Families of prudent self-avoiding walks

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Abstract. A self-avoiding walk on the square lattice is *prudent*, if it never takes a step towards a vertex it has already visited. Préa was the first to address the enumeration of these walks, in 1997. For 4 natural classes of prudent walks, he wrote a system of recurrence relations, involving the length of the walks and some additional "catalytic" parameters.

The generating function of the first class is easily seen to be rational. The second class was proved to have an algebraic (quadratic) generating function by Duchi (FPSAC'05). Here, we solve exactly the third class, which turns out to be much more complex: its generating function is not algebraic, nor even D-finite. The fourth class — general prudent walks — still defeats us. However, we design an isotropic family of prudent walks on the triangular lattice, which we count exactly. Again, the generating function is proved to be non-D-finite.

We also study the end-to-end distance of these walks and provide random generation procedures.

Résumé. Un chemin auto-évitant sur le réseau carré est *prudent*, s'il ne fait jamais un pas en direction d'un point qu'il a déjà visité. Préa est le premier à avoir cherché à énumérer ces chemins, en 1997. Pour 4 classes naturelles de chemins prudents, il donne un système de relations de récurrence, impliquant la longueur des chemins et plusieurs paramètres "catalytiques" supplémentaires.

La première classe a une série génératrice simple, rationnelle. La deuxième a une série algébrique (quadratique) (Duchi, FPSAC'05). Nous comptons ici les chemins de la troisième classe, et observons un saut de complexité: la série obtenue n'est ni algébrique, ni même différentiellement finie. La quatrième classe, celle des chemins prudents généraux, résiste encore. Cependant, nous définissons un modèle isotrope de chemins prudents sur réseau triangulaire, que nous résolvons de nouveau, la série obtenue n'est pas différentiellement finie.

Nous étudions aussi la vitesse d'éloignement de ces chemins, et proposons des algorithmes de génération aléatoire.

Keywords: enumeration, self-avoiding walks, D-finite generating functions. **AMS classification:** 05A15

1 Introduction

1.1 Families of self-avoiding walks

The study of self-avoiding walks is a famous "elementary" problem in combinatorics, which is also of interest in probability theory and in statistical physics [12]. Recall that, given a lattice with some origin O, a self-avoiding walk (SAW) is a path starting from O that does not visit twice the same vertex (Fig. 1).

It is strongly believed that, for two-dimensional lattices, the number $a(n)$ of n-step SAW and the average end-to-end distance of these walks satisfy

$$a(n) \sim \alpha \mu^n n^\gamma \quad \text{and} \quad \mathbb{E}(D_n) \sim \kappa n^\nu$$

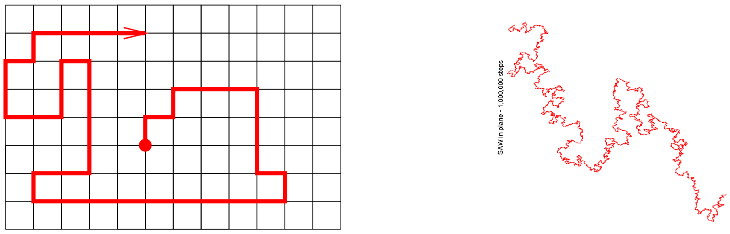

Fig. 1: A self-avoiding walk on the square lattice, and a (quasi-)random SAW of length 1,000,000.

where $\gamma = 11/32$ and $\nu = 3/4$. The growth constant μ is lattice-dependent. Several independent but so far non-rigorous methods predict that $\mu = \sqrt{2+\sqrt{2}}$ on the honeycomb lattice. Moreover, numerical studies suggest that μ may also be a quartic number for the square lattice [10]. On the probability side, it has been proved that, if the scaling limit of SAW exists and has some conformal invariance property, it must be described by the process SLE(8/3) (stochastic Loewner evolution) [11]. This would imply that the predicted values of γ and ν are correct.

The fact that all these conjectures only deal with *asymptotic* properties of SAW tells how far the problem is from the reach of *exact* enumeration. The followers of this discipline thus focus on the study of subclasses of SAW. A simple example is the family of *partially directed* walks, that is, SAW formed of North, East and West steps. It is easy to see that their generating function is rational [18, Example 4.1.2],

$$\sum_n p(n)t^n = \frac{1+t}{1-2t-t^2}, \qquad (1)$$

which gives $c(n) \sim \alpha\mu^n$, with $\mu = 1+\sqrt{2} = 2.41...$ The coordinates (X_n, Y_n) of the endpoint satisfy:

$$E(X_n) = 0, \quad E(X_n^2) \sim \alpha_1 n \quad \text{and} \quad \mathbb{E}(Y_n) \sim \alpha_2 n.$$

The prudent walks studied in this paper form a more general class of SAW which have a natural kinetic description: a walk is *prudent* if it never takes a step pointing towards a vertex it has already visited. In other words, the walk is so cautious that it only takes steps in directions where the road is perfectly clear. Various examples are shown on Fig. 2. In particular, partially directed walks are prudent.

These walks have already been studied in the past under different names [20, 17, 15, 7, 6]. The first two papers above deal with Monte-Carlo simulations. Préa [15] wrote recurrence relations for numbers $c(n; i, j, h)$ that count prudent walks according to their length (n) and to three additional *catalytic* parameters (i, j, h). By this, we mean that these parameters are essential to the existence of these recurrence

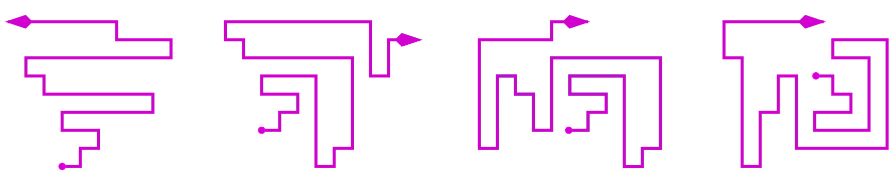

Fig. 2: Four prudent walks: the first one is 1-sided, the others are respectively 2-sided, 3-sided and 4-sided.

relations, and that it is far from obvious to derive from them a recursion, say, for some numbers $c(n; i, j)$ that would only take into account two of the catalytic parameters. Préa also defined four natural families of prudent walks of increasing generality, called k-*sided*, for k ranging from 1 to 4. In particular, 1-sided walks coincide with partially directed walks, and 4-sided walks coincide with general prudent walks. He wrote recurrence relations for each of these classes: three catalytic parameters are needed for general (4-sided) prudent walks, but two (resp. one, zero) suffice for 3-sided (resp 2-sided, 1-sided) walks. This reflects the increasing generality of these four classes of walks.

Recall that the generating function of 1-sided walks (partially directed walks) is rational (1). Then Duchi [7] proved that 2-sided walks have an algebraic (quadratic) generating function. She also found an algebraic generating function for 3-sided walks, but there was a subtle flaw in her derivation, and the latter result turned out to be wrong, as was detected by Guttmann. He and his co-authors performed a numerical study of prudent walks, in order to get an idea of their asymptotic enumeration, and of the properties of the associated generating functions [6]. In particular, they conjectured that the length generating function of general prudent walks is not *D-finite*, that is, does not satisfy any linear differential equation with polynomial coefficients. This implies that it is not algebraic.

1.2 Contents

In Section 2 of this paper, we collect functional equations that define the generating functions of the four classes of prudent walks introduced by Préa [15]. This is not really original, as these functional equations are basically equivalent to Préa's recurrence relations. Moreover, similar equations (or systems of equations) were written by Duchi [7]. Our approach may be a bit more systematic.

In Sections 3 to 5 we address the solution of these equations. The case of 1-sided walks is immediate and leads the above rational generating function. We then recall the systematic *kernel method*, which solves (linear) equations with *one* catalytic variable. In particular, it provides the generating function of 2-sided walks (Section 3). The extension of this method to equations with *two* catalytic variables is not yet completely understood, although a number of papers has been devoted to instances of such equations in the past few years [5, 3, 9, 14, 13]. Here, we solve the equation obtained for 3-sided walks and prove that their generating function is *not D-finite*, having a rather complex singularity structure (Section 4). We also prove that the growth constants of 3-sided walks and 2-sided walks are the same. The growth constant of *general* prudent walks is also predicted to be the same [6].

The final equation, which deals with general prudent walks and involves *three* catalytic variables, still defeats us. This is annoying, as the other classes are by definition anisotropic. However, we introduce a new isotropic class of prudent walks on the triangular lattice, which are described by (only) two catalytic variables. We solve the associated functional equation, and prove that the generating function of these triangular prudent walks (Fig. 3, left) is not D-finite, having a natural boundary (Section 5).

Fig. 3: Left: A triangular prudent walk in a box of size 7. Right: The two catalytic parameters.

We also refine our equations to take into account parameters related to the end-to-end distance of a prudent walk. Extending our solutions to these refined equations is harmless, but the conclusion we draw is somewhat disappointing: prudent walks drift away from the origin at a *positive speed*. In other words, the end-to-end distance grows linearly with the length of the walk.

We have also addressed the uniform random generation of prudent walks. Their step-by-step recursive structure allows for a standard recursive approach which we describe in terms of generating trees, in the spirit of [2]. Our procedures involve the precomputation of $O(n^4)$ numbers (for general prudent walks) so that the typical length we can reach is a few hundreds. This still provides interesting pictures (Fig. 4, 6, 7, 8). We do not give the details of the recursive procedures in this abstract. Moreover, most of the proofs are omitted. The complete version will be available soon on the ArXiv.

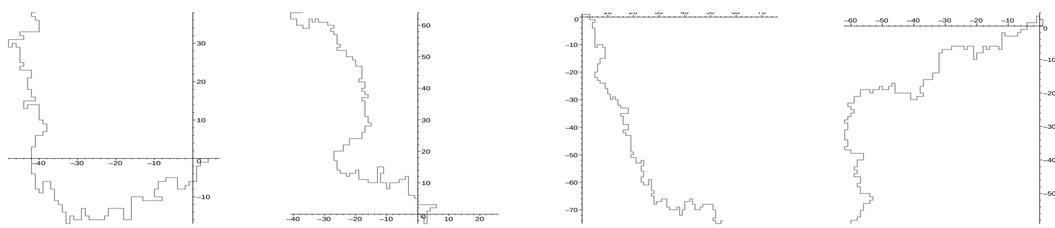

Fig. 4: Random prudent walks.

1.3 Families of prudent walks

The *box* of a square lattice walk is the smallest rectangle that contains it. It is not hard to see that the endpoint of a prudent walk is always on the border of the box. This means that every new step either walks on the border of the box, of moves one of its four sides.

Observe that a prudent walk is partially directed if its endpoint, as the walk grows, always lies on the top side of the box. This is why these walks are also called 1-*sided*. Similarly, a prudent walk is 2-*sided* if its endpoint lies always on the top or right side of the box. It is 3-*sided* if its endpoint is always on the top, right or left side of the box. Of course, 4-*sided* walks coincide with general prudent walks (Fig. 2).

Consider now a walk on the triangular lattice. Define the (triangular) *box* of the walk as the smallest triangle pointing North that contains the walk. The walk is a *triangular prudent walk* if each new step either inflates the box, or walks along one side of the box in a prudent way (that is, not pointing to an already visited vertex). An example is shown in Fig. 3.

Given a class of walks $\mathcal{C}$, the *generating function* of walks of $\mathcal{C}$, counted by their length, is

$$C(t) = \sum_{w \in \mathcal{C}} t^{|w|},$$

where $|w|$ denotes the length of the walk w. The generalization of this definition to the series $C(t; u_1, \ldots, u_k)$ counting walks according to their length and to k additional parameters is immediate. We will often drop the length variable t, denoting this series $C(u_1, \ldots, u_k)$. A one-variable series $C(t)$ is *D-finite* if it satisfies a linear ODE with polynomial coefficients, $P_k(t)C^{(k)}(t) + \cdots + P_1(t)C'(t) + P_0(t)C(t) = 0$. Every algebraic series is D-finite. We refer to [19] for generalities on these power series.

2 Functional equations

The construction of functional equations for all the families of prudent walks we study rely on the same principle, which we first describe on 1-sided (partially directed) walks.

Consider a 1-sided walk. If it ends with a horizontal step, we can extend it in two different ways: either we repeat the last step, or we change direction and add a North (N) step. Otherwise, the walk is either empty or ends with a North step, and we have three ways (N, E and W) to extend it. This shows that North steps, which move the top side of the box, play a special role. Our functional equation is obtained by answering the following question: where is the last North step, and what has happened since then?

More specifically, let $P(t)$ denote the length generating function of 1-sided walks. The contribution in $P(t)$ of walks that contain no North step (horizontal walks) is

$$1 + 2\sum_{n\geq 1} t^n = \frac{1+t}{1-t}.$$

The other walks are obtained by concatenating a 1-sided walk, a North step, and then a horizontal walk. Their contribution is thus

$$P(t)\, t\, \frac{1+t}{1-t}.$$

Adding these two contributions gives a linear equation for $P(t)$, from which we readily derive (1).

The principle of this recursive description extends to k-sided walks for each k. We say that a step of a k-sided walk is *inflating* if, at the time it was added to the walk, it shifted one of the k sides of the box that are relevant in the definition of k-sided walks. For instance, when $k = 2$, an inflating step moves the top or right side of the box. We write our equations by answering the question: where is the last inflating step, and what has happened since then?

Since then, the walk has grown *without creating a new inflating step*. What does it mean? Assume $k \geq 2$, that the last inflating step was North, and that, since then, the walk has taken m East steps. Then m cannot be arbitrarily large, otherwise one or several of these East steps would be inflating, having moved the right side of the box. This is why we have to take into account other "catalytic" parameters in our enumeration of prudent walks. For a 2-sided walk, we keep track of the distance between the endpoint and the NE corner of the box, using a new variable u. Fig. 5 shows the catalytic parameters involved in the enumeration of 3-sided walks ending on the top/right side of the box, and of 4-sided walks ending on the top of the box. They give rise to series with catalytic variables. For instance, for 4-sided walks ending on the top of their box, we will use the series

$$T(t; u, v, w) \equiv T(u, v, w) = \sum_{i,j,h} T_{i,j,h} u^i v^j w^h$$

where $T_{i,j,h} \equiv T_{i,j,h}(t)$ counts 4-sided walks ending on the top of their box, at distance i (resp. j) from the NE (resp. NW) corner, such that the height of the box is h.

Finally, for triangular prudent walks ending of the right side of their (triangular) box, we keep track of the distances between the endpoint and the SE and N corners of the box (variables u and v, see Fig. 3).

Lemma 1 (Two-sided prudent walks) *The generating function $T(t; u) \equiv T(u)$ of 2-sided walks ending on the top side of their box satisfies*

$$\left(1 - \frac{tu(1-t^2)}{(1-tu)(u-t)}\right) T(u) = \frac{1}{1-tu} + t\,\frac{u-2t}{u-t} T(t).$$

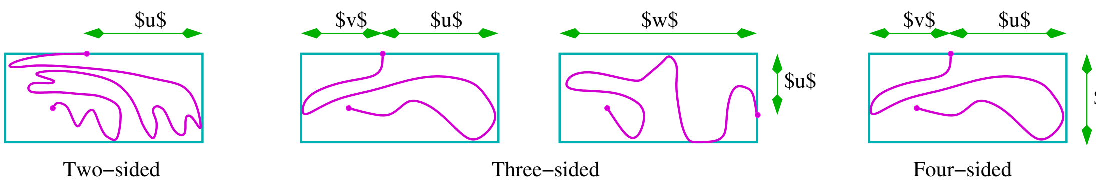

Fig. 5: Catalytic variables for k-sided walks, $k \geq 2$.

The generating function of 2*-sided walks, counted by the length and the distance of the endpoint to the NE corner of the box, is*

$$P(t;u) = 2T(t;u) - T(t;0).$$

Proof: We study separately 3 classes of 2-sided walks, according to the existence and direction of the last inflating step (LIS). This step has moved the right or top side of the box.

1. Neither the top nor the right side has ever moved: the walk is a sequence of West steps. The generating function for this class is $1/(1-tu)$.

2. The LIS goes East. This implies that the endpoint of the walk was on the right side of the box before that step. After that East step, the walk has made a sequence of North steps to reach the top side of the box. Observe that, by symmetry, the series $T(t;u)$ also counts walks ending on the right side of the box. These two observations give the generating function for this class as

$$t \sum_{i \geq 0} T_i t^i = tT(t).$$

3. The LIS goes North. After this step, there is either an (unbounded) sequence of West steps, or a bounded sequence of East steps. This gives the generating function for this class as

$$\frac{t^2 u}{1-tu} T(u) + t \sum_{i \geq 0} T_i \sum_{k=0}^{i} t^k u^{i-k} = \frac{t^2 u}{1-tu} T(u) + \frac{t}{u-t} \left(uT(u) - tT(t) \right).$$

Adding the 3 terms gives the functional equation satisfied by $T(u)$. The expression of $P(t;u)$ relies on an inclusion-exclusion argument: we first double the contribution of $T(u)$ to take into account walks ending on the right side of the box, and then subtract the series $T(0)$ counting those that end at the NE corner. □

The other classes of prudent walks are treated by similar arguments.

Lemma 2 (Three-sided prudent walks) *The generating functions* $T(t;u,v) \equiv T(u,v)$ *and* $R(t;u,w) \equiv R(u,w)$ *that count respectively* 3*-sided walks ending on the top side and on the right side of their box are related by*

$$\left(1 - \frac{tuv(1-t^2)}{(u-tv)(v-tu)}\right) T(u,v) = 1 + tuR(t,u) + tvR(t,v) - \tfrac{t^2 v}{u-tv} T(tv,v) - \tfrac{t^2 u}{v-tu} T(u,tu) \quad (2)$$

$$\left(1 - \frac{tuw(1-t^2)}{(u-t)(1-tu)}\right) R(u,w) = \tfrac{1}{1-tu} + tT(tw,w) - \tfrac{t^2 w}{u-t} R(t,w). \quad (3)$$

The generating function of 3-sided walks, counted by the length and by the width of the box, is

$$P(t;u) = T(t;u,u) + 2R(t;1,u) - 2T(t;u,0) - \frac{t}{1-t}.$$

Lemma 3 (General prudent walks) *The generating function $T(t;u,v,w) \equiv T(u,v,w)$ of prudent walks ending on the top side of their box satisfies*

$$\left(1 - \frac{tuvw(1-t^2)}{(u-tv)(v-tu)}\right) T(u,v,w) = 1 + \mathcal{G}(w,u) + \mathcal{G}(w,v) - \frac{tv}{u-tv}\mathcal{G}(v,w) - \frac{tu}{v-tu}\mathcal{G}(u,w)$$

with $\mathcal{G}(u,v) \equiv \mathcal{G}(t;u,v) = tvT(t;u,tu,v)$.

The generating function of prudent walks, counted by the length and the half-perimeter of the box, is

$$P(t;u) = 1 + 4T(t;u,u,u) - 4T(t;0,u,u).$$

Lemma 4 (Triangular prudent walks) *The generating function $R(t;u,v) \equiv R(u,v)$ of triangular prudent walks ending in the right side of their box satisfies*

$$\left(1 - \frac{tuv(1-t^2)(u+v)}{(u-tv)(v-tu)}\right) R(u,v) = 1 + tu(1+t)\frac{v-2tu}{v-tu}R(u,tu) + tv(1+t)\frac{u-2tv}{u-tv}R(tv,v). \quad (4)$$

The generating function of triangular prudent walks, counted by the length and the size of the box, is

$$P(t;u) = 1 + 3R(t;u,u) - 3R(t;u,0). \quad (5)$$

3 Enumeration and properties of 2-sided prudent walks

In this section, we recall how the *kernel method* works on linear equations with one catalytic variable [4, 1, 16], using the example of 2-sided walks. We first recover Duchi's algebraic generating function, and then refine our enumeration to keep track of other statistics like the end-to-end distance of the walk. This section is a sort of warm-up before the more difficult equations of Sections 4 and 5.

Proposition 5 *The generating function $P(t;u)$ of 2-sided walks, counted by their length and by the distance between the endpoint and the NE corner of the box, is*

$$P(t;u) = \frac{2(1-t^2)(1-t)U}{(1-uU)(1-tU)(2t-U)} - 1$$

where

$$U \equiv U(t) = \frac{1-t+t^2+t^3-\sqrt{(1-t^4)(1-2t-t^2)}}{2t}.$$

In particular, the length generating function is

$$P(t;1) = \frac{1}{1-2t-2t^2+2t^3}\left(1+t-t^3+t(1-t)\sqrt{\frac{1-t^4}{1-2t-t^2}}\right).$$

Proof: We start from the functional equation of Lemma 1, written as

$$\left((1-tu)(u-t)-tu(1-t^2)\right)T(u) = u-t+t(1-tu)(u-2t)T(t). \tag{6}$$

The series $U \equiv U(t)$ given in the proposition is the only power series in t that cancels the *kernel* of this equation, that is, the polynomial $\left((1-tu)(u-t)-tu(1-t^2)\right)$. The series $T(U) \equiv T(t;U)$ is well-defined. Replacing u by U in the equation cancels the left-hand side, and hence the right-hand side, giving a rational expression of $T(t)$ in terms of t and U. Then (6) gives $T(u)$, and the second equation of Lemma 1 provides $P(t;u)$. □

Proposition 6 (Asymptotic properties of 2-sided walks) *The number of n-step 2-sided walks is $p_n \sim \kappa\mu^n$ for some positive constant κ, with $1-2\rho-2\rho^2+2\rho^3=0$ and $\mu = 1/\rho = 2.48....$*

The average distance between the endpoint and the NE corner in a random n-step 2-sided walk is asymptotically constant. Let X_n (resp. Y_n) denote the abscissa (resp. ordinate) of the endpoint of a random n-step 2-sided walk. Then there exists a positive constant κ such that $\mathbb{E}(X_n+Y_n) \sim \kappa n$.

Proof: The singularities of $P(t;1)$ are found among the roots of $(1-2t-2t^2+2t^3)(1-t^4)(1-2t-t^2)$. It is not hard to see that the smallest one (in modulus) is a simple pole at ρ. We then apply the *singularity analysis* of [8] to derive the asymptotic behaviour of the number of paths.

We then differentiate $P(t;u)$ with respect to u and set $u=1$. The resulting series has still a simple pole at ρ, and the second statement follows.

Finally, we enrich the functional equations of Lemma 1 by taking into account the sum of the coordinates of the endpoint, using a new variable z. We solve the new equations in the same way we solved the case $z=1$. We differentiate $P(t,z;1)$ with respect to z and then set $z=1$. An elementary analysis of singularities shows the linearity of $\mathbb{E}(X_n+Y_n)$. □

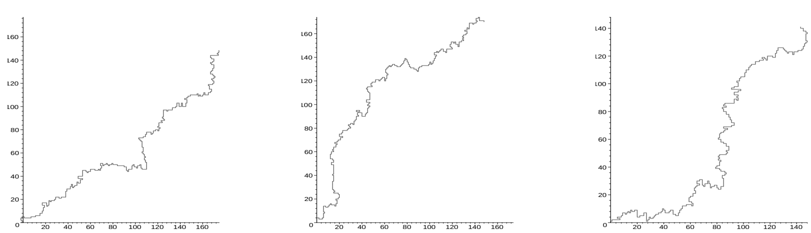

Fig. 6: Random 2-sided walks of length 500.

4 Enumeration and properties of 3-sided prudent walks

Proposition 7 *The generating function of 3-sided walks ending on the top side of their box satisfies*

$$T(t;u,tu) = \sum_{k\geq 0}(-1)^k \frac{\prod_{i=0}^{k-1}\left(\frac{t}{1-tq}-U(uq^{i+1})\right)}{\prod_{i=0}^{k}\left(\frac{tq}{q-t}-U(uq^{i})\right)}\left(1+\frac{U(uq^k)-t}{t(1-tU(uq^k))}+\frac{U(uq^{k+1})-t}{t(1-tU(uq^{k+1}))}\right)$$

where

$$U(w) \equiv U(t;w) = \frac{1 - tw + t^2 + t^3 w - \sqrt{(1-t^2)(1+t-tw+t^2w)(1-t-tw-t^2w)}}{2t}$$

is the only power series in t *satisfying* $(U-t)(1-tU) = twU(1-t^2)$*, and* $q \equiv q(t) = U(t;1)$.

The length generating function of 3*-sided walks is*

$$P(t;1) = \frac{1}{1-2t-t^2}\left(2\,t^2 q\,T(t;1,t) + \frac{(1+t)\left(2-t-t^2q\right)}{1-tq}\right) - \frac{1}{1-t}.$$

Proof: We first apply the standard kernel method to (3): setting $u = U(w)$ gives

$$twR(t,w) = \frac{U(w)-t}{t}\left(\frac{1}{1-tU(w)} + tT(tw,w)\right). \tag{7}$$

Recall that T is symmetric in its two (catalytic) variables. In particular, $T(tw,w) = T(w,tw)$. Equation (2) involves the series $R(t,u)$ and $R(t,v)$. We use (7) to express them in terms of $U(u)$, $U(v)$, $T(u,tu)$ and $T(v,tv)$ and thus obtain an equation that involves only the series T:

$$\left(1 - \frac{tuv(1-t^2)}{(u-tv)(v-tu)}\right)T(u,v) =$$
$$1 + \frac{U(u)-t}{t(1-tU(u))} + \frac{U(v)-t}{t(1-tU(v))} - \left(\frac{tv}{v-tu} - U(u)\right)T(u,tu) - \left(\frac{tu}{u-tv} - U(v)\right)T(v,tv).$$

The kernel of this new equation, $(u-tv)(v-tu) - tuv(1-t^2)$, is homogeneous in u and v. It vanishes for $v = qu$, where $q = U(1)$. Replacing v by qu in the equation gives

$$T(u,tu) = -\frac{\frac{t}{1-tq} - U(uq)}{\frac{tq}{q-t} - U(u)}T(uq,tuq) + \frac{1}{\frac{tq}{q-t} - U(u)}\left(1 + \frac{U(u)-t}{t(1-tU(u))} + \frac{U(uq)-t}{t(1-tU(uq))}\right).$$

Observe that $tq/(q-t) = \frac{1-tq}{1-t^2} = 1 + O(t)$, while $U(u) = O(t)$. Hence $\frac{tq}{q-t} - U(u)$ is invertible in the ring of power series in t with coefficients in $\mathbb{Q}[u]$. Moreover, $\frac{t}{1-tq} - U(uq) = O(t^3)$, so that we can iterate the above equation indefinitely. The net result is the expression of $T(u,tu)$ given in the proposition.

We then obtain easily the expression of $P(t;1)$ using Lemma 2. □

Proposition 8 (Nature of the g.f. and asymptotic properties of 3-sided walks) *The length generating function* $P(t;1)$ *of* 3*-sided walks is meromorphic in the disk* $\mathcal{D} = \{t : |t| < \sqrt{2}-1\}$*. In this disk, it has infinitely many poles, so that it cannot be D-finite.*

There is a unique dominant pole ρ*, which coincides with the radius of convergence of the series counting* 2*-sided walks. This is a simple pole of the series, so that the number of* n*-step* 3*-sided walks is*

$$p_n \sim \kappa\mu^n$$

for some positive constant κ*, with* $\mu = 1/\rho = 2.48\ldots$*.*

The average width of the box of random 3*-sided walks of (large) length* n *is equivalent to*

$$\frac{1+\rho}{2(1+3\rho)}n \sim 0.371\ldots n.$$

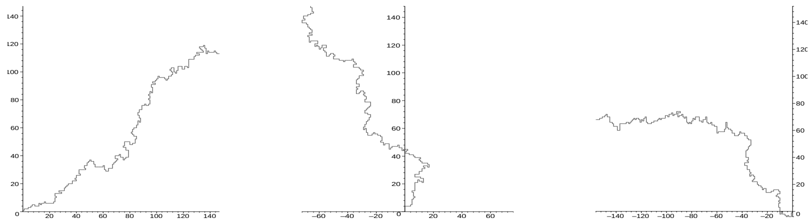

Fig. 7: Random 3-sided walks of length 400.

The proof is rather long. The poles come from the zeroes of $tq/(q-t) - U(q^i)$, $i \geq 0$. We prove that this series has a real positive root for all i, and that the residue at this point is non-zero.

5 Enumeration and properties of triangular prudent walks

We begin with a simple looking result that deals with the number of prudent walks contained in a box of a given size. If the box of a walk has size k, we say that the walk *spans a box* of size k.

Proposition 9 *The number of triangular prudent walks that span a box of size k is*

$$\tilde{p}_k = 2^{k-1}(k+1)(k+2)!.$$

More precisely, the number of triangular prudent walks that span a box of size k, and end on the right side of their box at distance i from the N corner (and thus at distance $j = k - i$ from the SW corner) is

$$\frac{2^k(k+2)!}{3} \quad \text{if } i = 0 \text{ or } j = 0, \qquad \frac{2^k(k+2)!}{6} \text{ otherwise.}$$

Proof. We specialize the functional equations of Lemma 4 to $t = 1$. Remarkably, the kernel of (4) reduces to 1. It is then easy to prove that this equation is satisfied by:

$$R(u, v) = \sum_{k \geq 0} \frac{2^k(k+2)!}{6} \left(u^k + v^k + \frac{u^{k+1} - v^{k+1}}{u - v} \right).$$

The fact that the numbers $\tilde{p}_k$ grow faster than exponentially is not unexpected. In particular, given a square of size k, the number of partially directed (1-sided) walks of height k that fit in this square is $(k+1)^{k+2}$.

We now move to the length enumeration of triangular prudent walks.

Proposition 10 *Set* $u = \frac{x(1-t)}{(1+tx)(1+t^2x)}$, *where x is a new variable. The generating function of triangular prudent walks ending on the right side of their box satisfies*

$$R(t; u, tu) = (1 + xt)(1 + xt^2) \sum_{k \geq 0} \frac{t^{\binom{k+1}{2}} \left(xt(1-2t^2)\right)^k}{(xt(1-2t^2); t)_{k+1}} \left(\frac{xt^3}{1-2t^2}; t \right)_k \tag{8}$$

where we have used the standard notation $(a; q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$. The generating function of triangular prudent walks, counted by the length and the size of the box, is

$$P(t; u) = 1 + \frac{6tu(1+t)}{1 - t - 2tu(1+t)} \left(1 + t\left(2u(1+t) - 1\right) R(t; u, tu)\right).$$

Remark. The parametrization of u in terms of the length variable t and another variable x is just a convenient way to write down $R(t; u, tu)$. Equivalently, we have

$$R(t; u, tu) = (1 + Xt)(1 + Xt^2) \sum_{k \geq 0} \frac{t^{\binom{k+1}{2}} \left(Xt(1 - 2t^2)\right)^k}{(Xt(1 - 2t^2); t)_{k+1}} \left(\frac{Xt^3}{1 - 2t^2}; t\right)_k$$

where $X \equiv X(u)$ is the only power series in t satisfying $u(1 + tX)(1 + t^2X) = X(1 - t)$.

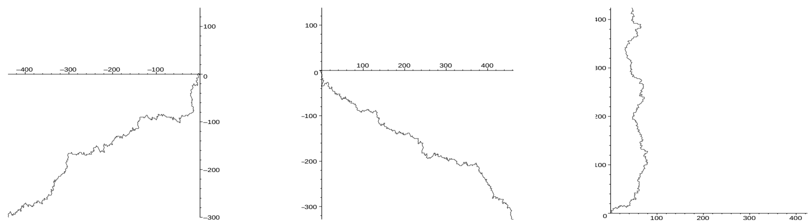

Fig. 8: Random triangular prudent walks walks of length 500.

Proposition 11 (Nature of the g.f. and asymptotic properties of triangular prudent walks)
The length generating function $P(t; 1)$ of triangular prudent walks is meromorphic in the domain $\mathcal{D} = \{t : |t| < 1\} \setminus [t_c, 1)$, where $t_c \simeq 0.29...$ is the real root of $1 - 3t - t^2 - t^3$. In this domain, it has infinitely many poles, so that it cannot be D-finite.

Moreover, $P(t; 1)$ has a unique pole of minimal modulus at $\rho = (\sqrt{17} - 3)/4 \simeq 0.28...$. The equation satisfied by ρ is $1 - 3\rho - 2\rho^2 = 0$. The number of triangular prudent walks of length n is

$$p_n \sim \kappa \left(\frac{3 + \sqrt{17}}{2}\right)^n$$

The average size of the box of random prudent walks of (large) length n is equivalent to

$$\left(1 + 1/\sqrt{17}\right) n/2.$$

Proof of Proposition 10. The kernel $K(u, v)$ of (4) admits a rational parametrization:

$$K(U(x), U(tx)) = 0 \quad \text{for} \quad U(x) = \frac{x(1 - t)}{(1 + tx)(1 + t^2x)}.$$

Setting $u = U(x)$ and $v = U(tx)$ in (4) gives

$$\Phi(x) = \frac{(1 + xt)(1 + xt^2)}{1 - xt(1 - 2t^2)} + \frac{xt^2(1 + xt)(1 - 2t^2 - xt^3)}{(1 + xt^3)(1 - xt(1 - 2t^2))} \Phi(xt).$$

with $\Phi(x) = R(u(x), tu(x))$. Iterating this equation gives the value (8) of $R(u, tu)$. The expression of $P(t; u)$ in terms of $R(u, tu)$ follows from (5), after specializing (4) to $v = u$ and then $v = 0$. □

Again, the proof of Proposition 11 is rather long, the main difficulty being to prove the existence of infinitely many poles in the domain $\mathcal{D}$. These poles accumulate on a portion of the unit circle.

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