by P.A. MacMahon in [19].

### 3.2 $q, t$ -Fuß-Catalan numbers for complex reflection groups

Recall that a complex reflection group  $W$  acts on the polynomial ring  $\mathbb{C}[\mathbf{x}, \mathbf{y}]$  diagonally as described in Section 2.2. Let  $I \trianglelefteq \mathbb{C}[\mathbf{x}, \mathbf{y}]$  be the ideal generated by all alternating polynomials and define the  $W$ -module  $M^{(m)} := I^m / \langle \mathbf{x}, \mathbf{y} \rangle I^m$ .

**Definition 3.8** The  $q, t$ -Fuß-Catalan numbers associated to  $W$  are defined as

$$\text{Cat}^{(m)}(W, q, t) := \mathcal{H}(M^{(m)}; q, t) = \sum_{i, j \geq 0} \dim(M_{ij}) q^i t^j.$$

**Remark 3.9** Using the computer algebra systems Singular and Macaulay 2, we computed the dimension of  $M^{(m)}$  as well as  $\text{Cat}^{(m)}(W, q, t)$  for the classical types at least up to rank 4 and small  $m$  and following exceptional types

$$I_2(k) \text{ for } k \in \{5, 6, 10, 12\}, \quad H_3, \quad G(k, 1, 1) \text{ for } k \leq 10, \quad G(4, 2, 2).$$

All following conjectures are based on these computations.

In [20], V. Reiner defined Fuß-Catalan numbers for classical reflection groups, and in [5] D. Bessis generalized this definition to well-generated complex reflection group  $W$ :

$$\text{Cat}^{(m)}(W) := \prod_{i=1}^l \frac{d_i + mh}{d_i},$$

where  $l$  is the rank of  $W$ ,  $d_1 \leq \dots \leq d_l$  are the degrees of the fundamental invariants and  $h = d_l$  is the Coxeter number. For definitions and further information see e.g. [2, Section 2.7] and [5].

For more or less general classes of reflection groups, this number counts a bunch of interesting combinatorial objects, see e.g. [2] and [3]. For  $W = \mathcal{S}_n$ , it reduces to  $\text{Cat}_n^{(m)}$  and for any real reflection group, it reduces for  $m = 1$  to the well-known Catalan numbers  $\text{Cat}(W)$  for real reflection groups, which are shown in Fig. 1.

$A_{n-1}$	$B_n$	$D_n$	$I_2(k)$	$H_3$	$H_4$	$F_4$	$E_6$	$E_7$	$E_8$
$\frac{1}{n+1} \binom{2n}{n}$	$\binom{2n}{n}$	$\binom{2n}{n} - \binom{2(n-1)}{n-1}$	$k+2$	32	280	105	833	4160	25080

**Fig. 1:**  $\text{Cat}(W)$  for the irreducible real reflection groups.

Our first conjecture concerns  $\text{Cat}^{(m)}(W, 1, 1)$ , the dimension of  $M$ :

**Conjecture 3.10** *Let  $W$  be a well-generated complex reflection group. Then*

$$\text{Cat}^{(m)}(W, 1, 1) = \text{Cat}^{(m)}(W).$$

In [6], D. Bessis and V. Reiner defined a  $q$ -extension of  $\text{Cat}^{(m)}(W)$  by

$$\prod_{i=1}^l \frac{[d_i + mh]_q}{[d_i]_q}.$$

The following conjecture, which is obviously stronger than Conjecture 3.10, would generalize Theorem 3.6 and would thereby give a new answer to a question of C. Kriloff and V. Reiner in [1, Problem 2.2]:

**Conjecture 3.11** *Let  $W$  be a well-generated complex reflection group. Then*

$$q^{mN} \text{Cat}^{(m)}(W, q, q^{-1}) = \prod_{i=1}^l \frac{[d_i + mh]_q}{[d_i]_q},$$

where  $N := \sum (d_i - 1)$ .

**Remark 3.12** *For a real reflection group  $W$ , the  $(d_i - 1)$ 's appearing in the conjecture are the exponents associated to  $W$  and  $N$  is equal to the number of positive roots.*

**Open Problem 3.13** *Are there statistics  $q\text{stat}$  and  $t\text{stat}$  on objects counted by  $\text{Cat}^{(m)}(W)$  which generalize area and bounce on Catalan paths  $\mathcal{D}_n^{(m)}$  such that*

$$\text{Cat}^{(m)}(W, q, t) = \sum_{\lambda} q^{\text{qstat}(\lambda)} t^{\text{tstat}(\lambda)}?$$

In the next section, we will present some conjectures concerning this open problem.

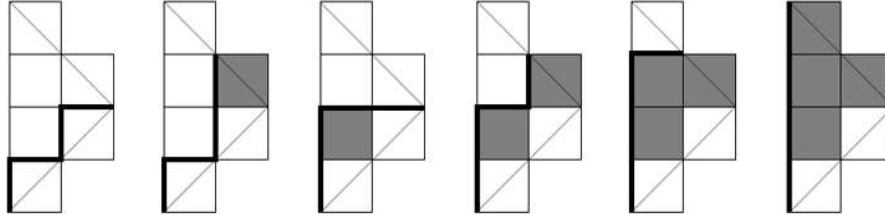
## 4 A generalization of the area statistic to crystallographic reflection groups

### 4.1 The area statistic for Catalan paths of type A

Specializing  $t = 1$  reduces  $\text{Cat}_n^{(m)}(q, t)$  to the well-known *Carlitz  $q$ -Fuß-Catalan numbers* defined by

$$\text{Cat}_n^{(m)}(q) := \text{Cat}_n^{(m)}(q, 1) = \sum_{\lambda \in \mathcal{D}_n^{(m)}} q^{\text{area}(\lambda)}.$$

They satisfy the following recurrence which can be deduced from a generating function identity proved by C. Krattenthaler in [17, Theorem 9]:



**Fig. 2:** All type  $B$  Catalan paths of length 2.

**Theorem 4.1**

$$\text{Cat}_{n+1}^{(m)}(q) = \sum_{k_1 + \dots + k_{m+1} = n} q^{n(\mathbf{k})} \text{Cat}_{k_1}^{(m)}(q) \dots \text{Cat}_{k_{m+1}}^{(m)}(q), \quad \text{Cat}_0^{(m)}(q) = 1,$$

where  $n(\mathbf{k}) = n(k_1, \dots, k_{m+1}) := \sum (m + 1 - i)k_i$ .

**4.2 An area statistic for Catalan paths of type  $B$**

For  $m = 1$ , we define an *area*-statistic on type  $B$  Catalan paths and establish an analogous recurrence.

**Definition 4.2** A type  $B$  Catalan path of length  $n$  is a lattice path of  $2n$  steps, either north or east, that starts at  $(0, 0)$  and stays above the diagonal  $x = y$ . For such a path  $\lambda$ , we define  $\text{area}(\lambda)$  to be the number of boxes in the region confined by the path, the diagonal  $x = y$  and the anti-diagonal  $x = 2n - y$ , not counting the halfboxes at the diagonal  $x = y$  but counting the halfboxes at the anti-diagonal  $x = 2n - y$ .

**Example 4.3** In Fig. 2, all Catalan paths of type  $B_2$  are shown, the boxes which contribute to the area are shaded.

In analogy to type  $A$ , we define  $q$ -Catalan numbers for type  $B$  in the following way:

**Definition 4.4**

$$\text{Cat}_{B_n}(q) := \sum q^{\text{area}(\lambda)},$$

where the sum ranges over all type  $B$  Catalan paths  $\lambda$  of length  $n$ .

**Example 4.5** As shown in Example 4.3, we have  $\text{Cat}_{B_2}(q) = 1 + 2q + q^2 + q^3 + q^4$ .

The definition is based on the following conjecture:

**Conjecture 4.6**

$$\text{Cat}^{(1)}(W_{B_n}, q, 1) = \text{Cat}_{B_n}(q).$$

$\text{Cat}_{B_n}(q)$  satisfy the following recurrence involving Catalan numbers of type  $A$ :

**Theorem 4.7**

$$\text{Cat}_{B_n}(q) = \text{Cat}_n(q) + \sum_{k=0}^{n-1} q^{2k+1} \text{Cat}_{B_k}(q) \text{Cat}_{n-k}(q), \quad \text{Cat}_{B_0}(q) = 1.$$



**Proof:** Let  $\lambda$  be a type  $B$  Catalan path of length  $n$ . Then either  $\lambda$  has as many east as north steps, which means  $\lambda$  is equal to a type  $A$  Catalan path of length  $n$ , or there exists a last point  $(k, k + 1)$  where the path touches the diagonal  $x + 1 = y$  and stays strictly above afterwards. Now, we have an initial type  $A$  like Catalan path of length  $k + 1$  (where the last step is a north step instead of an east step). After this north step, a type  $B$  Catalan path of length  $n - k - 1$  starts. This gives the following recurrence which is equivalent to the statement:

$$\text{Cat}_{B_n}(q) = \text{Cat}_n(q) + \sum_{k=0}^{n-1} q \text{Cat}_{k+1}(q) q^{2(n-k-1)} \text{Cat}_{B_{n-k-1}}(q).$$

□

**Corollary 4.8** *The  $\text{Cat}_{B_n}(q)$  satisfy the following generating function identity:*

$$\sum_{n \geq 0} \frac{x^n q^{-n(n-1)} (1 - qx)}{(-x; q^{-1})_{2n+1}} \text{Cat}_{B_n}(q) = 1,$$

where  $(a; q)_k := (1 - a)(1 - qa) \dots (1 - q^{k-1}a)$ .

We will see in the next section that both,  $m$ -Catalan paths of type  $A$  and Catalan paths of type  $B$  are special cases of a more general construction and that it is not possible to construct Catalan paths of type  $B$  for higher  $m$ 's as lattice paths (at least not in the manner of defining an area generating function equal to the specialization  $t = 1$  of  $q, t$ -Fuß-Catalan numbers). This is likely to be the reason why we were - so far - not able to find a recurrence in type  $B$  for higher  $m$ 's.

### 4.3 The extended Shi arrangement and the coheight statistic

Fix  $\Phi$  to be a crystallographic root system and let  $W = W_\Phi$  be the associated reflection group. The *root poset* of  $\Phi$  is given by the partial order on the set of positive roots  $\Phi^+$  defined by covering relation

$$\alpha < \beta := \beta - \alpha \text{ is a simple root.}$$

An *order ideal*  $I \trianglelefteq \Phi^+$  is a subset  $I \subseteq \Phi^+$  such that  $\alpha \leq \beta \in I$  implies  $\alpha \in I$ .

**Theorem 4.9 (V. Reiner [20])** *Let  $\Phi$  be a crystallographic root system. Then*

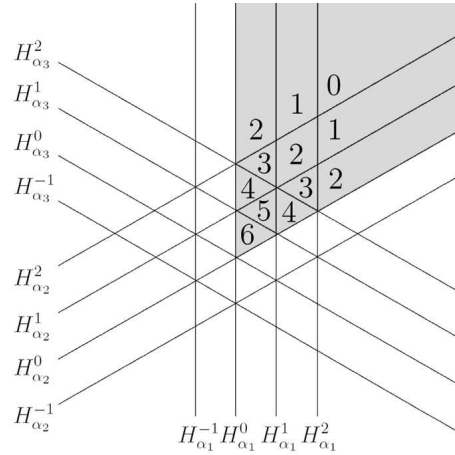
$$\#\{I \trianglelefteq \Phi^+\} = \text{Cat}(W).$$

In [3], C.A. Athanasiadis generalized this theorem to  $\text{Cat}^{(m)}(W)$  as follows: Let  $\mathcal{I}$  be an increasing chain of order ideals  $I_1 \subseteq \dots \subseteq I_m \subseteq \Phi^+$ . We call  $\mathcal{I}$  a *filtered chain* of length  $m$  if for  $i, j \geq 1$ ,

$$\begin{aligned} (I_i + I_j) \cap \Phi^+ &\subseteq I_{i+j} \quad \text{with } i + j \leq m, \\ (J_i + J_j) \cap \Phi^+ &\subseteq J_{i+j}, \end{aligned}$$

where  $J_i = \Phi^+ \setminus I_i$  and  $J_i = J_m$  for  $i > m$ .

Let  $V$  be the vector space spanned by  $\Phi$ , with inner product  $(\cdot, \cdot)$ . The *extended Shi arrangement*  $\text{Shi}^{(m)}(\Phi)$



**Fig. 3:** The extended Shi arrangement of type  $A_2$  and  $m = 2$ .

is given by the collection of hyperplanes in  $V$  defined by the affine equations  $(\alpha, x) = k$  for  $\alpha \in \Phi$  and  $-m < k \leq m$ . Thus  $\text{Shi}^{(m)}(\Phi)$  is a deformation of the Coxeter arrangement  $\mathcal{A}_\Phi$ . A *positive region* of  $\text{Shi}^{(m)}(\Phi)$  is a connected component of  $V \setminus \text{Shi}^{(m)}(\Phi)$  which lies in the fundamental chamber of  $\mathcal{A}_\Phi$ .

C.A. Athanasiadis defined the following map  $\Psi$  between positive regions of  $\text{Shi}^{(m)}(\Phi)$  and filtered chains in  $\Phi^+$  and showed that  $\Psi$  is a bijection: let  $R$  be a positive region and let  $x \in R$ . Then  $\Psi(R)$  is defined to be the filtered chain  $I_1 \subseteq \dots \subseteq I_m \subseteq \Phi^+$  such that

$$\begin{aligned} (\alpha, x) < i & \quad , \quad \text{if } \alpha \in I_i \\ (\alpha, x) > i & \quad , \quad \text{if } \alpha \in J_i = \Phi^+ \setminus I_i. \end{aligned}$$

**Theorem 4.10 (C.A. Athanasiadis [3])** *Both the number of filtered chains in  $\Phi^+$  and the number of positive regions of  $\text{Shi}^{(m)}(\Phi)$  is equal to  $\text{Cat}^{(m)}(W)$ .*

**Definition 4.11** *Let  $R^0$  be the fundamental region given by  $0 < (x, \alpha) < 1$  for all  $\alpha \in \Phi^+$ . For any region  $R$ , define the height of  $R$ , denoted  $h(R)$ , to be the number of hyperplanes in  $\text{Shi}^{(m)}(\Phi)$  that separate  $R$  from  $R^0$  and the coheight by  $\text{coh}(R) := mN - h(R)$ . Furthermore, we (combinatorially) define  $q$ -Fuß-Catalan numbers associated to  $W$  by*

$$\text{Cat}^{(m)}(W, q) := \sum_R q^{\text{coh}(R)},$$

where the sum ranges over all positive regions of  $\text{Shi}^{(m)}(\Phi)$ .

**Example 4.12** *Let  $W$  be the reflection group of type  $A_2$  and  $m = 2$ . In Fig. 3, the extended Shi arrangement of the given type is shown. The positive roots are denoted by  $\alpha_1, \alpha_2$  and  $\alpha_3 = \alpha_1 + \alpha_2$ , the fundamental chamber is shaded and the positive regions are labelled by their coheights. This gives*

$$\text{Cat}^{(2)}(W_{A_2}, q) = 1 + 2q + 3q^2 + 2q^3 + 2q^4 + q^5 + q^6.$$

**Remark 4.13** The poset of all regions, defined by the covering relation  $R \prec R'$  if  $h(R) = h(R') - 1$  and  $R, R'$  share a common face, is isomorphic to the poset  $NN^{(k)}(W)$  described in [2, Definition 5.1.20].

**Proposition 4.14** Let  $I_1 \subseteq \dots \subseteq I_m \subseteq \Phi^+$  be a filtered chain and let  $R$  be the associated region. Then the bijection given above implies that

$$\text{coh}(R) = \sum \#I_i.$$

The next theorem shows that the coheight on regions in the fundamental chamber reduces for type  $A$  to the area on  $m$ -Catalan paths of type  $A$  and for type  $B$  with  $m = 1$  to the area on Catalan paths of type  $B$ .

**Theorem 4.15** For all integers  $m \geq 1$ , we have

$$\text{Cat}^{(m)}(W_{A_{n-1}}, q) = \text{Cat}_n^{(m)}(q) \quad , \quad \text{Cat}^{(1)}(W_{B_n}, q) = \text{Cat}_{B_n}(q).$$

**Remark 4.16** Counting lattice paths consisting of north and east steps having a boundary can always be seen as counting order ideals in very special kinds of posets. The posets occurring for  $\text{Cat}^{(m)}(W, q)$  with  $W$  of type  $D_n$  with  $n \geq 4$  and of type  $B_n$  with  $n, m \geq 2$  fail to have this property.

The definition of *coheight* is motivated by the following conjecture which would partially answer Open Problem 3.13:

**Conjecture 4.17** Let  $W$  be a crystallographic reflection group. Then

$$\text{Cat}^{(m)}(W, q, 1) = \text{Cat}^{(m)}(W, q).$$

#### 4.4 Non-crystallographic reflection groups

So far, no definition for root posets for non-crystallographic reflection groups is known. In [2], D. Armstrong suggests, how these root posets *should* look like in types  $I_2(m)$  and  $H_3$ . Our computations confirm these ideas: Let  $W$  be the a reflection group of one of the following types:  $I_2(5)$ ,  $I_2(10)$ ,  $I_2(12)$ ,  $H_3$ . Then

$$\text{Cat}^{(m)}(W, q, 1) = \sum_{I \triangleleft \Phi^+} q^{\#I},$$

where  $\Phi^+$  is Armstrong's suggested root poset of type  $W$ . The situation in the cyclic group of order  $k$ ,  $W = G(k, 1, 1)$  is the following: The  $q, t$ -Fuß-Catalan numbers associated to  $G(k, 1, 1)$  are all equal. As we already know the classical case  $k = 2$ , we get

$$\text{Cat}^{(m)}(W, q, t) = \text{Cat}_2^{(m)}(q, t) = q + t.$$

We also computed  $\text{Cat}^{(m)}(W, q, t)$  for  $W$  the non-well-generated reflection group of type  $G(4, 2, 2)$  and the result was, up to  $m = 3$ ,

$$\text{Cat}^{(m)}(W, q, t) = \text{Cat}^{(m)}(W_{G_2}, q, t).$$

## 5 Connections to rational Cherednik algebras

Let  $W$  be a real reflection group or, equivalently, let  $W$  be a (finite) Coxeter group acting on the polynomial ring  $\mathbb{C}[\mathbf{x}, \mathbf{y}]$  and let  $M^{(m)}$  be the  $W$ -module defined in Section 3.2. It is easy to see that  $M^{(m)}$  is, except for the sign twist, equal to the alternating component of the module

$$\mathbf{R}[\mathbf{x}, \mathbf{y}] := (I^{m-1}/I^{m-1}J) \otimes \epsilon^{m-1},$$

where  $J$  is the ideal generated by all *invariant polynomials without constant term*,

$$M^{(m)} \cong (\mathbf{R}[\mathbf{x}, \mathbf{y}] \otimes \epsilon)^W = \mathbf{e}_\epsilon \mathbf{R}[\mathbf{x}, \mathbf{y}].$$

**Remark 5.1**  $\mathbf{R}[\mathbf{x}, \mathbf{y}]$  reduces for  $m = 1$  to the coinvariant ring  $\mathbb{C}[\mathbf{x}, \mathbf{y}]/J$  and was, in type  $A$ , introduced in [11].

The conjectured connection to *rational Cherednik algebras* is the following: in [7], C.F. Dunkl and E. Opdam constructed a certain  $W$ -module  $L$  depending on a non-negative integer  $m$ . This module carries a natural tensor product filtration and in [4, Theorem 1.6] Y. Berest, P. Etingof and V. Ginzburg showed that the Hilbert series of the trivial component of its associated graded module  $\text{gr}(L)$  is given by

$$\mathcal{H}(\mathbf{e}(\text{gr}(L)); q) = q^{-mN} \prod_{i=1}^l \frac{[d_i + mh]_q}{[d_i]_q}.$$

Using [13, Lemma 6.7 (2)] together with equalities (7.6) and (7.8) in [4], we obtain the following result which partially generalizes [12, Theorem 5].

**Theorem 5.2** *Let  $W$  be a real reflection group and let  $\mathbf{R}[\mathbf{x}, \mathbf{y}]$  be graded by degree in  $\mathbf{x}$  minus degree in  $\mathbf{y}$ . Then there exists a natural surjection of graded  $W$ -modules,*

$$\mathbf{R}[\mathbf{x}, \mathbf{y}] \otimes \epsilon \twoheadrightarrow \text{gr}(L).$$

**Conjecture 5.3** *The kernel of this surjection does not contain a copy of the trivial representation.*

This conjecture would imply the following corollary:

**Corollary 5.4** *Let  $M^{(m)}$  be graded by degree in  $\mathbf{x}$  minus degree in  $\mathbf{y}$ . Then*

$$M^{(m)} \cong \mathbf{e}(\text{gr}(L))$$

*as graded  $W$ -modules.*

**Remark 5.5** *Together with the above discussion, Corollary 5.4 would imply Conjecture 3.11.*

## References

- [1] D. Armstrong, *Braid groups, clusters and free probability: An outline from the AIM workshop* (2004), available at <http://www.math.cornell.edu/~armstron/>.
- [2] D. Armstrong, *Generalized noncrossing partitions and combinatorics of Coxeter groups*, Mem. Amer. Math. Soc. (to appear) (2007), available at arXiv:math/0611106v2.

- [3] C.A. Athanasiadis, *On a refinement of the generalized Catalan numbers for Weyl groups*, Trans. Amer. Math. Soc. **357** (2005), 179–196.
- [4] Y. Berest, P. Etingof, V. Ginzburg, *Finite dimensional representations of rational Cherednik algebras*, Int. Math. Res. Not. **19** (2003), 1053–1088.
- [5] D. Bessis, *Topology of complex reflection arrangements* (2004), available at arXiv:math/0411645v1.
- [6] D. Bessis, V. Reiner, *Cyclic sieving of noncrossing partitions for complex reflection groups* (2007), available at arXiv:math.CO/0701792v1.
- [7] C.F. Dunkl, E. Opdam, *Dunkl operators for complex reflection groups*, Proc. Lond. Math. Soc. **86** (2003), 70–108.
- [8] S. Fomin, N. Reading, *Root systems and generalized associahedra*, Lecture notes for the IAS/Park City Graduate Summer School in Combinatorics (2004).
- [9] J. Förlinger, J. Hofbauer, *q-Catalan Numbers*, J. Combin. Theory Ser. A **40**:2 (1985), 248–264.
- [10] A. Garsia, J. Haglund, *A proof of the q, t-Catalan positivity conjecture*, Discrete Math. **256** (2003), 677–717.
- [11] A. Garsia, M. Haiman, *A remarkable q, t-Catalan sequence and q-Lagrange inversion*, Algebraic Combin. **5** (1996), 191–244.
- [12] I. Gordon, *On the quotient ring by diagonal harmonics*, Invent. Math. **153** (2003), 503–518.
- [13] I. Gordon, J.T. Stafford, *Rational Cherednik algebras and Hilbert schemes*, Adv. Math. **198**:1 (2005), 222–274.
- [14] J. Haglund, *Conjectured statistics for the q, t-Catalan numbers*, Adv. Math. **175**:2 (2003), 319–334.
- [15] M. Haiman, *Vanishing theorems and character formulas for the Hilbert scheme of points in the plane*, Invent. Math. **149** (2002), 371–407.
- [16] M. Haiman, *t, q-Catalan numbers and the Hilbert scheme*, Discrete Math. **193** (2004), 201–224, Selected papers in honor of Adriano Garsia.
- [17] C. Krattenthaler, *Counting lattice paths with a linear boundary, Part 2: q-ballot and q-Catalan numbers*, Sitz.ber. d. ÖAW, Math.-naturwiss. Klasse **198** (1997), 171–199.
- [18] N. A. Loehr, *Conjectured Statistics for the Higher q, t-Catalan sequences*, Electron. J. Combin. **12** (2005).
- [19] P. A. MacMahon, *Combinatory Analysis Vol. 2*, Cambridge University Press, London (1960).
- [20] V. Reiner, *Non-crossing partitions for classical reflection groups*, Discrete Math. **177** (1997), 195–222.