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# The Sorting Order on a Coxeter Group

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**Abstract.** Let  $(W, S)$  be an arbitrary Coxeter system. For each sequence  $\omega = (\omega_1, \omega_2, \dots) \in S^*$  in the generators we define a partial order—called the  $\omega$ -sorting order—on the set of group elements  $W_\omega \subseteq W$  that occur as finite subwords of  $\omega$ . We show that the  $\omega$ -sorting order is a supersolvable join-distributive lattice and that it is strictly between the weak and strong Bruhat orders on the group. Moreover, the  $\omega$ -sorting order is a “maximal lattice” in the sense that the addition of any collection of edges from the Bruhat order results in a nonlattice.

Along the way we define a class of structures called supersolvable antimatroids and we show that these are equivalent to the class of supersolvable join-distributive lattices.

**Keywords:** Coxeter group, join-distributive lattice, supersolvable lattice, antimatroid, convex geometry

## Extended Abstract

Let  $(W, S)$  be an arbitrary Coxeter system and let  $\omega = (\omega_1, \omega_2, \dots) \in S^*$  be an arbitrary sequence in the generators, called the **sorting sequence**. We will identify a finite subword  $\alpha \subseteq \omega$  with the pair  $(\alpha, I(\alpha))$ , where  $I(\alpha) \subseteq I(\omega) = \{1, 2, \dots\}$  is the **index set** encoding the positions of the letters. Given a word  $\alpha = (\alpha_1, \dots, \alpha_k) \in S^*$ , let

$$\langle \alpha \rangle = \alpha_1 \cdots \alpha_k \in W$$

denote the corresponding group element. The subsets of the ground set  $I(\omega)$  are ordered lexicographically: if  $A$  and  $B$  are subsets of  $I(\omega)$  we say that  $A \leq_{\text{lex}} B$  if the minimum element of  $(A \cup B) \setminus (A \cap B)$  is contained in  $A$ .

**Definition 1** We say that a finite subword  $\alpha \subseteq \omega$  of the sorting sequence is  $\omega$ -sorted if

1.  $\alpha$  is a reduced word,
2.  $I(\alpha) = \min_{\leq_{\text{lex}}} \{I(\beta) \subseteq I(\omega) : \langle \beta \rangle = \langle \alpha \rangle\}$ .

That is,  $\alpha$  is  $\omega$ -sorted if it is the lexicographically-least reduced word for  $\langle \alpha \rangle$  among subwords of  $\omega$ .

Let  $W_\omega \subseteq W$  denote the set of group elements that occur as subwords of the sorting sequence. Then  $\omega$  induces a canonical reduced word for each element of  $W_\omega$ —its  $\omega$ -sorted word. This, in turn, induces a partial order on the set  $W_\omega$  by subword containment of sorted words.

**Definition 2** Given group elements  $u, w \in W_\omega$ , we write  $u \leq_\omega w$  if the index set of  $\omega$ -sort( $u$ ) is contained in the index set of  $\omega$ -sort( $w$ ). This is called the  $\omega$ -sorting order on  $W_\omega$ .

The sorting orders are closely related to other important orders on the group.

**Theorem 1** *Let  $\leq_R$  denote the right weak order and let  $\leq_B$  denote the Bruhat order on  $W$ . For all  $u, w \in W_\omega$  we have*

$$u \leq_R w \Rightarrow u \leq_\omega w \Rightarrow u \leq_B w.$$

For example, let  $W = \mathfrak{S}_4$  be the symmetric group of permutations of  $\{1, 2, 3, 4\}$  with the generating set of adjacent transpositions

$$S = \{s_1 = (12), s_2 = (23), s_3 = (34)\}.$$

Figure 1 displays the Hasse diagrams of the weak order,  $(s_1, s_2, s_3, s_2, s_1, s_2)$ -sorting order and strong order on the symmetric group  $\mathfrak{S}_4$ —the weak order is indicated by the shaded edges; solid edges indicate the sorting order; solid and broken edges together give the Bruhat order.

It turns out that the collection of  $\omega$ -sorted words has a remarkable structure. Given a ground set  $E$  and a collection of finite subsets  $\mathcal{F} \subseteq 2^E$ , the pair  $(E, \mathcal{F})$  is called a set system. A set system  $(E, \mathcal{F})$  is called an **antimatroid** (see (11)) if it satisfies

- For all nonempty  $A \in \mathcal{F}$  there exists  $x \in A$  such that  $A \setminus \{x\} \in \mathcal{F}$ ,
- For all  $A, B \in \mathcal{F}$  with  $B \not\subseteq A$  there exists  $x \in B \setminus A$  such that  $A \cup \{x\} \in \mathcal{F}$ .

Equivalently,  $\mathcal{F}$  is the collection of open sets for a closure operator  $\tau : 2^E \rightarrow 2^E$  that satisfies the anti-exchange property:

- If  $x, y \notin \tau(A)$  then  $x \in \tau(A \cup \{y\})$  implies  $x \notin \tau(A \cup \{x\})$ .

Such an operator  $\tau$  models the notion of “convex hull”, and so it is called a **convex closure**. Furthermore, we say that a lattice  $L$  is **join-distributive** if it satisfies:

- For each  $x \in L$ , the interval  $[x, y]$ , where  $y$  is the join of elements that cover  $x$ , is a boolean algebra.

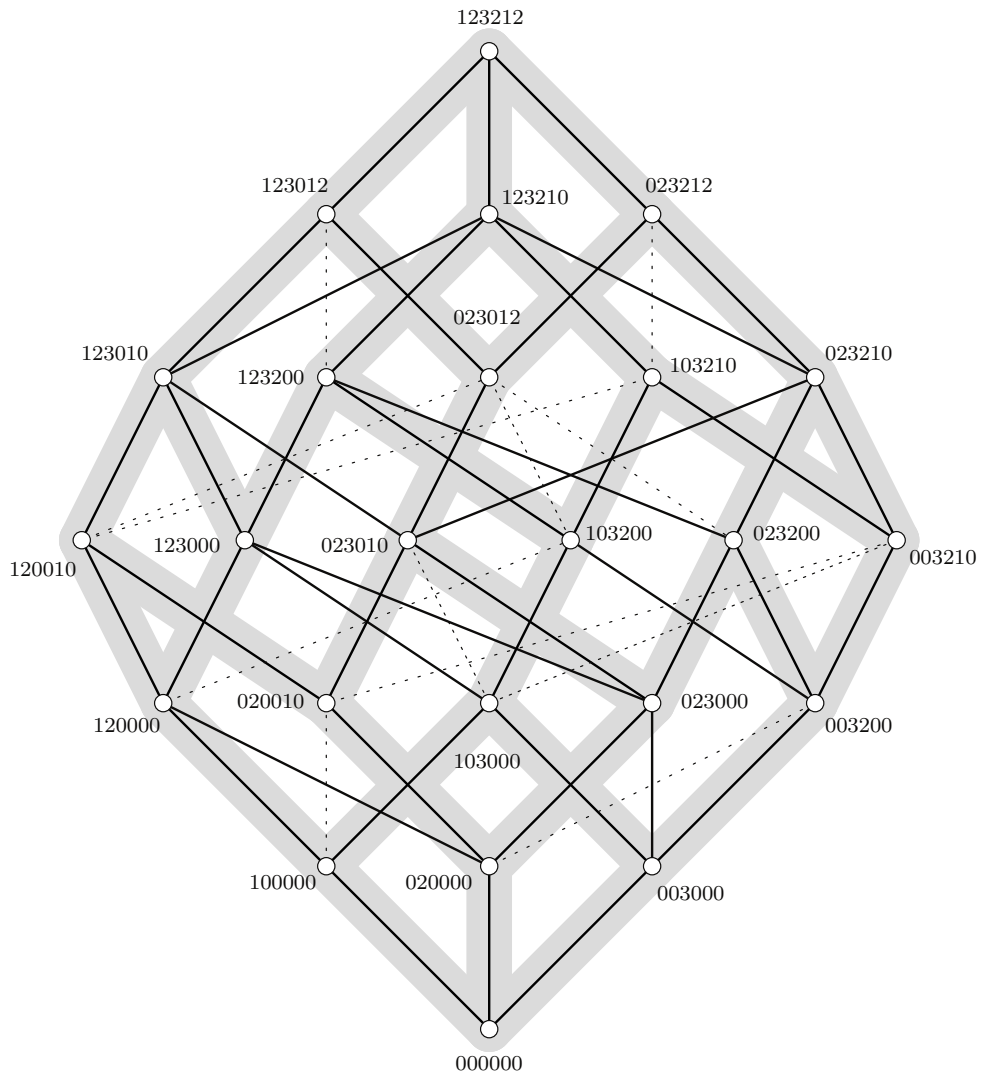
Edelman (5) proved that a finite lattice is join-distributive if and only if it arises as the lattice of open sets of a convex closure. We will generalize Edelman’s characterization to the case of supersolvable join-distributive lattices.

**Definition 3** *Consider a set system  $(E, \mathcal{F})$  on a totally ordered ground set  $(E, \leq_E)$ . We say that  $(E, \mathcal{F})$  is a **supersolvable antimatroid** if it satisfies:*

- $\emptyset \in \mathcal{F}$ .
- For all  $A, B \in \mathcal{F}$  with  $B \not\subseteq A$  and  $x = \min_{\leq_E} B \setminus A$  we have  $A \cup \{x\} \in \mathcal{F}$ .

**Theorem 2** *A (possibly infinite) lattice  $P$  is join-distributive and every interval in  $P$  is supersolvable if and only if  $P$  arises as the lattice of feasible sets of a supersolvable antimatroid.*

Our main result is the following.



**Fig. 1:** The weak order, 123212-sorting order and strong order on  $S_4$

**Theorem 3** *Let  $(W, S)$  be an arbitrary Coxeter system and consider an arbitrary sequence  $\omega \in S^*$ . The collection of index sets of  $\omega$ -sorted subwords*

$$\mathcal{F} = \{I(\alpha) \subseteq I(\omega) : \alpha \text{ is } \omega\text{-sorted}\}$$

*is a supersolvable antimatroid with respect to the natural order on the ground set  $E = I(\omega)$ .*

**Corollary 1** *The  $\omega$ -sorting order is a join-distributive lattice in which every interval is supersolvable and it is graded by the usual Coxeter length function  $\ell : W \rightarrow \mathbb{Z}$ .*

Note that this holds even when infinitely many group elements occur as subwords of the sorting sequence. This is remarkable because the weak order on an infinite group is *not* a lattice. Indeed, we do not know of any other natural source of lattice structures on the elements of an infinite Coxeter group.

We also have

**Corollary 2** *There exists a reduced sequence  $\omega' \subseteq \omega$  (that is, every prefix of  $\omega'$  is a reduced word) such that the  $\omega'$ -sorting order coincides with the  $\omega$ -sorting order.*

That is, we may assume that the sorting sequence is reduced. Finally, we have

**Lemma 1** *If  $\omega$  and  $\zeta$  are sequences that differ by the exchange of adjacent commuting generators, then the  $\omega$ -sorting order coincides with the  $\zeta$ -sorting order.*

In summary, for each commutation class of reduced sequences we obtain a supersolvable join-distributive lattice that is strictly between the weak and Bruhat orders. This is particularly interesting in the case that  $\omega$  represents a commutation class of reduced words for the longest element  $w_\circ$  in a finite Coxeter group.

We end by noting an important special case. Let  $(W, S)$  be a Coxeter system with generators  $S = \{s_1, \dots, s_n\}$ . Any word of the form  $(s_{\sigma(1)}, \dots, s_{\sigma(n)})$ —where  $\sigma \in \mathfrak{S}_n$  is a permutation—is called a **Coxeter word**, and the corresponding element  $\langle c \rangle \in W$  is a **Coxeter element**. We say that a **cyclic sequence** is any sequence of the form

$$c^\infty := ccc\dots$$

The case of  $c^\infty$ -sorted words was first considered by Reading (see (13; 14)), and this is the main motivation behind our work. However, Reading did not consider the structure of the collection of sorted words nor did he consider the sorting order.

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