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The quasiinvariants of the symmetric group

Jason Bandlow^{1†} and Gregg Musiker^{2‡}

¹*Department of Mathematics, University of California, One Shields Avenue, Davis, CA 95616, U.S.A.*

²*Department of Mathematics, Massachusetts Institute of Technology, 77 Massachusetts Avenue, Cambridge, MA 02139, U.S.A.*

Abstract. For m a non-negative integer and G a Coxeter group, we denote by $\mathbf{QI}_m(G)$ the ring of m -quasiinvariants of G , as defined by Chalykh, Feigin, and Veselov. These form a nested series of rings, with $\mathbf{QI}_0(G)$ the whole polynomial ring, and the limit $\mathbf{QI}_\infty(G)$ the usual ring of invariants. Remarkably, the ring $\mathbf{QI}_m(G)$ is freely generated over the ideal generated by the invariants of G without constant term, and the quotient is isomorphic to the left regular representation of G . However, even in the case of the symmetric group, no basis for $\mathbf{QI}_m(G)$ is known. We provide a new description of $\mathbf{QI}_m(S_n)$, and use this to give a basis for the isotypic component of $\mathbf{QI}_m(S_n)$ indexed by the shape $[n - 1, 1]$.

Résumé. Pour m un entier positif ou nul et G un groupe de Coxeter, nous notons $QI_m(G)$ l'anneau des quasiinvariants définis par Chalykh, Feigin et Veselov. On obtient ainsi une série d'anneaux emboîtés, $QI_0(G)$ étant l'anneau des polynômes, et la limite $QI_\infty(G)$ l'anneau des invariants usuels. Il est remarquable que l'anneau $QI_m(G)$ est librement généré sur l'idéal engendré par les invariants de G sans terme constant, et le quotient est isomorphe à la représentation régulière à gauche de G . Cependant, même dans le cas du groupe symétrique, aucune base de $QI_m(G)$ n'est connue. Nous donnons une nouvelle description de $QI_m(G)$ et l'utilisons pour obtenir une base du composant isotypique de $QI_m(S_n)$ indexée par la partition $(n - 1, 1)$.

Keywords: symmetric group, invariants, quasiinvariants

1 Introduction

A permutation $\sigma \in S_n$ acts on a polynomial in $\mathbf{R} = \mathbb{Q}[x_1, \dots, x_n]$ by permutation of indices:

$$\sigma P(x_1, \dots, x_n) = P(x_{\sigma(1)}, \dots, x_{\sigma(n)}). \quad (1)$$

The S_n -invariant polynomials are known as symmetric functions, and denoted by Λ_n . It is well known (e.g., [Sta99]) that the elements of Λ_n without constant term are generated by the elementary symmetric functions $\{e_1, \dots, e_n\}$ where

$$e_j = \sum_{i_1 < i_2 < \dots < i_j} x_{i_1} \dots x_{i_j}. \quad (2)$$

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The ring of coinvariants of S_n is the quotient

$$\mathbf{R}/\langle e_1, \dots, e_n \rangle. \quad (3)$$

As an S_n -module, the ring of coinvariants is known to be isomorphic to the left regular representation. It is also known that \mathbf{R} is freely generated over this ideal which implies that if we choose a basis of minimal degree elements $\mathcal{B} = \{b_1, \dots, b_n\}$ for the ring of coinvariants, any element $P \in \mathbf{R}$ has a unique expansion

$$P = \sum_{i=1}^{n!} b_i g_i \quad (4)$$

with the $g_i \in \Lambda_n$. More information is given by the Hilbert series for the isotypic component of \mathbf{R} corresponding to λ , namely

$$\frac{\sum_{T \in ST(\lambda)} f_\lambda q^{\text{cocharge}(T)}}{(1-q)(1-q^2) \dots (1-q^n)}. \quad (5)$$

Known bases for the ring of coinvariants with combinatorial descriptions include the Artin monomials and the Schubert polynomials.

In [CV90] and [FV02], Chalykh, Feigin and Veselov introduced a generalization of invariance known as “ m -quasiinvariance”. For the symmetric group, the m -quasiinvariants are the polynomials $P \in \mathbb{Q}[x_1, \dots, x_n]$ which have the divisibility property

$$(x_i - x_j)^{2m+1} \mid \left(1 - (i, j)\right) P \quad (6)$$

for every transposition (i, j) . We set

$$\mathbf{QI}_m = \{m\text{-quasiinvariants of } S_n\}. \quad (7)$$

A straightforward calculation (which we do not do here) shows that the m -quasiinvariants of S_n form a ring and an S_n module, and that we have the following containments:

$$\mathbf{R} = \mathbf{QI}_0 \supset \mathbf{QI}_1 \supset \dots \supset \mathbf{QI}_m \supset \dots \supset \Lambda_n. \quad (8)$$

For all m , the ring \mathbf{QI}_m was conjectured in [FV02] to be freely generated over the ideal generated by symmetric functions without constant term. The corresponding quotient $\mathbf{QI}_m/\langle e_1, \dots, e_n \rangle$ was conjectured to be isomorphic as an S_n module to the left regular representation. These facts were proved in [EG02]. Further information was given in [FV03], where they showed the Hilbert series of the isotypic component of \mathbf{QI}_m indexed by λ is given by

$$\frac{\sum_{T \in ST(\lambda)} f_\lambda q^{m\left(\binom{n}{2} - \text{content}(\lambda(T))\right) + \text{cocharge}(T)}}{(1-q)(1-q^2) \dots (1-q^n)}. \quad (9)$$

Here f_λ is the number of standard tableaux of shape λ and *content* and *cocharge* are two statistics on tableaux—we will not need the precise definitions, though we note that *content* only depends on the

shape of T and thus is actually a statistic on partitions. We should note that the definitions, theorems and conjectures cited above were all phrased in terms of general Coxeter groups; it is only for simplicity that we have restricted our attention to S_n .

In light of the combinatorially interesting bases for the coinvariants in the classical (*i.e.*, $m = 0$) case, the authors have looked for a basis for larger m . In [FV02], and later in [BM05], a basis was given for the case $n = 3$. (The work [FV02] specifically described the quasiinvariants for dihedral groups; in particular for $D_3 \cong S_3$.) Further, in [FV03], Felder and Veselov provide integral expressions, $\phi^{(j)}(x)$ for $2 \leq j \leq n$, for the lowest degree non-symmetric m -quasiinvariants, *i.e.* those of degree $mn + 1$. In the present work, we give a complete basis of the isotypic component indexed by the partition $[n - 1, 1]$. This is accomplished by means of a new characterization of \mathbf{QI}_m :

Theorem 1 *The vector space of quasiinvariants has the following direct sum decomposition:*

$$\mathbf{QI}_m = \bigoplus_{T \in ST(n)} (\gamma_T \mathbf{R} \cap V_T^{2m+1} \mathbf{R})$$

where $ST(n)$ is the set of standard tableaux of size n , γ_T is a projection operator due to Young (defined in full detail in the next section) and V_T is the polynomial given by the product over the columns of T of the associated “Vandermonde determinants” (this is also defined in detail below). This characterization is proved using completely elementary methods. We then give a basis for the $[n - 1, 1]$ isotypic component, using the previous characterization to show that our basis does, in fact, consist of quasiinvariants. Precisely, for T a standard Young tableau of shape $[n - 1, 1]$ with j the entry in the second row, we set

$$Q_T^{k,m} = \int_{x_1}^{x_j} t^k \prod_{i=1}^n (t - x_i)^m dt. \quad (10)$$

With this definition, we have

Theorem 2 *The set*

$$\{Q_T^{0,m}, Q_T^{1,m}, Q_T^{2,m}, \dots, Q_T^{n-2,m}\} \quad (11)$$

is a basis for $\gamma_T (\mathbf{QI}_m / \langle e_1, \dots, e_n \rangle)$.

The fact \mathbf{QI}_m is freely generated over $\langle e_1, \dots, e_n \rangle$ then gives us that the set

$$\{Q_T^{k,m} e_1^{p_1} e_2^{p_2} \dots e_n^{p_n}\} \quad 0 \leq k \leq n - 2, \quad p_i \in \mathbb{N}$$

is a basis for \mathbf{QI}_m .

2 Acknowledgements

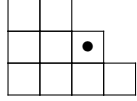
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3 Definitions and notation

Throughout this work, we will write elements of the symmetric group S_n using cycle notation. We will perform computations in the group algebra of S_n , and as such it will be useful to have shorthand notation for many commonly occurring elements. For a given subgroup A of S_n , we set

$$[A] = \sum_{\sigma \in A} \sigma \quad \text{and} \\ [A]' = \sum_{\sigma \in A} \text{sgn}(\sigma)\sigma.$$

The Young diagram of a partition λ consists of a collection of λ_i left-justified boxes in the i^{th} row of the positive integer lattice. These boxes are indexed by ordered pairs (i, j) , where i is the row index and j is the column index. For example, in the following Young diagram of $[4, 3, 2]$, the cell $(2, 3)$ is marked:



A *tableau* of shape $\lambda \vdash n$ is a function from the cells of the Young diagram of λ to the set $\{1, \dots, n\}$. We write the $T(i, j)$ for the value of T at the cell (i, j) . For example, if T is the following tableau, $T(2, 3) = 8$:

| | | |
|---|---|---|
| 6 | 7 | |
| 4 | 5 | 8 |
| 1 | 2 | 3 |

We call a tableau *standard* if it is injective and the entries increase across the rows and up the columns. For example, the tableau above is standard. We denote the set of standard tableaux of shape λ by $ST(\lambda)$ and the set of all standard tableaux with n boxes by $ST(n)$.

Given a tableau T we let C_i be the set of elements in the i^{th} column and we define R_i similarly for the rows. We also set

$$C(T) = \bigcup_i \{(a, b) : a, b \in C_i\} \quad (12)$$

$$N(T) = \prod_i [C_i]' \quad (13)$$

$$P(T) = \prod_i [R_i] \quad (14)$$

$$\gamma_T = \frac{f_\lambda N(T)P(T)}{n!} \quad (15)$$

$$\lambda(T) = \text{the shape of tableau } T. \quad (16)$$

Finally, we define the following useful polynomial associated with a tableau T :

$$V_T = \prod_{(i,j) \in C(T)} (x_i - x_j). \quad (17)$$

4 Useful Facts From Representation Theory

The following is a fundamental fact about the representation theory of the symmetric group.

Proposition 1 For W a finite dimensional S_n -module,

$$W \cong \bigoplus_{\lambda \vdash n} V_\lambda^{\oplus m_\lambda} \quad (18)$$

where the V_λ are the irreducible representations of S_n and the m_λ are non-negative integers.

The vector space and S_n -module $V_\lambda^{\oplus m_\lambda}$ is known as the *isotypic component* of V indexed by λ . Now, \mathbf{QI}_m is infinite dimensional, but it is the direct sum of homogeneous components, each of which are finite dimensional. So we have that each homogeneous component of \mathbf{QI}_m decomposes into the direct sum of irreducibles. The direct sum of all copies of V_λ occurring in this decomposition is still itself an S_n -module, and is still referred to as the isotypic component indexed by λ . However, we will find the following decomposition of V more useful.

Proposition 2 On any S_n module W , the group algebra elements $\{\gamma_T\}_{T \in ST(n)}$ act as projection operators. In symbols, we have the conditions

1. $\gamma_T^2 = \gamma_T$
2. $W = \bigoplus_{T \in ST(n)} \gamma_T W$.

Note that in this decomposition, unlike the previous one, the direct summands are not themselves S_n -modules. We do have the following proposition, however, nicely relating the previous two.

Proposition 3 For any S_n module W ,

$$\bigoplus_{T \in ST(\lambda)} \gamma_T W \quad (19)$$

is the isotypic component of W indexed by λ .

In the case of the quasiinvariants, we have the following

Proposition 4 The \mathbb{Q} -vector space of m -quasiinvariants has the following direct sum decomposition:

$$\mathbf{QI}_m = \bigoplus_{T \in ST(n)} \gamma_T \mathbf{QI}_m.$$

Our goal will be to use the decomposition $\mathbf{QI}_m / \langle e_1, \dots, e_n \rangle = \bigoplus_T \gamma_T (\mathbf{QI}_m / \langle e_1, \dots, e_n \rangle)$ to find a partial basis for this space.

5 A New Characterization of S_n -Quasiinvariants

In this section we prove the following theorem:

Theorem 1 The vector space of quasiinvariants has the following direct sum decomposition:

$$\mathbf{QI}_m = \bigoplus_{T \in ST(n)} (\gamma_T \mathbf{R} \cap V_T^{2m+1} \mathbf{R}).$$

We will prove this by showing

$$\gamma_T \mathbf{QI}_m = \gamma_T \mathbf{R} \cap V_T^{2m+1} \mathbf{R}. \quad (20)$$

Combining (20) with Proposition 4 will prove the theorem. We note first that one containment of (20) is easy:

$$\gamma_T \mathbf{QI}_m \subseteq \gamma_T \mathbf{R} \cap V_T^{2m+1} \mathbf{R}. \quad (21)$$

By definition, any element of $\gamma_T \mathbf{QI}_m$ is also in $\gamma_T \mathbf{R}$. Thus, we must only show that for $Q \in \mathbf{QI}_m$ we have

$$\gamma_T Q \in V_T^{2m+1} \mathbf{R}. \quad (22)$$

Let $P = \gamma_T Q = N(T)Q'$. P must be anti-symmetric with respect to all transpositions in $C(T)$ since it is in the image of $N(T)$. Thus, for any $(a, b) \in C(T)$, $\left(1 - (a, b)\right)P = 2P$. Hence $(x_a - x_b)^{2m+1}$ divides $2P$ (and also P) for all $(a, b) \in C(T)$. This establishes equation (21).

It remains to show that for all standard tableaux T and all $m \geq 0$, we have the following containment of vector spaces:

$$\gamma_T \mathbf{R} \cap V_T^{2m+1} \mathbf{R} \subseteq \gamma_T \mathbf{QI}_m. \quad (23)$$

Since γ_T is an idempotent, it suffices to show that for any polynomial P in the ideal $V_T^{2m+1} \mathbf{R}$, $\gamma_T P = P$ implies that P is m -quasiinvariant. With this in mind, we let P be such that $V_T^{2m+1} | P$ and $\gamma_T P = P$. We wish to show that $\left(1 - (a, b)\right)P$ is divisible by $(x_a - x_b)^{2m+1}$ for all transpositions (a, b) . We first consider the case where a and b are in the same column of T . In this case we have

$$(a, b)N(T) = -N(T)$$

and so

$$(a, b)P = (a, b)\gamma_T P = -\gamma_T P = -P.$$

Thus

$$\left(1 - (a, b)\right)P = 2P \in V_T^{2m+1} \mathbf{R}$$

which is divisible by the required factor of $(x_a - x_b)^{2m+1}$.

The remaining case, showing that $\left(1 - (a, b)\right)P$ is divisible by $(x_a - x_b)^{2m+1}$ when a and b are in different columns, is the only difficult part of the proof. Suppose without loss that a is in column i , and is to the left of b , which is in column j . We define $\alpha_{i,b} \in \mathbb{Q}[S_n]$ to be the sum of all transpositions (c, b) where c is an element of column i . The key property of this element is the following:

Lemma 1 *The operator $\alpha_{i,b}$ preserves γ_T . In symbols,*

$$\alpha_{i,b}\gamma_T = \gamma_T.$$

The proof of Lemma 1, though completely elementary, is lengthy and is omitted here.

With this fact in hand, it is not difficult to complete the proof. By Lemma 1, P is preserved by $\alpha_{i,b}$:

$$\alpha_{i,b}P = \alpha_{i,b}\gamma_TP = \gamma_TP = P. \quad (24)$$

This fact and the definition of $\alpha_{i,b}$ gives

$$\left(1 - (a, b)\right)P = P - (a, b)P \quad (25)$$

$$= \alpha_{i,b}P - (a, b)P \quad (26)$$

$$= \sum_{\substack{c \in C_i \\ c \neq a}} (c, b)P. \quad (27)$$

Since $P \in V_T^{2m+1}\mathbf{R}$, for any $c \in C_i$ with $c \neq a$ we can rewrite P as

$$P = (x_c - x_a)^{2m+1}(\text{other factors}). \quad (28)$$

Thus

$$(c, b)P = (x_b - x_a)^{2m+1}(\text{other factors}) \quad (29)$$

and we have

$$(x_b - x_a)^{2m+1} \text{ divides } (c, b)P \text{ for every } c \in C_i \text{ with } c \neq a. \quad (30)$$

Hence $(x_b - x_a)^{2m+1}$ divides the right-hand side of equation 27, which completes the proof.

6 A Basis For The Isotypic Component $\lambda(T) = [n - 1, 1]$

In this section, we refer to the quotient $\mathbf{QI}_m / \langle e_1, \dots, e_n \rangle$ by the symbol \mathbf{QI}_m^* . Our object here is to describe a basis for $\gamma_T \mathbf{QI}_m^*$ when T has a hook shape of the form $[n - 1, 1]$. Until otherwise specified, let λ be the partition $[n - 1, 1]$ and let T be one of the $(n - 1)$ standard tableaux of shape λ . In fact T is uniquely defined by the lone entry of the second row. Suppose it's $j \in \{2, 3, \dots, n\}$. We define

$$Q_T^{k,m} = \int_{x_1}^{x_j} t^k \prod_{i=1}^n (t - x_i)^m dt. \quad (31)$$

Theorem 2 *The polynomials $\{Q_T^{k,m}\}_{k=0}^{n-2}$ are a set of representatives for a basis of $\gamma_T \mathbf{QI}_m^*$.*

By equation (9) we know this consists of the right number of elements, and they are of the right degree. Thus we are faced with two tasks: showing the elements $Q_T^{k,m}$ are, in fact, m -quasiinvariant, and showing that they are linearly independent in \mathbf{QI}_m^* . To show the m -quasiinvariance of $Q_T^{k,m}$ we must, by Theorem 1, show these polynomials are preserved by γ_T and divisible by V_T^{2m+1} .

Lemma 2 *The polynomials $Q_T^{k,m}$ are preserved by the operator γ_T . That is,*

$$\gamma_T Q_T^{k,m} = Q_T^{k,m}. \quad (32)$$

Proof: We will prove this inductively, by showing that the $Q_T^{k,m}$ satisfy a recursion that respects the action of γ_T . In what follows, e_i will denote the i th elementary symmetric function in the variables x_1, \dots, x_n , with the convention that $e_0 = 1$. We first state for reference a classical symmetric function identity:

$$\prod_{i=1}^n (t - x_i) = \sum_{i=0}^n (-1)^i e_i t^{n-i}. \quad (33)$$

We now show that for $m > 1$, we have the recurrence:

$$Q_T^{k,m} = \sum_{i=0}^n (-1)^i e_i Q_T^{n-i+k, m-1}. \quad (34)$$

This can be seen by first expanding the definition of $Q_T^{k,m}$ to get

$$Q_T^{k,m} = \int_{x_1}^{x_j} \left(\prod_{i=1}^n (t - x_i) \right) t^k \prod_{l=1}^n (t - x_l)^{m-1} dt. \quad (35)$$

Substituting (33) into (35) and pulling out the factors not involving t gives

$$Q_T^{k,m} = \int_{x_1}^{x_j} \left(\sum_{i=0}^n (-1)^i e_i t^{n-i} \right) t^k \prod_{l=1}^n (t - x_l)^{m-1} dt \quad (36)$$

$$= \sum_{i=0}^n (-1)^i e_i \int_{x_1}^{x_j} t^{n-i+k} \prod_{l=1}^n (t - x_l)^{m-1} dt \quad (37)$$

$$= \sum_{i=0}^n (-1)^i e_i Q_T^{n-i+k, m-1}. \quad (38)$$

We now proceed with a proof of Lemma 2 by induction on m . From the definition of $Q_T^{k,m}$ we have

$$Q_T^{k,0} = \int_{x_1}^{x_j} t^k \prod_{i=1}^n (t - x_i)^0 dt \quad (39)$$

$$= \frac{1}{k+1} (x_j^{k+1} - x_1^{k+1}). \quad (40)$$

That $x_j^{k+1} - x_1^{k+1}$ is preserved by γ_T is a straightforward calculation which we omit here.

We can now complete the proof by using equation (34), the inductive hypothesis, and the fact that symmetric functions commute with group algebra elements:

$$\gamma_T Q_T^{k,m} = \sum_{i=0}^n (-1)^i \gamma_T (e_i Q_T^{n-i+k,m-1}) \quad (41)$$

$$= \sum_{i=0}^n (-1)^i e_i (Q_T^{n-i+k,m-1}) \quad (42)$$

$$= Q_T^{k,m}. \quad (43)$$

□

In order to complete our task of showing that $Q_T^{k,m} \in \mathbf{QI}_m$, we must show that $(x_j - x_1)^{2m+1}$ divides $Q_T^{k,m}$. We do so by proving the following stronger statement:

Lemma 3 For all k ,

$$\lim_{x_1 \rightarrow x_j} \frac{Q_T^{k,m}}{(x_j - x_1)^{2m+1}} = \frac{(-1)^m m!^2}{(2m+1)!} x_j^k \prod_{\substack{i=2 \\ i \neq j}}^n (x_j - x_i)^m.$$

Proof: This is a straightforward calculation. Expanding $Q_T^{k,m}$ according to the definition gives

$$\lim_{x_j \rightarrow x_1} \frac{Q_T^{k,m}}{(x_j - x_1)^{2m+1}} = \lim_{x_j \rightarrow x_1} \frac{\int_{x_1}^{x_j} t^k \prod_{i=1}^n (t - x_i)^m dt}{(x_j - x_1)^{2m+1}} \quad (44)$$

which is an indeterminate expression of the form $\frac{0}{0}$. Repeatedly applying L'Hopital's rule and evaluating the numerator with Leibniz's rule for differentiation under the integral gives that the expression in (44) equals

$$\lim_{x_j \rightarrow x_1} \frac{(-1)^m \cdot m! \int_{x_1}^{x_j} t^k \prod_{\substack{i=1 \\ i \neq j}}^n (t - x_i)^m dt}{(2m+1)(2m)(2m-1) \cdots (m+2)(x_j - x_1)^{m+1}}.$$

One more application of L'Hopital's rule, evaluated this time with the Fundamental Theorem of Calculus, yields

$$\lim_{x_j \rightarrow x_1} \frac{(-1)^m \cdot m! \cdot x_j^k \prod_{\substack{i=1 \\ i \neq j}}^n (x_j - x_i)^m}{(2m+1)(2m)(2m-1) \cdots (m+1)(x_j - x_1)^m}. \quad (45)$$

We now cancel the term $(x_j - x_1)^m$ from both numerator and denominator and the Lemma is proven. □

The polynomiality of $\lim_{x_1 \rightarrow x_j} \frac{Q_T^{k,m}}{(x_j - x_1)^{2m+1}}$ immediately gives that $(x_j - x_1)^{2m+1}$ divides $Q_T^{k,m}$. Thus we have established that $Q_T^{k,m}$ is m -quasiinvariant.

We need one more lemma to establish Theorem 2.

Lemma 4 *The set $\{Q_T^{k,m}\}_{k=0}^{n-2}$ is linearly independent in \mathbf{QI}_m^* .*

Proof: As these polynomials are of different degrees, it suffices to show that they are non-zero in \mathbf{QI}_m^* . Put another way, we must show that $Q_T^{k,m}$ is not in the ideal of $\gamma_T \mathbf{QI}_m$ generated by $\langle e_1, \dots, e_n \rangle$. Equivalently, we must show that polynomials of the form

$$P_k = Q_T^{k,m} + A_1 Q_T^{k-1,m} + \dots + A_{k-1} Q_T^{1,m} + A_k Q_T^{0,m} \quad (46)$$

(where the A_i are symmetric functions of degree i) can only equal 0 if $k \geq n-1$. To accomplish this, we use the explicit formulas for $\lim_{x_j \rightarrow x_1} Q_T^{k,m}/V_T^{2m+1}$ given by Lemma 3 to show the stronger statement

$$\lim_{x_j \rightarrow x_1} P_k/V_T^{2m+1} = 0 \implies k \geq n-1 \quad (47)$$

regardless of the choice of the symmetric functions. Letting \widetilde{A}_i denote the limit $x_j \rightarrow x_1$ applied to the symmetric function A_i , and assuming without loss of generality that $j = 2$, we have

$$\lim_{x_2 \rightarrow x_1} P_k/V_T^{2m+1} = 0 \quad (48)$$

$$\implies \left(\frac{(-1)^m m!^2}{(2m+1)!} \prod_{i=3}^n (x_1 - x_i)^m \right) \left(x_1^k + \widetilde{A}_1 x_1^{k-1} + \dots + \widetilde{A}_{k-1} x_1 + \widetilde{A}_k \right) = 0 \quad (49)$$

$$\implies x_1^k + \widetilde{A}_1 x_1^{k-1} + \dots + \widetilde{A}_{k-1} x_1 + \widetilde{A}_k = 0 \quad (50)$$

$$\implies \lim_{x_2 \rightarrow x_1} (x_1^k + A_1 x_1^{k-1} + \dots + A_k) = 0. \quad (51)$$

We set

$$Q(x_1, \dots, x_n) = x_1^k + A_1 x_1^{k-1} + \dots + A_k \quad (52)$$

and now equation (51) implies that

$$Q(x_1, \dots, x_n) = (x_2 - x_1) \cdot R(x_1, \dots, x_n). \quad (53)$$

However, Q must be symmetric with respect to all pairs of variables not involving x_1 . Thus, for any $\sigma \in S_{\{2,3,\dots,n\}}$, $\sigma Q = Q$ and so

$$Q(x_1, \dots, x_n) = \sigma Q(x_1, \dots, x_n) = (x_{\sigma(2)} - x_1) \cdot \sigma R(x_1, \dots, x_n). \quad (54)$$

Hence $\prod_{i=2}^n (x_i - x_1)$ divides $Q(x_1, \dots, x_n)$, and k , which is the degree of $Q(x_1, \dots, x_n)$, must be greater than or equal to $n-1$. \square

The proof of Theorem 2 now follows immediately from Lemmas 2, 3, and 4.

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