



HAL
open science

Invariant and coinvariant spaces for the algebra of symmetric polynomials in non-commuting variables

François Bergeron, Aaron Lauve

► **To cite this version:**

François Bergeron, Aaron Lauve. Invariant and coinvariant spaces for the algebra of symmetric polynomials in non-commuting variables. 20th Annual International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2008), 2008, Viña del Mar, Chile. pp.543-556, 10.46298/dmtcs.3606 . hal-01185140

HAL Id: hal-01185140

<https://inria.hal.science/hal-01185140v1>

Submitted on 19 Aug 2015

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Invariant and coinvariant spaces for the algebra of symmetric polynomials in non-commuting variables

François Bergeron and Aaron Lauve

LaCIM, Université du Québec à Montréal, CP 8888, Succ. Centre-ville, Montréal (Québec) H3C 3P8, CANADA

Abstract. We analyze the structure of the algebra $\mathbb{K}\langle \mathbf{x} \rangle^{\mathfrak{S}_n}$ of symmetric polynomials in non-commuting variables in so far as it relates to $\mathbb{K}[\mathbf{x}]^{\mathfrak{S}_n}$, its commutative counterpart. Using the “place-action” of the symmetric group, we are able to realize the latter as the invariant polynomials inside the former. We discover a tensor product decomposition of $\mathbb{K}\langle \mathbf{x} \rangle^{\mathfrak{S}_n}$ analogous to the classical theorems of Chevalley, Shephard-Todd on finite reflection groups. In the case $|\mathbf{x}| = \infty$, our techniques simplify to a form readily generalized to many other familiar pairs of combinatorial Hopf algebras.

Résumé. Nous analysons la structure de l’algèbre $\mathbb{K}\langle \mathbf{x} \rangle^{\mathfrak{S}_n}$ des polynômes symétriques en des variables non-commutatives pour obtenir des analogues des résultats classiques concernant la structure de l’anneau $\mathbb{K}[\mathbf{x}]^{\mathfrak{S}_n}$ des polynômes symétriques en des variables commutatives. Plus précisément, au moyen de “l’action par positions”, on réalise $\mathbb{K}[\mathbf{x}]^{\mathfrak{S}_n}$ comme sous-module de $\mathbb{K}\langle \mathbf{x} \rangle^{\mathfrak{S}_n}$. On découvre alors une nouvelle décomposition de $\mathbb{K}\langle \mathbf{x} \rangle^{\mathfrak{S}_n}$ comme produit tensoriel, obtenant ainsi un analogues des théorèmes classiques de Chevalley et Shephard-Todd. Dans le cas $|\mathbf{x}| = \infty$, nos techniques se simplifient en une forme aisément généralisables à beaucoup d’autres paires d’algèbres de Hopf familières.

Keywords: Chevalley theorem, symmetric group, noncommutative symmetric polynomials, set partitions

1 Introduction

One of the more striking results of the invariant theory of reflection groups is certainly the following: if W is a finite group of $n \times n$ matrices, then there is a graded W -module decomposition of the polynomial ring $S = \mathbb{K}[\mathbf{x}]$, in variables $\mathbf{x} = \{x_1, x_2, \dots, x_n\}$, as a tensor product⁽ⁱ⁾

$$S \simeq S_W \otimes S^W, \quad (1)$$

if and only if W is a group generated by (pseudo) reflections. As usual, S affords the natural W -module structure obtained by considering it as the symmetric space on the defining vector space X^* for W , e.g.,

⁽ⁱ⁾ We assume throughout that \mathbb{K} is a field containing \mathbb{Q} .

$w \cdot f(\mathbf{x}) = f(w \cdot \mathbf{x})$. It is customary to denote by S^W the ring of W -invariant polynomials for this action. To finish parsing (1), recall that S_W stands for the **coinvariant space**, i.e., the W -module defined as

$$S_W := S / \langle S_+^W \rangle, \tag{2}$$

the quotient of S by the ideal generated by constant-term free W -invariant polynomials. We give S , S^W , and S_W a grading by polynomial degree in \mathbf{x} (the latter being well-defined because $\langle S_+^W \rangle$ is a homogeneous ideal). The motivation behind the quotient in (2) is to eliminate redundant copies of irreducible W -modules inside S . Indeed, if \mathcal{V} is such a module and $f(\mathbf{x})$ is any W -invariant polynomial with no constant term, then $\mathcal{V}f(\mathbf{x})$ is an isomorphic copy of \mathcal{V} living within $\langle S_+^W \rangle$. As a result, the coinvariant space S_W is the interesting part of the story.

The context for the present paper is the algebra $T = \mathbb{K}\langle \mathbf{x} \rangle$ of noncommutative polynomials, with W -module structure on T obtained by considering it as the tensor space on the defining space X^* for W . In the special case when W is the symmetric group \mathfrak{S}_n , we elucidate a relationship between the space S^W and the subalgebra T^W of W -invariants in T . The subalgebra T^W was first studied in [14, 5] with the aim of obtaining noncommutative analogs of classical results concerning symmetric function theory. Recent work in [12, 3] has extended a large part of the story surrounding (1) to this noncommutative context. In particular, there is an explicit \mathfrak{S}_n -module decomposition of the form $T \simeq T_{\mathfrak{S}_n} \otimes T^{\mathfrak{S}_n}$, cf. [3, Theorem 8.7].

By contrast, our work proceeds in a somewhat complementary direction. We consider $\mathcal{N} = T^{\mathfrak{S}_n}$ as a tower of \mathfrak{S}_d -modules under the “place-action” and realize $S^{\mathfrak{S}_n}$ inside \mathcal{N} as a subspace Λ of invariants for this action. This leads to a decomposition of \mathcal{N} analogous to (1). More explicitly, our main result is as follows.

Theorem 1 *There is an explicitly constructed subspace \mathcal{C} of \mathcal{N} so that \mathcal{C} and the place-action invariants Λ exhibit a graded vector space isomorphism*

$$\mathcal{N} \simeq \mathcal{C} \otimes \Lambda. \tag{3}$$

As an immediate corollary we derive the Hilbert series formula

$$\text{Hilb}_t(\mathcal{C}) = \text{Hilb}_t(\mathcal{N}) \prod_{i=1}^n (1 - t^i). \tag{4}$$

Here, as usual, the **Hilbert series** of a graded space $\mathcal{V} = \bigoplus_{d \geq 0} \mathcal{V}_d$ is the formal power series defined as

$$\text{Hilb}_t(\mathcal{V}) = \sum_{d \geq 0} \dim \mathcal{V}_d t^d,$$

where \mathcal{V}_d is the **homogeneous degree d component** of \mathcal{V} . The fact that (4) expands as a series in $\mathbb{N}[t]$ is not at all obvious, as one may check that the Hilbert series of \mathcal{N} is

$$\text{Hilb}_t(\mathcal{N}) = 1 + \sum_{k=1}^n \frac{t^k}{(1-t)(1-2t) \cdots (1-kt)} \tag{5}$$

(taking $n = |\mathbf{x}|$). We underline that the harder part of our work lies in working out the case $n < \infty$. This is accomplished in Section 6. If we restrict ourselves to the case $n = \infty$, both \mathcal{N} and Λ become Hopf

algebras and things are much simpler. Our results are then consequences of a general theorem of Blattner, Cohen and Montgomery. As we will see in Section 5, stronger results hold in this simpler context. For example, (4) may be refined to a statement about “shape” enumeration.

2 The algebra $S^\mathfrak{S}$ of symmetric polynomials

2.1 Vector space structure of $S^\mathfrak{S}$

We specialize our introductory discussion to the group $W = \mathfrak{S}_n$ of permutation matrices. The action on $S = \mathbb{K}[\mathbf{x}]$ is simply the **permutation action** $\sigma \cdot x_i = x_{\sigma(i)}$ and $S^{\mathfrak{S}_n}$ comprises the usual symmetric polynomials. We suppress n in the notation and denote the subring of symmetric polynomials by $S^\mathfrak{S}$. (Note that upon sending n to ∞ , the elements of $S^\mathfrak{S}$ become formal series in $\mathbb{K}[[\mathbf{x}]]$ of bounded degree; we still call them polynomials to affect a uniform discussion.) A monomial in S of degree d may be written as follows: given an r -subset $\mathbf{y} = \{y_1, y_2, \dots, y_r\}$ of \mathbf{x} and a **composition** of d into r parts, $\mathbf{a} = (a_1, a_2, \dots, a_r)$ ($a_i > 0$), we write $\mathbf{y}^\mathbf{a}$ for $y_1^{a_1} y_2^{a_2} \dots y_r^{a_r}$. We assume that the variables y_i are naturally ordered, so that whenever $y_i = x_j$ and $y_{i+1} = x_k$ we have $j < k$. Reordering the entries of a composition \mathbf{a} in decreasing order results in a partition $\lambda(\mathbf{a})$ called the **shape** of \mathbf{a} . Summing over monomials $\mathbf{y}^\mathbf{a}$ with the same shape leads to the monomial symmetric polynomial

$$m_\mu = m_\mu(\mathbf{x}) := \sum_{\lambda(\mathbf{a})=\mu, \mathbf{y} \subseteq \mathbf{x}} \mathbf{y}^\mathbf{a}.$$

Letting $\mu = (\mu_1, \dots, \mu_r)$ run over all partitions of $d = |\mu| = \mu_1 + \dots + \mu_r$ gives a basis for $S_d^\mathfrak{S}$. As usual, we set $m_0 := 1$ and agree that $m_\mu = 0$ if μ has too many parts (i.e., $n < r$).

2.2 Dimension enumeration

A fundamental result in the invariant theory of \mathfrak{S}_n is that $S^\mathfrak{S}$ is generated by a family $\{f_k\}_{1 \leq k \leq n}$ of algebraically independent symmetric polynomials, having respective degrees $\deg f_k = k$. (One may choose $\{m_k\}_{1 \leq k \leq n}$ for such a family.) It follows immediately that the Hilbert series of $S^\mathfrak{S}$ is

$$\text{Hilb}_t(S^\mathfrak{S}) = \prod_{i=1}^n \frac{1}{1-t^i}. \tag{6}$$

Recalling that the Hilbert series of S is $(1-t)^{-n}$, we see from (1) and (6) that the Hilbert series for the coinvariant space $S_\mathfrak{S}$ is the well-known t -analog of $n!$:

$$\prod_{i=1}^n \frac{1-t^i}{1-t} = \prod_{i=1}^n (1+t+\dots+t^{i-1}). \tag{7}$$

In particular, contrary to the situation in (4), the series $\text{Hilb}_t(S)/\text{Hilb}_t(S^\mathfrak{S})$ in $\mathbb{Z}[[t]]$ is *obviously* positive.

2.3 Algebra and coalgebra structures of $S^\mathfrak{S}$

Given partitions μ and ν , there is an explicit formula for computing the product $m_\mu \cdot m_\nu$. In lieu of giving the formula, we refer the reader to [3, §4.1] and simply give an example:

$$m_{21} \cdot m_{11} = 3m_{2111} + 2m_{221} + 2m_{311} + m_{32}. \tag{8}$$

The extremal terms above are relevant to our coming discussion. Note that if $n < 4$, then the first term disappears. However, if n is sufficiently large then analogs of these terms always appear with positive integer coefficients for a given pair (μ, ν) . If $\mu = (\mu_1, \dots, \mu_r)$ and $\nu = (\nu_1, \dots, \nu_s)$ with $r \leq s$, then the partition indexing the left-most term is denoted by $\mu \cup \nu$ and is given by sorting the list $(\mu_1, \dots, \mu_r, \nu_1, \dots, \nu_s)$ in increasing order; the right-most term is indexed by $\mu + \nu := (\mu_1 + \nu_1, \dots, \mu_r + \nu_r, \nu_{r+1}, \dots, \nu_s)$. Taking $\mu = 31$ and $\nu = 221$, we would have $\mu \cup \nu = 32211$ and $\mu + \nu = 531$.

The ring $S^{\mathfrak{S}}$ is also afforded a coalgebra structure with coproduct $\Delta : S_d^{\mathfrak{S}} \rightarrow \bigoplus_{k=0}^d S_k^{\mathfrak{S}} \otimes S_{d-k}^{\mathfrak{S}}$ and counit $\varepsilon : S^{\mathfrak{S}} \rightarrow \mathbb{K}$ given, respectively, by

$$\Delta(m_\mu) = \sum_{\theta \cup \nu = \mu} m_\theta \otimes m_\nu \quad \text{and} \quad \varepsilon(m_\mu) = \delta_{\mu,0}.$$

In the case $n = \infty$, Δ and ε are algebra maps, making $S^{\mathfrak{S}}$ a connected graded (by degree) Hopf algebra.

3 The algebra \mathcal{N} of noncommutative symmetric polynomials

3.1 Vector space structure of \mathcal{N}

Suppose now that \mathbf{x} denotes a set of non-commuting variables. The algebra $T = \mathbb{K}\langle \mathbf{x} \rangle$ of noncommutative polynomials is graded by degree. A degree d **noncommutative monomial** $\mathbf{z} \in T_d$ is simply a length- d “word”:

$$\mathbf{z} = z_1 z_2 \cdots z_d, \quad \text{with each } z_i \in \mathbf{x}.$$

In other terms, \mathbf{z} is a function $\mathbf{z} : [d] \rightarrow \mathbf{x}$, with $[d]$ denoting the set $\{1, \dots, d\}$. The permutation-action on \mathbf{x} clearly extends to T , giving rise to the subspace $\mathcal{N} = T^{\mathfrak{S}}$ of noncommutative \mathfrak{S} -invariants. With the aim of describing a linear basis for the homogeneous component \mathcal{N}_d , we next introduce set partitions of $[d]$ and the type of a monomial $\mathbf{z} : [d] \rightarrow \mathbf{x}$. We write $\mathbf{A} \vdash [d]$ when $\mathbf{A} = \{A_1, \dots, A_r\}$ is a **set partition** of $[d]$, i.e., $A_1 \cup \dots \cup A_r = [d]$, with $A_i \neq \emptyset$ and $A_i \cap A_j = \emptyset$ whenever $i \neq j$. The **type** $\tau(\mathbf{z})$ of a degree d monomial $\mathbf{z} : [d] \rightarrow \mathbf{x}$ is the set partition

$$\tau(\mathbf{z}) := \{z^{-1}(x) \mid x \in \mathbf{x}\} \setminus \{\emptyset\} \quad \text{of } [d],$$

whose parts are the non-empty fibers of the function \mathbf{z} . For instance,

$$\tau(x_1 x_8 x_1 x_5 x_8) = \{\{1, 3\}, \{2, 5\}, \{4\}\}.$$

In the sequel, we lighten the heavy notation for set partitions, writing, e.g., $\{\{1, 3\}, \{2, 5\}, \{4\}\}$ as 13.25.4. Clearly the type of a monomial is a finite set partition with at most n parts. Note that we may always order the parts in increasing order of their minimum elements. The **shape** $\lambda(\mathbf{A})$ of a set partition $\mathbf{A} = \{A_1, \dots, A_r\}$ is the (integer) partition $\lambda(|A_1|, \dots, |A_r|)$ obtained by sorting the part sizes of \mathbf{A} in increasing order. Observing that the permutation-action is *type preserving*, we are led to consider the **monomial** linear basis for the space \mathcal{N}_d :

$$m_{\mathbf{A}} = m_{\mathbf{A}}(\mathbf{x}) := \sum_{\tau(\mathbf{z})=\mathbf{A}} \mathbf{z}$$

For example, with $n = 2$, we have $m_{\emptyset} = 1$, $m_1 = x_1 + x_2$, $m_{12} = x_1^2 + x_2^2$, $m_{1.2} = x_1 x_2 + x_2 x_1$, $m_{123} = x_1^3 + x_2^3$, $m_{12.3} = x_1^2 x_2 + x_2^2 x_1$, $m_{13.2} = x_1 x_2 x_1 + x_2 x_1 x_2$, $m_{1.23} = x_1 x_2^2 + x_2 x_1^2$, $m_{1.2.3} = 0, \dots$ (Note that we set $m_{\emptyset} := 1$, taking \emptyset as the unique set partition of the empty set, and we agree that $m_{\mathbf{A}} = 0$ if \mathbf{A} is a set partition with more than n parts.)

3.2 Dimension enumeration and shape grading

Above, we determined that $\dim \mathcal{N}_d$ is the number of set partitions of d into at most n parts. These are counted by the (length restricted) **Bell numbers** $B_d^{(n)}$. Then (5) follows from the fact that its right-hand side is the ordinary generating function for length restricted Bell numbers. See [9, §2]. We next highlight a finer enumeration, where we grade \mathcal{N} by shape rather than degree.

For each partition μ , we may consider the submodule \mathcal{N}_μ spanned by those $m_{\mathbf{A}}$ for which $\lambda(\mathbf{A}) = \mu$. This results in a direct sum decomposition $\mathcal{N}_d = \bigoplus_{\mu \vdash d} \mathcal{N}_\mu$. A simple dimension description for \mathcal{N}_d takes the form of a **shape Hilbert series** in the following manner. View commuting variables q_i as marking parts of size i and set $\mathbf{q}_\mu := q_{\mu_1} q_{\mu_2} \cdots q_{\mu_r}$. Then

$$\text{Hilb}_{\mathbf{q}}(\mathcal{N}_d) = \sum_{\mu \vdash d} \dim \mathcal{N}_\mu \mathbf{q}_\mu = \sum_{\mathbf{A} \vdash [d]} q_{\lambda(\mathbf{A})}. \tag{9}$$

Here, \mathbf{q}_μ is a marker for set partitions of shape $\lambda(\mathbf{A}) = \mu$ and the sum is over all partitions into at most n parts. Such a shape grading also makes sense for $S_d^{\mathfrak{S}}$. Summing over all $d \geq 0$ and all μ , we get

$$\text{Hilb}_{\mathbf{q}}(S^{\mathfrak{S}}) = \sum_{\mu} \mathbf{q}_\mu = \prod_{i \geq 1} \frac{1}{1 - q_i}. \tag{10}$$

Using classical combinatorial arguments (cf. Chapter 2.3 of [2], Example 13), we see that the enumerator polynomials $\text{Hilb}_{\mathbf{q}}(\mathcal{N}_d)$ are naturally collected in the **exponential generating function**

$$\sum_{d=0}^{\infty} \text{Hilb}_{\mathbf{q}}(\mathcal{N}_d) \frac{t^d}{d!} = \sum_{m=0}^n \frac{1}{m!} \left(\sum_{k=1}^{\infty} q_k \frac{t^k}{k!} \right)^m. \tag{11}$$

For example, with $n = 3$, we have

$$\text{Hilb}_{\mathbf{q}}(\mathcal{N}_6) = q_6 + 6 q_5 q_1 + 15 q_4 q_2 + 15 q_4 q_1^2 + 10 q_3^2 + 60 q_3 q_2 q_1 + 15 q_2^3,$$

thus $\dim \mathcal{N}_{222} = 15$ when $n \geq 3$. Evidently, the \mathbf{q} -polynomials $\text{Hilb}_{\mathbf{q}}(\mathcal{N}_d)$ specialize to the length restricted Bell numbers $B_d^{(n)}$ when we set all q_k equal to 1.

In view of (10), (11), and Theorem 1, we are led to claim the following refinement of (4).

Corollary 2 For $n = \infty$, the shape Hilbert series of the space \mathcal{C} is given by the expression

$$\text{Hilb}_{\mathbf{q}}(\mathcal{C}) = \sum_{d \geq 0} d! \exp \left(\sum_{k=1}^{\infty} q_k \frac{t^k}{k!} \right) \Big|_{t^d} \prod_{i \geq 1} (1 - q_i), \tag{12}$$

with $(-)|_{t^d}$ standing for the operation of taking the coefficient of t^d .

Thus we have the expansion

$$\begin{aligned} \text{Hilb}_{\mathbf{q}}(\mathcal{C}) = & 1 + 2 q_2 q_1 + (3 q_3 q_1 + 2 q_2^2 + 3 q_2 q_1^2) \\ & + (4 q_4 q_1 + 9 q_3 q_2 + 6 q_3 q_1^2 + 10 q_2^2 q_1 + 4 q_2 q_1^3) + \dots \end{aligned}$$

Corollary 2 will follow immediately from the explicit description of \mathcal{C} and the isomorphism $\mathcal{C} \otimes \Lambda \rightarrow \mathcal{N}$ in Section 5, which is not only degree preserving, but shape preserving as well.

3.3 Algebra and coalgebra structures of \mathcal{N}

Since the action of \mathfrak{S} on T is multiplicative, it is straightforward to see that \mathcal{N} is an subalgebra of T . The *multiplication rule* in \mathcal{N} , expressing a product $m_{\mathbf{A}} \cdot m_{\mathbf{B}}$ as a sum of basis vectors $\sum_{\mathbf{C}} m_{\mathbf{C}}$, is easy to describe. Since we make heavy use of the rule later, we develop it carefully here. We begin with an example (the digits corresponding to $\mathbf{B} = \mathbf{1.2}$ appear in bold):

$$m_{13.2} \cdot m_{\mathbf{1.2}} = m_{13.2.4.5} + m_{134.2.5} + m_{135.2.4} + m_{13.24.5} + m_{13.25.4} + m_{135.24} + m_{134.25} \tag{13}$$

Compare this to (8). Notice that the shapes indexing the first and last terms in (13) are the partitions $\lambda(13.2) \cup \lambda(1.2)$ and $\lambda(13.2) + \lambda(1.2)$. As was the case in $S^{\mathfrak{S}}$, one of these shapes, namely $\lambda(\mathbf{A}) + \lambda(\mathbf{B})$, will always appear in the product, while appearance of the shape $\lambda(\mathbf{A}) \cup \lambda(\mathbf{B})$ depends on the cardinality of \mathbf{x} .

Let us now describe the multiplication rule. Given any $D \subseteq \mathbb{N}$ and $k \in \mathbb{N}$, we write D^{+k} for the set

$$D^{+k} := \{a + k \mid a \in D\}.$$

By extension, for any set partition $\mathbf{A} = \{A_1, \dots, A_r\}$ we set $\mathbf{A}^{+k} := \{A_1^{+k}, A_2^{+k}, \dots, A_r^{+k}\}$. These definitions allow for the introduction of a bilinear (non-commutative) operation denoted by “ ω ” on formal linear combinations of set partitions. Given partitions $\mathbf{A} = \{A_1, A_2, \dots, A_r\}$ of $[c]$ and a partition $\mathbf{B} = \{B_1, B_2, \dots, B_s\}$ of $[d]$, the summands of $\mathbf{A} \omega \mathbf{B}$ are set partitions of $[c + d]$. The operation ω is recursively defined by the rules:

(a) $\mathbf{A} \omega \emptyset = \emptyset \omega \mathbf{A} = \mathbf{A}$, with \emptyset denoting the unique set partition of the empty set;

(b) $\mathbf{A} \omega \mathbf{B} = \{A_1\} \cup (\mathbf{A}' \omega \mathbf{B}^{+c}) + \sum_{i=1}^s \{A_1 \cup B_i^{+c}\} \cup (\mathbf{A}' \omega (\mathbf{B} \setminus \{B_i\})^{+c})$,

with union \cup extended bilinearly and \mathbf{A}' denoting $\{A_2, \dots, A_r\}$.

As shown in [3, Prop. 3.2], the multiplication rule for $m_{\mathbf{A}}$ and $m_{\mathbf{B}}$ in \mathcal{N} , is

$$m_{\mathbf{A}} \cdot m_{\mathbf{B}} = \sum_{\mathbf{C} \in \mathbf{A} \omega \mathbf{B}} m_{\mathbf{C}}. \tag{14}$$

The subalgebra \mathcal{N} , like its commutative analog, is freely generated by certain monomial symmetric polynomials $\{m_{\mathbf{A}}\}_{\mathbf{A} \in \mathcal{A}}$, where \mathcal{A} is some carefully chosen collection of set partitions. This is the main theorem of Wolf [14]. See also [3, §7]. We use two such collections later, our choice depending on whether or not $n < \infty$.

The operation $(-)^{+k}$ has a left inverse called the **standardization** operator and denoted by “ $(-)^{\downarrow}$ ”. It maps set partitions \mathbf{A} of any cardinality- d subset $D \subseteq \mathbb{N}$ to set partitions of $[d]$, with \mathbf{A}^{\downarrow} defined as the pullback of \mathbf{A} along the unique increasing bijection from $[d]$ to D . For example, $(18.4)^{\downarrow} = 13.2$ and $(18.4.67)^{\downarrow} = 15.2.34$. The coproduct Δ and counit ε on \mathcal{N} are given, respectively, by

$$\Delta(m_{\mathbf{A}}) = \sum_{\mathbf{B} \cup \mathbf{C} = \mathbf{A}} m_{\mathbf{B}^{\downarrow}} \otimes m_{\mathbf{C}^{\downarrow}} \quad \text{and} \quad \varepsilon(m_{\mathbf{A}}) = \delta_{\mathbf{A}, \emptyset},$$

where $\mathbf{B} \cup \mathbf{C} = \mathbf{A}$ means that \mathbf{B} and \mathbf{C} form complementary subsets of \mathbf{A} . In the case $n = \infty$, the maps Δ and ε are algebra maps, making \mathcal{N} a graded connected Hopf algebra.

4 The place-action of \mathfrak{S} on \mathcal{N}

4.1 Swapping places in T_d and \mathcal{N}_d

On top of the permutation-action of the symmetric group \mathfrak{S}_x on T , we also consider the “place-action” of \mathfrak{S}_d on the degree d homogeneous component T_d . Observe that the permutation-action of $\sigma \in \mathfrak{S}_x$ on a monomial \mathbf{z} corresponds to the functional composition

$$\sigma \circ \mathbf{z} : [d] \xrightarrow{\mathbf{z}} \mathbf{x} \xrightarrow{\sigma} \mathbf{x}.$$

By contrast, the **place-action** of $\rho \in \mathfrak{S}_d$ on \mathbf{z} gives the monomial

$$\mathbf{z} \circ \rho : [d] \xrightarrow{\rho} [d] \xrightarrow{\mathbf{z}} \mathbf{x}$$

composing ρ with \mathbf{z} on the right. In the linear extension of this action to all of T_d , it is easily seen that \mathcal{N}_d (even each \mathcal{N}_μ) is an invariant subspace of T_d . Indeed, for any set partition $\mathbf{A} = \{A_1, \dots, A_r\} \vdash [d]$ and $\rho \in \mathfrak{S}_d$, one has (see [12, §2])

$$m_{\mathbf{A}} \cdot \rho = m_{\rho^{-1} \cdot \mathbf{A}}, \tag{15}$$

where as usual $\rho^{-1} \cdot \mathbf{A} := \{\rho^{-1}(A_1), \rho^{-1}(A_2), \dots, \rho^{-1}(A_r)\}$.

4.2 The place-action structure of \mathcal{N}

Notice that the action in (15) is transitive on set partitions and is shape-preserving. It follows that a basis for the place-action invariants in \mathcal{N}_d is indexed by partitions. For such a basis we choose the polynomials

$$\mathbf{m}_\mu := \frac{1}{(\dim \mathcal{N}_\mu) \mu!} \sum_{\lambda(\mathbf{A})=\mu} m_{\mathbf{A}}, \tag{16}$$

with $\mu! = a_1! a_2! \dots$ whenever $\mu = 1^{a_1} 2^{a_2} \dots$. The normalizing coefficient will be explained in (19).

To simplify our discussion of the structure of \mathcal{N} in this context, we will say that \mathfrak{S} acts on \mathcal{N} rather than being fastidious about underlying in each situation that individual \mathcal{N}_d 's are being acted upon on the right by the corresponding group \mathfrak{S}_d . We also denote the set $\mathcal{N}^\mathfrak{S}$ of **place-invariants** by Λ . To summarize,

$$\Lambda = \text{span}\{\mathbf{m}_\mu : \mu \text{ a partition of } d, d \in \mathbb{N}\}. \tag{17}$$

The pair (\mathcal{N}, Λ) begins to look like the pair $(S, S^\mathfrak{S})$ from the introduction. This was the observation that originally motivated our search for Theorem 1.

We next decompose \mathcal{N} into irreducible place-action representations. Although this can be worked out for any value of n , the results are more elegant when we send n to infinity. Recall that the **Frobenius characteristic** of a \mathfrak{S}_d -module \mathcal{V} is the symmetric function

$$\text{Frob}(\mathcal{V}) = \sum_{\mu \vdash d} v_\mu s_\mu,$$

where s_μ is a Schur function—the character of “the” irreducible \mathfrak{S}_d representation \mathcal{V}_μ indexed by μ —and v_μ is the multiplicity of \mathcal{V}_μ in \mathcal{V} . To reveal the \mathfrak{S}_d -module structure of \mathcal{N}_μ we may use (15) and standard techniques from the theory of combinatorial species, cf. [2]. The Frobenius characteristic of \mathcal{N}_μ is given by the following lemma.

Lemma 3 For a partition $\mu = 1^{a_1} 2^{a_2} \cdots k^{a_k}$, having a_i parts of size i , we have

$$\text{Frob}(\mathcal{N}_\mu) = h_{d_1}[h_1] h_{d_2}[h_2] \cdots h_{d_k}[h_k], \tag{18}$$

with $f[g]$ denoting plethysm of f and g , and h_i denoting the i^{th} homogeneous symmetric function.

Recall that the **plethysm** $f[g]$ of two symmetric functions is obtained by linear and multiplicative extension of the rule $p_k[p_\ell] := p_{k\ell}$, where the p_k 's denote the usual power sum symmetric functions (see [10, I.8] for notations and more details). For instance, one finds that $h_3[h_2] = s_6 + s_{42} + s_{222}$. That is, \mathcal{N}_{222} decomposes into 3 irreducible components, with the trivial representation s_6 coming from \mathfrak{m}_{222} inside Λ .

4.3 Λ meets $S^\mathfrak{S}$

We begin by explaining the choice of coefficient in (16). From [12, Thm. 2.1], one learns that the restriction to \mathcal{N} of the **abelianization** map $\mathbf{ab} : T \rightarrow S$ (the map making the variables commute) satisfies:

- (a) $\mathbf{ab}(\mathcal{N}) = S^\mathfrak{S}$, and
- (b) $\mathbf{ab}(m_{\mathbf{A}})$ is a multiple of $m_{\lambda(\mathbf{A})}$ depending only on $\mu = \lambda(\mathbf{A})$, more precisely

$$\mathbf{ab}(m_\mu) = m_\mu. \tag{19}$$

Formula (19) suggests that a natural right-inverse to $\mathbf{ab}(-)$ is given by

$$\iota : S^\mathfrak{S} \hookrightarrow \mathcal{N}, \quad \text{with} \quad \iota(m_\mu) := m_\mu. \tag{20}$$

The fact that the image of $S^\mathfrak{S}$ in \mathcal{N} is exactly the subspace Λ affords us a quick proof of Theorem 1 in the case $n = \infty$. The isomorphism we construct for $n < \infty$ still uses the map ι , but in a less essential way.

5 The coinvariant space of \mathcal{N} (Case: $n = \infty$)

5.1 Proof of main result

Suppose $n = \infty$. Combining results of [3] and a theorem of Blattner, Cohen, and Montgomery [6], we may immediately deduce the existence of a subspace \mathcal{C} of \mathcal{N} together with a vector space isomorphism $\mathcal{N} \simeq \mathcal{C} \otimes \Lambda$. Indeed, from Propositions 4.3 and 4.5 of [3], we get that the map ι is a **coalgebra splitting** of $\mathbf{ab} : \mathcal{N} \rightarrow S^\mathfrak{S} \rightarrow 0$, i.e.,

$$\mathbf{ab} \circ \iota = \text{id} \quad \text{and} \quad \Delta_{\mathcal{N}} \circ \iota = (\iota \otimes \iota) \circ \Delta_{S^\mathfrak{S}}.$$

Moreover \mathbf{ab} is a morphism of Hopf algebras. In this context, Theorem 4.14 of [6] suggests that we let \mathcal{C} be the **left Hopf kernel** of the Hopf map \mathbf{ab} ,

$$\mathcal{C} = \{h \in \mathcal{N} : (\text{id} \otimes \mathbf{ab}) \circ \Delta(h) = h \otimes 1\}.$$

This theorem gives an algebra isomorphism between \mathcal{N} and the *crossed product* $\mathcal{C} \#_\sigma S^\mathfrak{S}$. In fact, since $\Delta_{\mathcal{N}}$ is cocommutative, it is an isomorphism of Hopf algebras. We refer the interested reader to [6, §4] for the technical details. We mention only that: (i) the space \mathcal{C} is actually a Hopf subalgebra of \mathcal{N} by construction; (ii) the crossed product $\mathcal{C} \#_\sigma S^\mathfrak{S}$ is a certain algebra structure built on the tensor product $\mathcal{C} \otimes S^\mathfrak{S}$ using a cocycle $\sigma : S^\mathfrak{S} \times S^\mathfrak{S} \rightarrow \mathcal{C}$; and (iii) the isomorphism amounts to a cocycle twisting of simple multiplication: $\mathcal{C} \otimes S^\mathfrak{S} \mapsto \mathcal{C} \cdot \Lambda$. This completes the proof of Theorem 1. Moreover, since all spaces and morphisms are graded by degree, the Hilbert series for \mathcal{C} is the quotient of that for \mathcal{N} by that for Λ . This demonstrates (4).

5.2 Atomic set partitions.

Recall the result of Wolf that \mathcal{N} is a polynomial algebra, i.e., \mathcal{N} is freely generated by some collection of polynomials. We announce our first choice for this collection now, following the terminology of [4]. Let Π denote the set of all set partitions (of $[d]$, $\forall d \geq 0$). We introduce the **atomic set partitions** $\dot{\Pi}$. A set partition $\mathbf{A} = \{A_1, \dots, A_r\}$ of $[d]$ is atomic if there does not exist a pair (s, c) ($1 \leq s < r, 1 \leq c < d$) such that $\{A_1, \dots, A_s\}$ is a set partition of $[c]$. Conversely, \mathbf{A} is not atomic if there are set partitions \mathbf{B} of $[d']$ and \mathbf{C} of $[d'']$ splitting \mathbf{A} in two: $\mathbf{A} = \mathbf{B} \cup \mathbf{C}^{+d'}$. We write $\mathbf{A} = \mathbf{B}|\mathbf{C}$ in this situation. A **maximal splitting** $\mathbf{A} = \mathbf{A}'|\mathbf{A}''|\dots|\mathbf{A}^{(r)}$ of \mathbf{A} is one where each $\mathbf{A}^{(i)}$ is atomic. For example, the partition 17.235.4.68 is atomic, while 12.346.57.8 is not. The maximal splitting of the latter would be 12|124.35|1, but we abuse notation and write 12|346.57|8 to improve legibility.

It is proven in [4] that \mathcal{N} is freely generated by the atomic polynomials. To get a better sense of the structure, let us order Π by giving $\dot{\Pi}$ a total order “ \prec ” and then extending lexicographically. Given two atomic set partitions \mathbf{A} and \mathbf{B} , we demand that $\mathbf{A} \prec \mathbf{B}$ if $\mathbf{A} \vdash [c]$ and $\mathbf{B} \vdash [d]$ with $c < d$. In case \mathbf{A}, \mathbf{B} are partitions of the same set $[d]$, then any ordering will do for the current purpose... one interesting choice is to order \mathbf{A} and \mathbf{B} by ordering lexicographically their associated **rhyme scheme words**.⁽ⁱⁱ⁾ Our convention for writing set partitions provides a bijection between set partitions and this special class of words, sending $\mathbf{A} = \{A_1, A_2, \dots, A_r\} \in \Pi_d$ to $w(\mathbf{A}) = w_1 w_2 \dots w_d$ defined by $w_i := k$ if and only if $i \in A_k$. For example, $w(13.2) = 121$ and $w(17.235.4.68) = 12232414$. Using this ordering on $\dot{\Pi}$, we have the following chain within the set partitions of shape 3221:

$$1|23|45|678 \prec 13.2|456|78 \prec 13.24|578.6 \prec 14.23|578.6 \prec 17.235.4.68 \prec 17.236.4.58.$$

In fact, 1|23|45|678 is the unique minimal element of $\Pi_{(3221)}$.

Define the **leading term** of a sum $\sum_{\mathbf{C}} \alpha_{\mathbf{C}} m_{\mathbf{C}}$ to be the monomial $m_{\mathbf{C}_0}$ such that \mathbf{C}_0 is lexicographically least among all \mathbf{C} with $\alpha_{\mathbf{C}} \neq 0$. Combined with (14), our choice for \prec makes it clear that the leading term of $m_{\mathbf{A}} \cdot m_{\mathbf{B}}$ is $m_{\mathbf{A}|\mathbf{B}}$. That is, multiplication in \mathcal{N} is *shape-filtered*. Since the left Hopf kernel \mathcal{C} is a subalgebra, it is shape-filtered as well. Finally, the isomorphism $\mathcal{C} \otimes \Lambda \rightarrow \mathcal{N}$ respects the shape structures on either side. This completes the proof of Corollary 2.

It is proven in [8] that \mathcal{N} is not only freely generated by the *atomic polynomials* $\{m_{\mathbf{A}}|\mathbf{A} \in \dot{\Pi}\}$, but co-freely generated by them as well. By a classic theorem of Milnor and Moore [11], this means that \mathcal{N} is isomorphic to the universal enveloping algebra $\mathfrak{U}(\mathcal{L}(\dot{\Pi}))$ of the free Lie algebra $\mathcal{L}(\dot{\Pi})$ on the set $\dot{\Pi}$. This description will be useful in the next subsection. Let us finish this section with a few final remarks on atomic set partitions. First, note that set partitions with one part are trivially atomic. The set of these is denoted by $\dot{\Pi}_b$. They are analogs of the generators m_k for the algebra $S^{\mathfrak{S}}$. The remaining atomic set partitions

$$\dot{\Pi}_d := \left\{ \{A_1, \dots, A_r\} \in \dot{\Pi} : r > 1 \right\}$$

are more interesting. They index a large portion of the generators for \mathcal{C} . They are also the subject of an open question formulated at the end of Section 5.3.

⁽ⁱⁱ⁾ Quoting Bill Blewett from [13, A000110], “a rhyme scheme is a string of letters (eg, *abba*) such that the leftmost letter is always *a* and no letter may be greater than one more than the greatest letter to its left. Thus *aac* is not valid since *c* is more than one greater than *a*. For example, $[\#\Pi_3 = 5]$ because there are 5 rhyme schemes on 3 letters: *aaa, aab, aba, abb, abc*.”

5.3 Explicit description of the Hopf algebra structure of \mathcal{C}

It is not too hard to find elements in the left Hopf kernel of the abelianization map \mathbf{ab} . Consider the following simple calculation. The sum of monomials $\tilde{m}_{13.2} := m_{13.2} - m_{12.3}$ is primitive. Indeed,

$$\begin{aligned} \Delta(\tilde{m}_{13.2}) &= 1 \otimes m_{13.2} + m_{12} \otimes m_1 + m_1 \otimes m_{12} + m_{13.2} \otimes 1 \\ &\quad - 1 \otimes m_{12.3} - m_{12} \otimes m_1 - m_1 \otimes m_{12} - m_{12.3} \otimes 1 \\ &= 1 \otimes \tilde{m}_{13.2} + \tilde{m}_{13.2} \otimes 1. \end{aligned}$$

We conclude that $(\text{id} \otimes \mathbf{ab}) \circ \Delta(\tilde{m}_{13.2}) = \tilde{m}_{13.2} \otimes 1$. In other terms, $\tilde{m}_{13.2} \in \mathcal{C}$. The linear map Δ may be split as $\Delta = \Delta^p + \Delta^i$, the sum of its **primitive** and **imprimitive** parts respectively. What we have just done in the example is to find a modification $\tilde{m}_{13.2}$ of $m_{13.2}$ satisfying $\Delta^i(\tilde{m}_{13.2}) = 0$. This suggests the following proposition.

Proposition 4 *There is a primitive element*

$$\tilde{m}_{\mathbf{A}} = m_{\mathbf{A}} + \sum_{\mathbf{B} : \lambda(\mathbf{B}) = \lambda(\mathbf{A})} \alpha_{\mathbf{B}} m_{\mathbf{B}}$$

associated to each $\mathbf{A} \in \dot{\Pi}_{\sharp}$ such that $\sum_{\mathbf{B}} \alpha_{\mathbf{B}} = -1$ and $\mathbf{B} \in \dot{\Pi} \Rightarrow \alpha_{\mathbf{B}} = 0$.

The existence of primitives comes from the Milnor-Moore isomorphism of \mathcal{N} with $\mathfrak{U}(\mathfrak{L}(\dot{\Pi}))$. Showing that they can be chosen with the above properties is a simple calculation, inducting on the number of parts r of an atomic set partition $\mathbf{A} = \{A_1, \dots, A_r\}$ and applying $(\Delta^i)^r$.

The ideas behind the proposition and the preceding example yield several immediate corollaries: (i) each $\tilde{m}_{\mathbf{A}}$ from Proposition 4 belongs to \mathcal{C} ; (ii) \mathcal{C} is shape-graded, i.e., if $h \in \mathcal{C}$ is written as $\sum_{\mu} h_{\mu}$, then each h_{μ} belongs to \mathcal{C} as well; (iii) for any $g \in \mathcal{N}$ and $h \in \mathcal{C}$, we have that $[g, h] = gh - hg$ also belongs to \mathcal{C} ; (iv) if \mathbf{A} and \mathbf{B} belong to $\dot{\Pi}_{\flat}$, then $[m_{\mathbf{A}}, m_{\mathbf{B}}]$ belongs to \mathcal{C} . These points essentially account for all of \mathcal{C} , as the next result suggests. First, recall that $S^{\mathfrak{G}}$ is also a universal enveloping algebra of a Lie algebra. Namely, the abelian Lie algebra $\mathfrak{A}(\{m_1, m_2, \dots\})$, where all Lie brackets $[m_j, m_k]$ are zero. Since the integers $k = 1, 2, \dots$ are in 1-1 correspondence with $\dot{\Pi}_{\flat}$, we have a natural map from $\mathfrak{L}(\dot{\Pi})$ to $\mathfrak{A}(\{m_1, m_2, \dots\})$. Our final characterization of \mathcal{C} is as follows.

Corollary 5 *Let \mathcal{C} be the kernel of the map π from the free Lie algebra on $\dot{\Pi}$ to the free abelian Lie algebra on $\dot{\Pi}_{\flat}$. Then the coinvariant space \mathcal{C} is the universal enveloping algebra of the Lie algebra \mathcal{C} .*

Before turning to the case $n < \infty$, we remark that we have left unanswered the question of finding a systematic procedure (e.g., a closed formula in the spirit of Möbius inversion) that constructs a primitive element $\tilde{m}_{\mathbf{A}}$ for each $\mathbf{A} \in \dot{\Pi}_{\sharp}$.

6 The coinvariant space of \mathcal{N} (Case: $n < \infty$)

We repeat our example of Section 3.3 in the case $n = 3$. The leading term with respect to our previous order would be $m_{13.2.4.5}$, except that this term does not appear because 13.2.4.5 has more than $n = 3$ parts. Fortunately, the rhyme scheme bijection w reveals a more useful leading term:

$$m_{121} \cdot m_{12} = 0 + m_{12113} + m_{12131} + m_{12123} + m_{12132} + m_{12121} + m_{12112}.$$

The concatenation $121|12$ is the lexicographically smallest word appearing above. This is generally true: if $w(\mathbf{A}) = u$ and $w(\mathbf{B}) = v$, then uv is the smallest element of $w(\mathbf{A} \cup \mathbf{B})$. Let us call a rhyme scheme word a **verse** if it cannot be written as the concatenation of two shorter rhyme schemes. The **splitting** of a rhyme scheme w is the maximal deconcatenation $w = w'|w''|\dots|w^{(r)}$ of w into verses $w^{(i)}$. For example, 12314 is a verse while 11232411 is a string of four verses $1|12324|1|1$. It is easy to see that if a, b, c , and d are verses, then $a|c = b|d$ if and only if $a = b$ and $c = d$. The preceding observations make it clear that \mathcal{N} is *verse-filtered* and that \mathcal{N} is freely generated by the monomials $\{m_{w(\mathbf{A})} \mid w(\mathbf{A}) \text{ is a verse}\}$. This is the collection of monomials originally chosen by Wolf, cf. [3, §7] for details.

Toward locating \mathcal{C} within \mathcal{N} , we first locate $S^{\mathfrak{S}}$. Consider the partition $\mu = 32211$. Note that the lexicographically least rhyme scheme word of shape μ is $w(123.45.67.8.9) = 111223345$. We are led to introduce the words

$$w(\mu) := 1^{\mu_1} 2^{\mu_2} \dots k^{\mu_k}$$

associated to partitions $\mu = (\mu_1, \mu_2, \dots, \mu_k)$; we call these **descending rhymes** since $\mu_1 \geq \dots \geq \mu_k$. Finally, we want to view \mathcal{C} as the rhymes that don't involve a descending rhyme. Then, by the fact that \mathcal{N} is verse-filtered, we will get an easy vector space isomorphism $\mathcal{C} \otimes \Lambda \rightarrow \mathcal{N}$ given by multiplication. Toward that end, we introduce the notion of vexillary rhymes.

A **vexillary rhyme** is a word that begins with a maximal (but possibly empty) descending rhyme, followed by one extra verse. The **vexillary decomposition** of a rhyme scheme w is the expression of w as a product $w = w'|w''|\dots|w^{(r)}|w^{(r+1)}$, where $w', \dots, w^{(r)}$ are vexillary rhymes and $w^{(r+1)}$ is a possibly empty descending rhyme (which we call a **tail**). For a given word w , this decomposition is accomplished by first splitting w into verses, then recombining, from left to right, consecutive verses to form vexillary rhymes. For instance, the splitting of 1122212 is $1|1222|12$. The first two factors combine to make one vexillary rhyme; the last factor is a descending tail: $1122212 \mapsto \widehat{1|1222}|12$. Similarly,

$$1231231411122311 \mapsto 123|12314|1|1|1223|1|1 \mapsto \widehat{123|12314}|1|1|1223|1|1$$

Suppose now that u and v are rhyme schemes and that the vexillary decomposition of u is tail-free. Then by construction, the vexillary decomposition of uv is the concatenation of the respective vexillary decompositions of u and v . We are ready to identify \mathcal{C} as a subalgebra of \mathcal{N} .

Theorem 6 *Let \mathcal{C} be the subalgebra of \mathcal{N} generated by vexillary rhymes. Then \mathcal{C} has a basis indexed by rhyme scheme words w whose vexillary decompositions are tail-free. Moreover, the map $\mathcal{C} \otimes \Lambda \rightarrow \mathcal{N}$ given by $m_{w'}m_{w''}\dots m_{w^{(r)}} \otimes m_{(\mu_1\dots\mu_k)} \mapsto m_{w'|w''|\dots|w^{(r)}|w(\mu)}$ is a vector space isomorphism.*

7 Other directions

We conclude with another advertisement for the Blattner-Cohen-Montgomery theorem. The authors' present investigation into coinvariant spaces began by moving vertically within the commuting diagram (cube) of Hopf algebras depicted in Figure 1 (whereas in previous work, it was customary to move from left to right, cf. [1]). One may just as well move in other directions within the cube. To illustrate, we apply the Blattner-Cohen-Montgomery theorem to two other edges of interest (leaving aside any comments on group actions). The first of these concerns the downward arrow on the front-right side of the cube. Recall that, from a purely combinatorial perspective, bases in $\mathbb{K}\langle \mathbf{x} \rangle^{\sim \mathfrak{S}}$ are indexed by "set compositions" (ordered set partitions), and those in $\mathbb{K}[\mathbf{x}]^{\sim \mathfrak{S}}$ by integer compositions (here " \sim " indicates the quasi-action

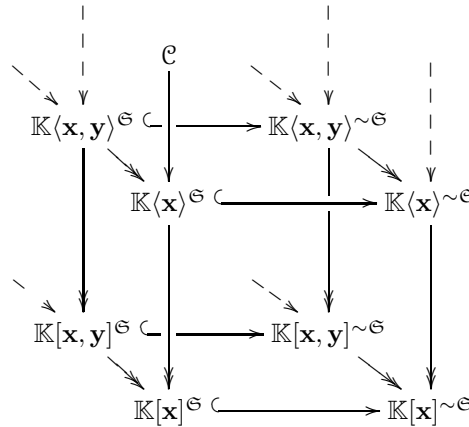


FIG. 1: The Hopf algebras of symmetric and quasisymmetric functions in one and two sets of commuting and noncommuting variables.

of Hivert, cf. [7, §3]). One may find a coalgebra splitting from $\mathbb{K}[x]^{~\mathfrak{S}}$ to $\mathbb{K}\langle x \rangle^{~\mathfrak{S}}$ and an associated coinvariant subalgebra in the spirit of our $(\mathcal{N}, S^{\mathfrak{S}})$ investigation.

Another direction is to consider the Hopf algebra morphism $\mathbf{sp} : \mathbb{K}[x, y]^{~\mathfrak{S}} \rightarrow \mathbb{K}[x]^{~\mathfrak{S}}$ (the bottom-right arrow going from NW to SE in Figure 1). These are the **diagonally quasi-symmetric functions** and **quasi-symmetric functions** respectively. For details omitted below, we refer the reader to [1]. The space $\mathbb{K}[x, y]^{\mathfrak{S}}$ is defined as the \mathfrak{S} -invariants, inside $\mathbb{K}[x, y]$, under the diagonal embedding of \mathfrak{S} in $\mathfrak{S} \times \mathfrak{S}$. (The quasi-action of Hivert passes easily through this diagonal embedding.) A basis for $\mathbb{K}[x, y]^{~\mathfrak{S}}$ is given by the “monomial functions” $m_{\mathbf{a}, \mathbf{b}}$, indexed by “bicompositions”, i.e., elements (\mathbf{a}, \mathbf{b}) in $\mathbb{N}^{2 \times r}$ such that $a_i + b_i > 0$. These $m_{\mathbf{a}, \mathbf{b}}$ conveniently map to the quasi-symmetric function $m_{\mathbf{a} + \mathbf{b}}$ under the specialization map \mathbf{sp} sending y_i to x_i . It is straightforward to show that the map sending $m_{\mathbf{a}}$ to $m_{\mathbf{a}, \mathbf{0}}$, is a coalgebra splitting. We may thus analyze this situation in a manner analogous to our main result. Perhaps more surprising than the fact that the quotient

$$\text{Hilb}_t(\mathbb{K}[x, y]^{~\mathfrak{S}}) / \text{Hilb}_t(\mathbb{K}[x]^{~\mathfrak{S}})$$

belongs to $\mathbb{N}[[t]]$ is the fact that the objects it counts have already been named. We discover a connection between compositions, set compositions, and “L-convex polyominoes.” See [13, A003480].

References

- [1] J.-C. Aval, F. Bergeron, and N. Bergeron. Diagonal Temperley-Lieb invariants and harmonics. *Séminaire Lotharingien de Combinatoire*, 54A (2005/07), Art. B54Aq, 19 pp. (electronic).
- [2] F. Bergeron, G. Labelle, P. Leroux. *Combinatorial Species and Tree-like Structures*. Cambridge University Press, 1998.
- [3] N. Bergeron, C. Reutenauer, M. Rosas, and M. Zabrocki. Invariants and coinvariants of the symmetric group in noncommuting variables. preprint, arXiv: math.CO/0502082.

- [4] N. Bergeron and M. Zabrocki. The Hopf algebras of symmetric functions and quasisymmetric functions in non-commutative variables are free and cofree. preprint, arXiv: math.CO/0509265.
- [5] G. M. Bergman and P. M. Cohn. Symmetric elements in free powers of rings. *J. London Math. Soc.* (2), 1:525–534, 1969.
- [6] R. J. Blattner, M. Cohen, and S. Montgomery. Crossed products and inner actions of Hopf algebras. *Trans. Amer. Math. Soc.*, 298(2):671–711, 1986.
- [7] F. Hivert. Hecke algebras, difference operators, and quasi-symmetric functions. *Adv. Math.*, 155(2):181–238, 2000.
- [8] F. Hivert, J.-C. Novelli, and J.-Y. Thibon. Commutative Hopf algebras of permutations and trees. preprint, arXiv: math.CO/0502456.
- [9] M. Klazar, Bell numbers, their relatives, and algebraic differential equations, *J. Combin. Theory Ser. A*, 102(1):63–87, 2003.
- [10] I. G. Macdonald, *Symmetric functions and Hall polynomials*, second ed., *Oxford Mathematical Monographs*. The Clarendon Press Oxford University Press, New York, 1995, With contributions by A. Zelevinsky, Oxford Science Publications.
- [11] J. W. Milnor and J. C. Moore. On the structure of Hopf algebras. *Ann. of Math.* (2), 81:211–264, 1965.
- [12] M. H. Rosas and B. E. Sagan. Symmetric functions in noncommuting variables. *Trans. Amer. Math. Soc.*, 358(1):215–232 (electronic), 2006.
- [13] N. J. A. Sloane. The On-Line Encyclopedia of Integer Sequences (2007). published electronically at www.research.att.com/njas/sequences/.
- [14] M. C. Wolf. Symmetric functions of non-commutative elements. *Duke Math. J.*, 2(4):626–637, 1936.

