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Invariant and coinvariant spaces for the algebra of symmetric polynomials in non-commuting variables

François Bergeron and Aaron Lauve

LaCIM, Université du Québec à Montréal, CP 8888, Succ. Centre-ville, Montréal (Québec) H3C 3P8, CANADA

Abstract. We analyze the structure of the algebra $\mathbb{K}\langle \mathbf{x} \rangle^{\mathfrak{S}_n}$ of symmetric polynomials in non-commuting variables in so far as it relates to $\mathbb{K}[\mathbf{x}]^{\mathfrak{S}_n}$, its commutative counterpart. Using the “place-action” of the symmetric group, we are able to realize the latter as the invariant polynomials inside the former. We discover a tensor product decomposition of $\mathbb{K}\langle \mathbf{x} \rangle^{\mathfrak{S}_n}$ analogous to the classical theorems of Chevalley, Shephard-Todd on finite reflection groups. In the case $|\mathbf{x}| = \infty$, our techniques simplify to a form readily generalized to many other familiar pairs of combinatorial Hopf algebras.

Résumé. Nous analysons la structure de l’algèbre $\mathbb{K}\langle \mathbf{x} \rangle^{\mathfrak{S}_n}$ des polynômes symétriques en des variables non-commutatives pour obtenir des analogues des résultats classiques concernant la structure de l’anneau $\mathbb{K}[\mathbf{x}]^{\mathfrak{S}_n}$ des polynômes symétriques en des variables commutatives. Plus précisément, au moyen de “l’action par positions”, on réalise $\mathbb{K}[\mathbf{x}]^{\mathfrak{S}_n}$ comme sous-module de $\mathbb{K}\langle \mathbf{x} \rangle^{\mathfrak{S}_n}$. On découvre alors une nouvelle décomposition de $\mathbb{K}\langle \mathbf{x} \rangle^{\mathfrak{S}_n}$ comme produit tensoriel, obtenant ainsi un analogues des théorèmes classiques de Chevalley et Shephard-Todd. Dans le cas $|\mathbf{x}| = \infty$, nos techniques se simplifient en une forme aisément généralisables à beaucoup d’autres paires d’algèbres de Hopf familières.

Keywords: Chevalley theorem, symmetric group, noncommutative symmetric polynomials, set partitions

1 Introduction

One of the more striking results of the invariant theory of reflection groups is certainly the following: if W is a finite group of $n \times n$ matrices, then there is a graded W -module decomposition of the polynomial ring $S = \mathbb{K}[\mathbf{x}]$, in variables $\mathbf{x} = \{x_1, x_2, \dots, x_n\}$, as a tensor product⁽ⁱ⁾

$$S \simeq S_W \otimes S^W, \quad (1)$$

if and only if W is a group generated by (pseudo) reflections. As usual, S affords the natural W -module structure obtained by considering it as the symmetric space on the defining vector space X^* for W , e.g.,

⁽ⁱ⁾ We assume throughout that \mathbb{K} is a field containing \mathbb{Q} .

$w \cdot f(\mathbf{x}) = f(w \cdot \mathbf{x})$. It is customary to denote by S^W the ring of W -invariant polynomials for this action. To finish parsing (1), recall that S_W stands for the **coinvariant space**, i.e., the W -module defined as

$$S_W := S / \langle S_+^W \rangle, \tag{2}$$

the quotient of S by the ideal generated by constant-term free W -invariant polynomials. We give S , S^W , and S_W a grading by polynomial degree in \mathbf{x} (the latter being well-defined because $\langle S_+^W \rangle$ is a homogeneous ideal). The motivation behind the quotient in (2) is to eliminate redundant copies of irreducible W -modules inside S . Indeed, if \mathcal{V} is such a module and $f(\mathbf{x})$ is any W -invariant polynomial with no constant term, then $\mathcal{V}f(\mathbf{x})$ is an isomorphic copy of \mathcal{V} living within $\langle S_+^W \rangle$. As a result, the coinvariant space S_W is the interesting part of the story.

The context for the present paper is the algebra $T = \mathbb{K}\langle \mathbf{x} \rangle$ of noncommutative polynomials, with W -module structure on T obtained by considering it as the tensor space on the defining space X^* for W . In the special case when W is the symmetric group \mathfrak{S}_n , we elucidate a relationship between the space S^W and the subalgebra T^W of W -invariants in T . The subalgebra T^W was first studied in [14, 5] with the aim of obtaining noncommutative analogs of classical results concerning symmetric function theory. Recent work in [12, 3] has extended a large part of the story surrounding (1) to this noncommutative context. In particular, there is an explicit \mathfrak{S}_n -module decomposition of the form $T \simeq T_{\mathfrak{S}_n} \otimes T^{\mathfrak{S}_n}$, cf. [3, Theorem 8.7].

By contrast, our work proceeds in a somewhat complementary direction. We consider $\mathcal{N} = T^{\mathfrak{S}_n}$ as a tower of \mathfrak{S}_d -modules under the “place-action” and realize $S^{\mathfrak{S}_n}$ inside \mathcal{N} as a subspace Λ of invariants for this action. This leads to a decomposition of \mathcal{N} analogous to (1). More explicitly, our main result is as follows.

Theorem 1 *There is an explicitly constructed subspace \mathcal{C} of \mathcal{N} so that \mathcal{C} and the place-action invariants Λ exhibit a graded vector space isomorphism*

$$\mathcal{N} \simeq \mathcal{C} \otimes \Lambda. \tag{3}$$

As an immediate corollary we derive the Hilbert series formula

$$\text{Hilb}_t(\mathcal{C}) = \text{Hilb}_t(\mathcal{N}) \prod_{i=1}^n (1 - t^i). \tag{4}$$

Here, as usual, the **Hilbert series** of a graded space $\mathcal{V} = \bigoplus_{d \geq 0} \mathcal{V}_d$ is the formal power series defined as

$$\text{Hilb}_t(\mathcal{V}) = \sum_{d \geq 0} \dim \mathcal{V}_d t^d,$$

where \mathcal{V}_d is the **homogeneous degree d component** of \mathcal{V} . The fact that (4) expands as a series in $\mathbb{N}[t]$ is not at all obvious, as one may check that the Hilbert series of \mathcal{N} is

$$\text{Hilb}_t(\mathcal{N}) = 1 + \sum_{k=1}^n \frac{t^k}{(1-t)(1-2t) \cdots (1-kt)} \tag{5}$$

(taking $n = |\mathbf{x}|$). We underline that the harder part of our work lies in working out the case $n < \infty$. This is accomplished in Section 6. If we restrict ourselves to the case $n = \infty$, both \mathcal{N} and Λ become Hopf

algebras and things are much simpler. Our results are then consequences of a general theorem of Blattner, Cohen and Montgomery. As we will see in Section 5, stronger results hold in this simpler context. For example, (4) may be refined to a statement about “shape” enumeration.

2 The algebra $S^\mathfrak{S}$ of symmetric polynomials

2.1 Vector space structure of $S^\mathfrak{S}$

We specialize our introductory discussion to the group $W = \mathfrak{S}_n$ of permutation matrices. The action on $S = \mathbb{K}[\mathbf{x}]$ is simply the **permutation action** $\sigma \cdot x_i = x_{\sigma(i)}$ and $S^{\mathfrak{S}_n}$ comprises the usual symmetric polynomials. We suppress n in the notation and denote the subring of symmetric polynomials by $S^\mathfrak{S}$. (Note that upon sending n to ∞ , the elements of $S^\mathfrak{S}$ become formal series in $\mathbb{K}[[\mathbf{x}]]$ of bounded degree; we still call them polynomials to affect a uniform discussion.) A monomial in S of degree d may be written as follows: given an r -subset $\mathbf{y} = \{y_1, y_2, \dots, y_r\}$ of \mathbf{x} and a **composition** of d into r parts, $\mathbf{a} = (a_1, a_2, \dots, a_r)$ ($a_i > 0$), we write $\mathbf{y}^\mathbf{a}$ for $y_1^{a_1} y_2^{a_2} \dots y_r^{a_r}$. We assume that the variables y_i are naturally ordered, so that whenever $y_i = x_j$ and $y_{i+1} = x_k$ we have $j < k$. Reordering the entries of a composition \mathbf{a} in decreasing order results in a partition $\lambda(\mathbf{a})$ called the **shape** of \mathbf{a} . Summing over monomials $\mathbf{y}^\mathbf{a}$ with the same shape leads to the monomial symmetric polynomial

$$m_\mu = m_\mu(\mathbf{x}) := \sum_{\lambda(\mathbf{a})=\mu, \mathbf{y} \subseteq \mathbf{x}} \mathbf{y}^\mathbf{a}.$$

Letting $\mu = (\mu_1, \dots, \mu_r)$ run over all partitions of $d = |\mu| = \mu_1 + \dots + \mu_r$ gives a basis for $S_d^\mathfrak{S}$. As usual, we set $m_0 := 1$ and agree that $m_\mu = 0$ if μ has too many parts (i.e., $n < r$).

2.2 Dimension enumeration

A fundamental result in the invariant theory of \mathfrak{S}_n is that $S^\mathfrak{S}$ is generated by a family $\{f_k\}_{1 \leq k \leq n}$ of algebraically independent symmetric polynomials, having respective degrees $\deg f_k = k$. (One may choose $\{m_k\}_{1 \leq k \leq n}$ for such a family.) It follows immediately that the Hilbert series of $S^\mathfrak{S}$ is

$$\text{Hilb}_t(S^\mathfrak{S}) = \prod_{i=1}^n \frac{1}{1-t^i}. \tag{6}$$

Recalling that the Hilbert series of S is $(1-t)^{-n}$, we see from (1) and (6) that the Hilbert series for the coinvariant space $S_\mathfrak{S}$ is the well-known t -analog of $n!$:

$$\prod_{i=1}^n \frac{1-t^i}{1-t} = \prod_{i=1}^n (1+t+\dots+t^{i-1}). \tag{7}$$

In particular, contrary to the situation in (4), the series $\text{Hilb}_t(S)/\text{Hilb}_t(S^\mathfrak{S})$ in $\mathbb{Z}[[t]]$ is *obviously* positive.

2.3 Algebra and coalgebra structures of $S^\mathfrak{S}$

Given partitions μ and ν , there is an explicit formula for computing the product $m_\mu \cdot m_\nu$. In lieu of giving the formula, we refer the reader to [3, §4.1] and simply give an example:

$$m_{21} \cdot m_{11} = 3m_{2111} + 2m_{221} + 2m_{311} + m_{32}. \tag{8}$$

The extremal terms above are relevant to our coming discussion. Note that if $n < 4$, then the first term disappears. However, if n is sufficiently large then analogs of these terms always appear with positive integer coefficients for a given pair (μ, ν) . If $\mu = (\mu_1, \dots, \mu_r)$ and $\nu = (\nu_1, \dots, \nu_s)$ with $r \leq s$, then the partition indexing the left-most term is denoted by $\mu \cup \nu$ and is given by sorting the list $(\mu_1, \dots, \mu_r, \nu_1, \dots, \nu_s)$ in increasing order; the right-most term is indexed by $\mu + \nu := (\mu_1 + \nu_1, \dots, \mu_r + \nu_r, \nu_{r+1}, \dots, \nu_s)$. Taking $\mu = 31$ and $\nu = 221$, we would have $\mu \cup \nu = 32211$ and $\mu + \nu = 531$.

The ring $S^{\mathfrak{S}}$ is also afforded a coalgebra structure with coproduct $\Delta : S_d^{\mathfrak{S}} \rightarrow \bigoplus_{k=0}^d S_k^{\mathfrak{S}} \otimes S_{d-k}^{\mathfrak{S}}$ and counit $\varepsilon : S^{\mathfrak{S}} \rightarrow \mathbb{K}$ given, respectively, by

$$\Delta(m_\mu) = \sum_{\theta \cup \nu = \mu} m_\theta \otimes m_\nu \quad \text{and} \quad \varepsilon(m_\mu) = \delta_{\mu, 0}.$$

In the case $n = \infty$, Δ and ε are algebra maps, making $S^{\mathfrak{S}}$ a connected graded (by degree) Hopf algebra.

3 The algebra \mathcal{N} of noncommutative symmetric polynomials

3.1 Vector space structure of \mathcal{N}

Suppose now that \mathbf{x} denotes a set of non-commuting variables. The algebra $T = \mathbb{K}\langle \mathbf{x} \rangle$ of noncommutative polynomials is graded by degree. A degree d **noncommutative monomial** $\mathbf{z} \in T_d$ is simply a length- d “word”:

$$\mathbf{z} = z_1 z_2 \cdots z_d, \quad \text{with each } z_i \in \mathbf{x}.$$

In other terms, \mathbf{z} is a function $\mathbf{z} : [d] \rightarrow \mathbf{x}$, with $[d]$ denoting the set $\{1, \dots, d\}$. The permutation-action on \mathbf{x} clearly extends to T , giving rise to the subspace $\mathcal{N} = T^{\mathfrak{S}}$ of noncommutative \mathfrak{S} -invariants. With the aim of describing a linear basis for the homogeneous component \mathcal{N}_d , we next introduce set partitions of $[d]$ and the type of a monomial $\mathbf{z} : [d] \rightarrow \mathbf{x}$. We write $\mathbf{A} \vdash [d]$ when $\mathbf{A} = \{A_1, \dots, A_r\}$ is a **set partition** of $[d]$, i.e., $A_1 \cup \dots \cup A_r = [d]$, with $A_i \neq \emptyset$ and $A_i \cap A_j = \emptyset$ whenever $i \neq j$. The **type** $\tau(\mathbf{z})$ of a degree d monomial $\mathbf{z} : [d] \rightarrow \mathbf{x}$ is the set partition

$$\tau(\mathbf{z}) := \{z^{-1}(x) \mid x \in \mathbf{x}\} \setminus \{\emptyset\} \quad \text{of } [d],$$

whose parts are the non-empty fibers of the function \mathbf{z} . For instance,

$$\tau(x_1 x_8 x_1 x_5 x_8) = \{\{1, 3\}, \{2, 5\}, \{4\}\}.$$

In the sequel, we lighten the heavy notation for set partitions, writing, e.g., $\{\{1, 3\}, \{2, 5\}, \{4\}\}$ as 13.25.4. Clearly the type of a monomial is a finite set partition with at most n parts. Note that we may always order the parts in increasing order of their minimum elements. The **shape** $\lambda(\mathbf{A})$ of a set partition $\mathbf{A} = \{A_1, \dots, A_r\}$ is the (integer) partition $\lambda(|A_1|, \dots, |A_r|)$ obtained by sorting the part sizes of \mathbf{A} in increasing order. Observing that the permutation-action is *type preserving*, we are led to consider the **monomial** linear basis for the space \mathcal{N}_d :

$$m_{\mathbf{A}} = m_{\mathbf{A}}(\mathbf{x}) := \sum_{\tau(\mathbf{z})=\mathbf{A}} \mathbf{z}$$

For example, with $n = 2$, we have $m_{\emptyset} = 1$, $m_1 = x_1 + x_2$, $m_{12} = x_1^2 + x_2^2$, $m_{1.2} = x_1 x_2 + x_2 x_1$, $m_{123} = x_1^3 + x_2^3$, $m_{12.3} = x_1^2 x_2 + x_2^2 x_1$, $m_{13.2} = x_1 x_2 x_1 + x_2 x_1 x_2$, $m_{1.23} = x_1 x_2^2 + x_2 x_1^2$, $m_{1.2.3} = 0, \dots$ (Note that we set $m_{\emptyset} := 1$, taking \emptyset as the unique set partition of the empty set, and we agree that $m_{\mathbf{A}} = 0$ if \mathbf{A} is a set partition with more than n parts.)

3.2 Dimension enumeration and shape grading

Above, we determined that $\dim \mathcal{N}_d$ is the number of set partitions of d into at most n parts. These are counted by the (length restricted) **Bell numbers** $B_d^{(n)}$. Then (5) follows from the fact that its right-hand side is the ordinary generating function for length restricted Bell numbers. See [9, §2]. We next highlight a finer enumeration, where we grade \mathcal{N} by shape rather than degree.

For each partition μ , we may consider the submodule \mathcal{N}_μ spanned by those $m_{\mathbf{A}}$ for which $\lambda(\mathbf{A}) = \mu$. This results in a direct sum decomposition $\mathcal{N}_d = \bigoplus_{\mu \vdash d} \mathcal{N}_\mu$. A simple dimension description for \mathcal{N}_d takes the form of a **shape Hilbert series** in the following manner. View commuting variables q_i as marking parts of size i and set $\mathbf{q}_\mu := q_{\mu_1} q_{\mu_2} \cdots q_{\mu_r}$. Then

$$\text{Hilb}_{\mathbf{q}}(\mathcal{N}_d) = \sum_{\mu \vdash d} \dim \mathcal{N}_\mu \mathbf{q}_\mu = \sum_{\mathbf{A} \vdash [d]} q_{\lambda(\mathbf{A})}. \tag{9}$$

Here, \mathbf{q}_μ is a marker for set partitions of shape $\lambda(\mathbf{A}) = \mu$ and the sum is over all partitions into at most n parts. Such a shape grading also makes sense for $S_d^{\mathfrak{S}}$. Summing over all $d \geq 0$ and all μ , we get

$$\text{Hilb}_{\mathbf{q}}(S^{\mathfrak{S}}) = \sum_{\mu} \mathbf{q}_\mu = \prod_{i \geq 1} \frac{1}{1 - q_i}. \tag{10}$$

Using classical combinatorial arguments (cf. Chapter 2.3 of [2], Example 13), we see that the enumerator polynomials $\text{Hilb}_{\mathbf{q}}(\mathcal{N}_d)$ are naturally collected in the **exponential generating function**

$$\sum_{d=0}^{\infty} \text{Hilb}_{\mathbf{q}}(\mathcal{N}_d) \frac{t^d}{d!} = \sum_{m=0}^n \frac{1}{m!} \left(\sum_{k=1}^{\infty} q_k \frac{t^k}{k!} \right)^m. \tag{11}$$

For example, with $n = 3$, we have

$$\text{Hilb}_{\mathbf{q}}(\mathcal{N}_6) = q_6 + 6 q_5 q_1 + 15 q_4 q_2 + 15 q_4 q_1^2 + 10 q_3^2 + 60 q_3 q_2 q_1 + 15 q_2^3,$$

thus $\dim \mathcal{N}_{222} = 15$ when $n \geq 3$. Evidently, the \mathbf{q} -polynomials $\text{Hilb}_{\mathbf{q}}(\mathcal{N}_d)$ specialize to the length restricted Bell numbers $B_d^{(n)}$ when we set all q_k equal to 1.

In view of (10), (11), and Theorem 1, we are led to claim the following refinement of (4).

Corollary 2 For $n = \infty$, the shape Hilbert series of the space \mathcal{C} is given by the expression

$$\text{Hilb}_{\mathbf{q}}(\mathcal{C}) = \sum_{d \geq 0} d! \exp \left(\sum_{k=1}^{\infty} q_k \frac{t^k}{k!} \right) \Big|_{t^d} \prod_{i \geq 1} (1 - q_i), \tag{12}$$

with $(-)|_{t^d}$ standing for the operation of taking the coefficient of t^d .

Thus we have the expansion

$$\begin{aligned} \text{Hilb}_{\mathbf{q}}(\mathcal{C}) = & 1 + 2 q_2 q_1 + (3 q_3 q_1 + 2 q_2^2 + 3 q_2 q_1^2) \\ & + (4 q_4 q_1 + 9 q_3 q_2 + 6 q_3 q_1^2 + 10 q_2^2 q_1 + 4 q_2 q_1^3) + \dots \end{aligned}$$

Corollary 2 will follow immediately from the explicit description of \mathcal{C} and the isomorphism $\mathcal{C} \otimes \Lambda \rightarrow \mathcal{N}$ in Section 5, which is not only degree preserving, but shape preserving as well.

3.3 Algebra and coalgebra structures of \mathcal{N}

Since the action of \mathfrak{S} on T is multiplicative, it is straightforward to see that \mathcal{N} is an subalgebra of T . The *multiplication rule* in \mathcal{N} , expressing a product $m_{\mathbf{A}} \cdot m_{\mathbf{B}}$ as a sum of basis vectors $\sum_{\mathbf{C}} m_{\mathbf{C}}$, is easy to describe. Since we make heavy use of the rule later, we develop it carefully here. We begin with an example (the digits corresponding to $\mathbf{B} = \mathbf{1.2}$ appear in bold):

$$m_{13.2} \cdot m_{\mathbf{1.2}} = m_{13.2.4.5} + m_{134.2.5} + m_{135.2.4} + m_{13.24.5} + m_{13.25.4} + m_{135.24} + m_{134.25} \tag{13}$$

Compare this to (8). Notice that the shapes indexing the first and last terms in (13) are the partitions $\lambda(13.2) \cup \lambda(1.2)$ and $\lambda(13.2) + \lambda(1.2)$. As was the case in $S^{\mathfrak{S}}$, one of these shapes, namely $\lambda(\mathbf{A}) + \lambda(\mathbf{B})$, will always appear in the product, while appearance of the shape $\lambda(\mathbf{A}) \cup \lambda(\mathbf{B})$ depends on the cardinality of \mathbf{x} .

Let us now describe the multiplication rule. Given any $D \subseteq \mathbb{N}$ and $k \in \mathbb{N}$, we write D^{+k} for the set

$$D^{+k} := \{a + k \mid a \in D\}.$$

By extension, for any set partition $\mathbf{A} = \{A_1, \dots, A_r\}$ we set $\mathbf{A}^{+k} := \{A_1^{+k}, A_2^{+k}, \dots, A_r^{+k}\}$. These definitions allow for the introduction of a bilinear (non-commutative) operation denoted by “ ω ” on formal linear combinations of set partitions. Given partitions $\mathbf{A} = \{A_1, A_2, \dots, A_r\}$ of $[c]$ and a partition $\mathbf{B} = \{B_1, B_2, \dots, B_s\}$ of $[d]$, the summands of $\mathbf{A} \omega \mathbf{B}$ are set partitions of $[c + d]$. The operation ω is recursively defined by the rules:

(a) $\mathbf{A} \omega \emptyset = \emptyset \omega \mathbf{A} = \mathbf{A}$, with \emptyset denoting the unique set partition of the empty set;

(b) $\mathbf{A} \omega \mathbf{B} = \{A_1\} \cup (\mathbf{A}' \omega \mathbf{B}^{+c}) + \sum_{i=1}^s \{A_1 \cup B_i^{+c}\} \cup (\mathbf{A}' \omega (\mathbf{B} \setminus \{B_i\})^{+c})$,

with union \cup extended bilinearly and \mathbf{A}' denoting $\{A_2, \dots, A_r\}$.

As shown in [3, Prop. 3.2], the multiplication rule for $m_{\mathbf{A}}$ and $m_{\mathbf{B}}$ in \mathcal{N} , is

$$m_{\mathbf{A}} \cdot m_{\mathbf{B}} = \sum_{\mathbf{C} \in \mathbf{A} \omega \mathbf{B}} m_{\mathbf{C}}. \tag{14}$$

The subalgebra \mathcal{N} , like its commutative analog, is freely generated by certain monomial symmetric polynomials $\{m_{\mathbf{A}}\}_{\mathbf{A} \in \mathcal{A}}$, where \mathcal{A} is some carefully chosen collection of set partitions. This is the main theorem of Wolf [14]. See also [3, §7]. We use two such collections later, our choice depending on whether or not $n < \infty$.

The operation $(-)^{+k}$ has a left inverse called the **standardization** operator and denoted by “ $(-)^{\downarrow}$ ”. It maps set partitions \mathbf{A} of any cardinality- d subset $D \subseteq \mathbb{N}$ to set partitions of $[d]$, with \mathbf{A}^{\downarrow} defined as the pullback of \mathbf{A} along the unique increasing bijection from $[d]$ to D . For example, $(18.4)^{\downarrow} = 13.2$ and $(18.4.67)^{\downarrow} = 15.2.34$. The coproduct Δ and counit ε on \mathcal{N} are given, respectively, by

$$\Delta(m_{\mathbf{A}}) = \sum_{\mathbf{B} \cup \mathbf{C} = \mathbf{A}} m_{\mathbf{B}^{\downarrow}} \otimes m_{\mathbf{C}^{\downarrow}} \quad \text{and} \quad \varepsilon(m_{\mathbf{A}}) = \delta_{\mathbf{A}, \emptyset},$$

where $\mathbf{B} \cup \mathbf{C} = \mathbf{A}$ means that \mathbf{B} and \mathbf{C} form complementary subsets of \mathbf{A} . In the case $n = \infty$, the maps Δ and ε are algebra maps, making \mathcal{N} a graded connected Hopf algebra.

4 The place-action of \mathfrak{S} on \mathcal{N}

4.1 Swapping places in T_d and \mathcal{N}_d

On top of the permutation-action of the symmetric group \mathfrak{S}_x on T , we also consider the “place-action” of \mathfrak{S}_d on the degree d homogeneous component T_d . Observe that the permutation-action of $\sigma \in \mathfrak{S}_x$ on a monomial \mathbf{z} corresponds to the functional composition

$$\sigma \circ \mathbf{z} : [d] \xrightarrow{\mathbf{z}} \mathbf{x} \xrightarrow{\sigma} \mathbf{x}.$$

By contrast, the **place-action** of $\rho \in \mathfrak{S}_d$ on \mathbf{z} gives the monomial

$$\mathbf{z} \circ \rho : [d] \xrightarrow{\rho} [d] \xrightarrow{\mathbf{z}} \mathbf{x}$$

composing ρ with \mathbf{z} on the right. In the linear extension of this action to all of T_d , it is easily seen that \mathcal{N}_d (even each \mathcal{N}_μ) is an invariant subspace of T_d . Indeed, for any set partition $\mathbf{A} = \{A_1, \dots, A_r\} \vdash [d]$ and $\rho \in \mathfrak{S}_d$, one has (see [12, §2])

$$m_{\mathbf{A}} \cdot \rho = m_{\rho^{-1} \cdot \mathbf{A}}, \tag{15}$$

where as usual $\rho^{-1} \cdot \mathbf{A} := \{\rho^{-1}(A_1), \rho^{-1}(A_2), \dots, \rho^{-1}(A_r)\}$.

4.2 The place-action structure of \mathcal{N}

Notice that the action in (15) is transitive on set partitions and is shape-preserving. It follows that a basis for the place-action invariants in \mathcal{N}_d is indexed by partitions. For such a basis we choose the polynomials

$$\mathbf{m}_\mu := \frac{1}{(\dim \mathcal{N}_\mu) \mu!} \sum_{\lambda(\mathbf{A})=\mu} m_{\mathbf{A}}, \tag{16}$$

with $\mu! = a_1! a_2! \dots$ whenever $\mu = 1^{a_1} 2^{a_2} \dots$. The normalizing coefficient will be explained in (19).

To simplify our discussion of the structure of \mathcal{N} in this context, we will say that \mathfrak{S} acts on \mathcal{N} rather than being fastidious about underlying in each situation that individual \mathcal{N}_d 's are being acted upon on the right by the corresponding group \mathfrak{S}_d . We also denote the set $\mathcal{N}^\mathfrak{S}$ of **place-invariants** by Λ . To summarize,

$$\Lambda = \text{span}\{\mathbf{m}_\mu : \mu \text{ a partition of } d, d \in \mathbb{N}\}. \tag{17}$$

The pair (\mathcal{N}, Λ) begins to look like the pair $(S, S^\mathfrak{S})$ from the introduction. This was the observation that originally motivated our search for Theorem 1.

We next decompose \mathcal{N} into irreducible place-action representations. Although this can be worked out for any value of n , the results are more elegant when we send n to infinity. Recall that the **Frobenius characteristic** of a \mathfrak{S}_d -module \mathcal{V} is the symmetric function

$$\text{Frob}(\mathcal{V}) = \sum_{\mu \vdash d} v_\mu s_\mu,$$

where s_μ is a Schur function—the character of “the” irreducible \mathfrak{S}_d representation \mathcal{V}_μ indexed by μ —and v_μ is the multiplicity of \mathcal{V}_μ in \mathcal{V} . To reveal the \mathfrak{S}_d -module structure of \mathcal{N}_μ we may use (15) and standard techniques from the theory of combinatorial species, cf. [2]. The Frobenius characteristic of \mathcal{N}_μ is given by the following lemma.

Lemma 3 For a partition $\mu = 1^{a_1} 2^{a_2} \cdots k^{a_k}$, having a_i parts of size i , we have

$$\text{Frob}(\mathcal{N}_\mu) = h_{d_1}[h_1] h_{d_2}[h_2] \cdots h_{d_k}[h_k], \tag{18}$$

with $f[g]$ denoting plethysm of f and g , and h_i denoting the i^{th} homogeneous symmetric function.

Recall that the **plethysm** $f[g]$ of two symmetric functions is obtained by linear and multiplicative extension of the rule $p_k[p_\ell] := p_{k\ell}$, where the p_k 's denote the usual power sum symmetric functions (see [10, I.8] for notations and more details). For instance, one finds that $h_3[h_2] = s_6 + s_{42} + s_{222}$. That is, \mathcal{N}_{222} decomposes into 3 irreducible components, with the trivial representation s_6 coming from \mathfrak{m}_{222} inside Λ .

4.3 Λ meets $S^\mathfrak{S}$

We begin by explaining the choice of coefficient in (16). From [12, Thm. 2.1], one learns that the restriction to \mathcal{N} of the **abelianization** map $\mathbf{ab} : T \rightarrow S$ (the map making the variables commute) satisfies:

- (a) $\mathbf{ab}(\mathcal{N}) = S^\mathfrak{S}$, and
- (b) $\mathbf{ab}(m_{\mathbf{A}})$ is a multiple of $m_{\lambda(\mathbf{A})}$ depending only on $\mu = \lambda(\mathbf{A})$, more precisely

$$\mathbf{ab}(m_\mu) = m_\mu. \tag{19}$$

Formula (19) suggests that a natural right-inverse to $\mathbf{ab}(-)$ is given by

$$\iota : S^\mathfrak{S} \hookrightarrow \mathcal{N}, \quad \text{with} \quad \iota(m_\mu) := m_\mu. \tag{20}$$

The fact that the image of $S^\mathfrak{S}$ in \mathcal{N} is exactly the subspace Λ affords us a quick proof of Theorem 1 in the case $n = \infty$. The isomorphism we construct for $n < \infty$ still uses the map ι , but in a less essential way.

5 The coinvariant space of \mathcal{N} (Case: $n = \infty$)

5.1 Proof of main result

Suppose $n = \infty$. Combining results of [3] and a theorem of Blattner, Cohen, and Montgomery [6], we may immediately deduce the existence of a subspace \mathcal{C} of \mathcal{N} together with a vector space isomorphism $\mathcal{N} \simeq \mathcal{C} \otimes \Lambda$. Indeed, from Propositions 4.3 and 4.5 of [3], we get that the map ι is a **coalgebra splitting** of $\mathbf{ab} : \mathcal{N} \rightarrow S^\mathfrak{S} \rightarrow 0$, i.e.,

$$\mathbf{ab} \circ \iota = \text{id} \quad \text{and} \quad \Delta_{\mathcal{N}} \circ \iota = (\iota \otimes \iota) \circ \Delta_{S^\mathfrak{S}}.$$

Moreover \mathbf{ab} is a morphism of Hopf algebras. In this context, Theorem 4.14 of [6] suggests that we let \mathcal{C} be the **left Hopf kernel** of the Hopf map \mathbf{ab} ,

$$\mathcal{C} = \{h \in \mathcal{N} : (\text{id} \otimes \mathbf{ab}) \circ \Delta(h) = h \otimes 1\}.$$

This theorem gives an algebra isomorphism between \mathcal{N} and the *crossed product* $\mathcal{C} \#_\sigma S^\mathfrak{S}$. In fact, since $\Delta_{\mathcal{N}}$ is cocommutative, it is an isomorphism of Hopf algebras. We refer the interested reader to [6, §4] for the technical details. We mention only that: (i) the space \mathcal{C} is actually a Hopf subalgebra of \mathcal{N} by construction; (ii) the crossed product $\mathcal{C} \#_\sigma S^\mathfrak{S}$ is a certain algebra structure built on the tensor product $\mathcal{C} \otimes S^\mathfrak{S}$ using a cocycle $\sigma : S^\mathfrak{S} \times S^\mathfrak{S} \rightarrow \mathcal{C}$; and (iii) the isomorphism amounts to a cocycle twisting of simple multiplication: $\mathcal{C} \otimes S^\mathfrak{S} \mapsto \mathcal{C} \cdot \Lambda$. This completes the proof of Theorem 1. Moreover, since all spaces and morphisms are graded by degree, the Hilbert series for \mathcal{C} is the quotient of that for \mathcal{N} by that for Λ . This demonstrates (4).

5.2 Atomic set partitions.

Recall the result of Wolf that \mathcal{N} is a polynomial algebra, i.e., \mathcal{N} is freely generated by some collection of polynomials. We announce our first choice for this collection now, following the terminology of [4]. Let Π denote the set of all set partitions (of $[d]$, $\forall d \geq 0$). We introduce the **atomic set partitions** $\dot{\Pi}$. A set partition $\mathbf{A} = \{A_1, \dots, A_r\}$ of $[d]$ is atomic if there does not exist a pair (s, c) ($1 \leq s < r, 1 \leq c < d$) such that $\{A_1, \dots, A_s\}$ is a set partition of $[c]$. Conversely, \mathbf{A} is not atomic if there are set partitions \mathbf{B} of $[d']$ and \mathbf{C} of $[d'']$ splitting \mathbf{A} in two: $\mathbf{A} = \mathbf{B} \cup \mathbf{C}^{+d'}$. We write $\mathbf{A} = \mathbf{B}|\mathbf{C}$ in this situation. A **maximal splitting** $\mathbf{A} = \mathbf{A}'|\mathbf{A}''|\dots|\mathbf{A}^{(r)}$ of \mathbf{A} is one where each $\mathbf{A}^{(i)}$ is atomic. For example, the partition 17.235.4.68 is atomic, while 12.346.57.8 is not. The maximal splitting of the latter would be 12|124.35|1, but we abuse notation and write 12|346.57|8 to improve legibility.

It is proven in [4] that \mathcal{N} is freely generated by the atomic polynomials. To get a better sense of the structure, let us order Π by giving $\dot{\Pi}$ a total order “ \prec ” and then extending lexicographically. Given two atomic set partitions \mathbf{A} and \mathbf{B} , we demand that $\mathbf{A} \prec \mathbf{B}$ if $\mathbf{A} \vdash [c]$ and $\mathbf{B} \vdash [d]$ with $c < d$. In case \mathbf{A}, \mathbf{B} are partitions of the same set $[d]$, then any ordering will do for the current purpose... one interesting choice is to order \mathbf{A} and \mathbf{B} by ordering lexicographically their associated **rhyme scheme words**.⁽ⁱⁱ⁾ Our convention for writing set partitions provides a bijection between set partitions and this special class of words, sending $\mathbf{A} = \{A_1, A_2, \dots, A_r\} \in \Pi_d$ to $w(\mathbf{A}) = w_1 w_2 \dots w_d$ defined by $w_i := k$ if and only if $i \in A_k$. For example, $w(13.2) = 121$ and $w(17.235.4.68) = 12232414$. Using this ordering on $\dot{\Pi}$, we have the following chain within the set partitions of shape 3221:

$$1|23|45|678 \prec 13.2|456|78 \prec 13.24|578.6 \prec 14.23|578.6 \prec 17.235.4.68 \prec 17.236.4.58.$$

In fact, 1|23|45|678 is the unique minimal element of $\Pi_{(3221)}$.

Define the **leading term** of a sum $\sum_{\mathbf{C}} \alpha_{\mathbf{C}} m_{\mathbf{C}}$ to be the monomial $m_{\mathbf{C}_0}$ such that \mathbf{C}_0 is lexicographically least among all \mathbf{C} with $\alpha_{\mathbf{C}} \neq 0$. Combined with (14), our choice for \prec makes it clear that the leading term of $m_{\mathbf{A}} \cdot m_{\mathbf{B}}$ is $m_{\mathbf{A}|\mathbf{B}}$. That is, multiplication in \mathcal{N} is *shape-filtered*. Since the left Hopf kernel \mathcal{C} is a subalgebra, it is shape-filtered as well. Finally, the isomorphism $\mathcal{C} \otimes \Lambda \rightarrow \mathcal{N}$ respects the shape structures on either side. This completes the proof of Corollary 2.

It is proven in [8] that \mathcal{N} is not only freely generated by the *atomic polynomials* $\{m_{\mathbf{A}}|\mathbf{A} \in \dot{\Pi}\}$, but co-freely generated by them as well. By a classic theorem of Milnor and Moore [11], this means that \mathcal{N} is isomorphic to the universal enveloping algebra $\mathfrak{U}(\mathcal{L}(\dot{\Pi}))$ of the free Lie algebra $\mathcal{L}(\dot{\Pi})$ on the set $\dot{\Pi}$. This description will be useful in the next subsection. Let us finish this section with a few final remarks on atomic set partitions. First, note that set partitions with one part are trivially atomic. The set of these is denoted by $\dot{\Pi}_b$. They are analogs of the generators m_k for the algebra $S^{\mathfrak{S}}$. The remaining atomic set partitions

$$\dot{\Pi}_d := \left\{ \{A_1, \dots, A_r\} \in \dot{\Pi} : r > 1 \right\}$$

are more interesting. They index a large portion of the generators for \mathcal{C} . They are also the subject of an open question formulated at the end of Section 5.3.

⁽ⁱⁱ⁾ Quoting Bill Blewett from [13, A000110], “a rhyme scheme is a string of letters (eg, *abba*) such that the leftmost letter is always *a* and no letter may be greater than one more than the greatest letter to its left. Thus *aac* is not valid since *c* is more than one greater than *a*. For example, $[\#\Pi_3 = 5]$ because there are 5 rhyme schemes on 3 letters: *aaa, aab, aba, abb, abc*.”

5.3 Explicit description of the Hopf algebra structure of \mathcal{C}

It is not too hard to find elements in the left Hopf kernel of the abelianization map \mathbf{ab} . Consider the following simple calculation. The sum of monomials $\tilde{m}_{13.2} := m_{13.2} - m_{12.3}$ is primitive. Indeed,

$$\begin{aligned} \Delta(\tilde{m}_{13.2}) &= 1 \otimes m_{13.2} + m_{12} \otimes m_1 + m_1 \otimes m_{12} + m_{13.2} \otimes 1 \\ &\quad - 1 \otimes m_{12.3} - m_{12} \otimes m_1 - m_1 \otimes m_{12} - m_{12.3} \otimes 1 \\ &= 1 \otimes \tilde{m}_{13.2} + \tilde{m}_{13.2} \otimes 1. \end{aligned}$$

We conclude that $(\text{id} \otimes \mathbf{ab}) \circ \Delta(\tilde{m}_{13.2}) = \tilde{m}_{13.2} \otimes 1$. In other terms, $\tilde{m}_{13.2} \in \mathcal{C}$. The linear map Δ may be split as $\Delta = \Delta^p + \Delta^i$, the sum of its **primitive** and **imprimitive** parts respectively. What we have just done in the example is to find a modification $\tilde{m}_{13.2}$ of $m_{13.2}$ satisfying $\Delta^i(\tilde{m}_{13.2}) = 0$. This suggests the following proposition.

Proposition 4 *There is a primitive element*

$$\tilde{m}_{\mathbf{A}} = m_{\mathbf{A}} + \sum_{\mathbf{B} : \lambda(\mathbf{B}) = \lambda(\mathbf{A})} \alpha_{\mathbf{B}} m_{\mathbf{B}}$$

associated to each $\mathbf{A} \in \dot{\Pi}_{\sharp}$ such that $\sum_{\mathbf{B}} \alpha_{\mathbf{B}} = -1$ and $\mathbf{B} \in \dot{\Pi} \Rightarrow \alpha_{\mathbf{B}} = 0$.

The existence of primitives comes from the Milnor-Moore isomorphism of \mathcal{N} with $\mathfrak{U}(\mathfrak{L}(\dot{\Pi}))$. Showing that they can be chosen with the above properties is a simple calculation, inducting on the number of parts r of an atomic set partition $\mathbf{A} = \{A_1, \dots, A_r\}$ and applying $(\Delta^i)^r$.

The ideas behind the proposition and the preceding example yield several immediate corollaries: (i) each $\tilde{m}_{\mathbf{A}}$ from Proposition 4 belongs to \mathcal{C} ; (ii) \mathcal{C} is shape-graded, i.e., if $h \in \mathcal{C}$ is written as $\sum_{\mu} h_{\mu}$, then each h_{μ} belongs to \mathcal{C} as well; (iii) for any $g \in \mathcal{N}$ and $h \in \mathcal{C}$, we have that $[g, h] = gh - hg$ also belongs to \mathcal{C} ; (iv) if \mathbf{A} and \mathbf{B} belong to $\dot{\Pi}_{\flat}$, then $[m_{\mathbf{A}}, m_{\mathbf{B}}]$ belongs to \mathcal{C} . These points essentially account for all of \mathcal{C} , as the next result suggests. First, recall that $S^{\mathfrak{G}}$ is also a universal enveloping algebra of a Lie algebra. Namely, the abelian Lie algebra $\mathfrak{A}(\{m_1, m_2, \dots\})$, where all Lie brackets $[m_j, m_k]$ are zero. Since the integers $k = 1, 2, \dots$ are in 1-1 correspondence with $\dot{\Pi}_{\flat}$, we have a natural map from $\mathfrak{L}(\dot{\Pi})$ to $\mathfrak{A}(\{m_1, m_2, \dots\})$. Our final characterization of \mathcal{C} is as follows.

Corollary 5 *Let \mathcal{C} be the kernel of the map π from the free Lie algebra on $\dot{\Pi}$ to the free abelian Lie algebra on $\dot{\Pi}_{\flat}$. Then the coinvariant space \mathcal{C} is the universal enveloping algebra of the Lie algebra \mathcal{C} .*

Before turning to the case $n < \infty$, we remark that we have left unanswered the question of finding a systematic procedure (e.g., a closed formula in the spirit of Möbius inversion) that constructs a primitive element $\tilde{m}_{\mathbf{A}}$ for each $\mathbf{A} \in \dot{\Pi}_{\sharp}$.

6 The coinvariant space of \mathcal{N} (Case: $n < \infty$)

We repeat our example of Section 3.3 in the case $n = 3$. The leading term with respect to our previous order would be $m_{13.2.4.5}$, except that this term does not appear because 13.2.4.5 has more than $n = 3$ parts. Fortunately, the rhyme scheme bijection w reveals a more useful leading term:

$$m_{121} \cdot m_{12} = 0 + m_{12113} + m_{12131} + m_{12123} + m_{12132} + m_{12121} + m_{12112}.$$

The concatenation $121|12$ is the lexicographically smallest word appearing above. This is generally true: if $w(\mathbf{A}) = u$ and $w(\mathbf{B}) = v$, then uv is the smallest element of $w(\mathbf{A} \cup \mathbf{B})$. Let us call a rhyme scheme word a **verse** if it cannot be written as the concatenation of two shorter rhyme schemes. The **splitting** of a rhyme scheme w is the maximal deconcatenation $w = w'|w''|\dots|w^{(r)}$ of w into verses $w^{(i)}$. For example, 12314 is a verse while 11232411 is a string of four verses $1|12324|1|1$. It is easy to see that if a, b, c , and d are verses, then $a|c = b|d$ if and only if $a = b$ and $c = d$. The preceding observations make it clear that \mathcal{N} is *verse-filtered* and that \mathcal{N} is freely generated by the monomials $\{m_{w(\mathbf{A})} \mid w(\mathbf{A}) \text{ is a verse}\}$. This is the collection of monomials originally chosen by Wolf, cf. [3, §7] for details.

Toward locating \mathcal{C} within \mathcal{N} , we first locate $S^\mathfrak{S}$. Consider the partition $\mu = 32211$. Note that the lexicographically least rhyme scheme word of shape μ is $w(123.45.67.8.9) = 111223345$. We are led to introduce the words

$$w(\mu) := 1^{\mu_1} 2^{\mu_2} \dots k^{\mu_k}$$

associated to partitions $\mu = (\mu_1, \mu_2, \dots, \mu_k)$; we call these **descending rhymes** since $\mu_1 \geq \dots \geq \mu_k$. Finally, we want to view \mathcal{C} as the rhymes that don't involve a descending rhyme. Then, by the fact that \mathcal{N} is verse-filtered, we will get an easy vector space isomorphism $\mathcal{C} \otimes \Lambda \rightarrow \mathcal{N}$ given by multiplication. Toward that end, we introduce the notion of vexillary rhymes.

A **vexillary rhyme** is a word that begins with a maximal (but possibly empty) descending rhyme, followed by one extra verse. The **vexillary decomposition** of a rhyme scheme w is the expression of w as a product $w = w'|w''|\dots|w^{(r)}|w^{(r+1)}$, where $w', \dots, w^{(r)}$ are vexillary rhymes and $w^{(r+1)}$ is a possibly empty descending rhyme (which we call a **tail**). For a given word w , this decomposition is accomplished by first splitting w into verses, then recombining, from left to right, consecutive verses to form vexillary rhymes. For instance, the splitting of 112212 is $1|1222|12$. The first two factors combine to make one vexillary rhyme; the last factor is a descending tail: $112212 \mapsto \widehat{1|1222}|12$. Similarly,

$$1231231411122311 \mapsto 123|12314|1|1|1223|1|1 \mapsto \widehat{123|12314|1|1}|1223|1|1$$

Suppose now that u and v are rhyme schemes and that the vexillary decomposition of u is tail-free. Then by construction, the vexillary decomposition of uv is the concatenation of the respective vexillary decompositions of u and v . We are ready to identify \mathcal{C} as a subalgebra of \mathcal{N} .

Theorem 6 *Let \mathcal{C} be the subalgebra of \mathcal{N} generated by vexillary rhymes. Then \mathcal{C} has a basis indexed by rhyme scheme words w whose vexillary decompositions are tail-free. Moreover, the map $\mathcal{C} \otimes \Lambda \rightarrow \mathcal{N}$ given by $m_{w'}m_{w''}\dots m_{w^{(r)}} \otimes m_{(\mu_1\dots\mu_k)} \mapsto m_{w'|w''|\dots|w^{(r)}|w(\mu)}$ is a vector space isomorphism.*

7 Other directions

We conclude with another advertisement for the Blattner-Cohen-Montgomery theorem. The authors' present investigation into coinvariant spaces began by moving vertically within the commuting diagram (cube) of Hopf algebras depicted in Figure 1 (whereas in previous work, it was customary to move from left to right, cf. [1]). One may just as well move in other directions within the cube. To illustrate, we apply the Blattner-Cohen-Montgomery theorem to two other edges of interest (leaving aside any comments on group actions). The first of these concerns the downward arrow on the front-right side of the cube. Recall that, from a purely combinatorial perspective, bases in $\mathbb{K}\langle \mathbf{x} \rangle^{\sim \mathfrak{S}}$ are indexed by "set compositions" (ordered set partitions), and those in $\mathbb{K}[\mathbf{x}]^{\sim \mathfrak{S}}$ by integer compositions (here " \sim " indicates the quasi-action

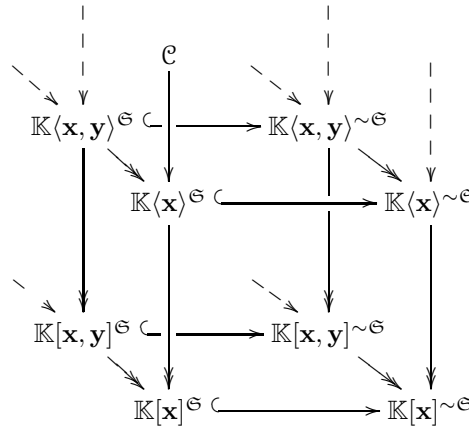


FIG. 1: The Hopf algebras of symmetric and quasisymmetric functions in one and two sets of commuting and noncommuting variables.

of Hivert, cf. [7, §3]). One may find a coalgebra splitting from $\mathbb{K}[\mathbf{x}]^{\sim \mathfrak{S}}$ to $\mathbb{K}\langle \mathbf{x} \rangle^{\sim \mathfrak{S}}$ and an associated coinvariant subalgebra in the spirit of our $(\mathcal{N}, S^{\mathfrak{S}})$ investigation.

Another direction is to consider the Hopf algebra morphism $\mathbf{sp} : \mathbb{K}[\mathbf{x}, \mathbf{y}]^{\sim \mathfrak{S}} \rightarrow \mathbb{K}[\mathbf{x}]^{\sim \mathfrak{S}}$ (the bottom-right arrow going from NW to SE in Figure 1). These are the **diagonally quasi-symmetric functions** and **quasi-symmetric functions** respectively. For details omitted below, we refer the reader to [1]. The space $\mathbb{K}[\mathbf{x}, \mathbf{y}]^{\mathfrak{S}}$ is defined as the \mathfrak{S} -invariants, inside $\mathbb{K}[\mathbf{x}, \mathbf{y}]$, under the diagonal embedding of \mathfrak{S} in $\mathfrak{S} \times \mathfrak{S}$. (The quasi-action of Hivert passes easily through this diagonal embedding.) A basis for $\mathbb{K}[\mathbf{x}, \mathbf{y}]^{\sim \mathfrak{S}}$ is given by the “monomial functions” $m_{\mathbf{a}, \mathbf{b}}$, indexed by “bicompositions”, i.e., elements (\mathbf{a}, \mathbf{b}) in $\mathbb{N}^{2 \times r}$ such that $a_i + b_i > 0$. These $m_{\mathbf{a}, \mathbf{b}}$ conveniently map to the quasi-symmetric function $m_{\mathbf{a} + \mathbf{b}}$ under the specialization map \mathbf{sp} sending y_i to x_i . It is straightforward to show that the map sending $m_{\mathbf{a}}$ to $m_{\mathbf{a}, \mathbf{0}}$, is a coalgebra splitting. We may thus analyze this situation in a manner analogous to our main result. Perhaps more surprising than the fact that the quotient

$$\text{Hilb}_t(\mathbb{K}[\mathbf{x}, \mathbf{y}]^{\sim \mathfrak{S}}) / \text{Hilb}_t(\mathbb{K}[\mathbf{x}]^{\sim \mathfrak{S}})$$

belongs to $\mathbb{N}[[t]]$ is the fact that the objects it counts have already been named. We discover a connection between compositions, set compositions, and “L-convex polyominoes.” See [13, A003480].

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