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Yu. Baryshnikov

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# On expected number of maximal points in polytopes

Yu. Baryshnikov<sup>1†</sup>

<sup>1</sup>Mathematical Sciences, Bell Laboratories,  
Murray Hill, NJ, USA

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We answer an old question: what are possible growth rates of the expected number of vector-maximal points in a uniform sample from a polytope.

**Keywords:** -

## 1 introduction

### 1.1 setup

We consider the following setup

- Fix a closed convex polyhedral cone  $K$  with nonempty interior in  $d$ -dimensional Euclidean space  $V = \mathbb{R}^d$ ;
- The cone  $K$  defines a partial ordering on  $V$ : for  $x, y \in V$ ,

$$x >_K y \quad \text{iff} \quad x - y \in K$$

(we say that  $x$  *dominates*  $y$  or that  $y$  is *dominated* by  $x$ .)

- a point  $x \in X$  is said to be  $K$ -*maximal*, or simply *maximal* if there are no further point  $x' \in X$  dominating  $x$ .

(We will be using notation

$$\max_K(X).$$

for the set of maximal points in a set  $X \subset V$ .)

- Assume further that a convex (compact) polyhedron  $P$  is given, and that

<sup>†</sup>Supported in part by ARO

- $X$  is a uniform size  $n$  sample from  $P$ .

The following question arises in various contexts:

**Question 1** Given  $P$  and  $K$ , find the expected size

$$M_n = \mathbb{E}|max_K(X)|$$

of  $K$ -maximal elements in  $X$ , as a function of the sample size  $n = |X|$ .

I will not try to survey here all situations where computing  $M_n$  can be useful, just mention some keywords – *multicriterial optimization, geometric algorithms, convex hulls* – and refer the reader to [6, 8, 4, 5, ?]

### 1.1.1 Convention

We will use  $f \approx g$  as a synonym for

$$\frac{f}{g} \rightarrow c, 0 < c < \infty.$$

## 1.2 what was known so far

The Question 1 was addressed by many authors having different applications in mind; consequently, they arrived at partial answers. Two of the possible setting studied most occupy in some sense opposite corners of the space of all problems:

- If  $P$  is the unit square, and  $K$  is the positive quarter plane (in  $d = 2$ ), the number of maximal point is the same as the number of records in an *iid* sample, i.e. *harmonic number*

$$H_n = \sum_{i=1}^n \frac{1}{i}.$$

More generally, in higher dimensions, if  $P$  is the unit cube, and  $K$  is the positive orthant (“Pareto cone”), the problem still is essentially combinatorial, and the expected number of maximal elements is the *incomplete polyzeta*: thus, in dimension  $d$ , the expected number of maximal elements is

$$M_n = \frac{1}{n} \sum_{1 \leq i_1 \leq \dots \leq i_{d-1} \leq n} \frac{1}{i_1 i_2 \dots i_{d-1}}.$$

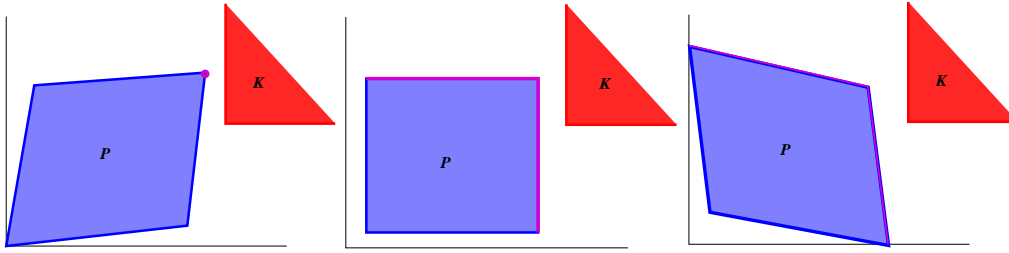
(this result was first established, it seems, in [2], and reproduced by many authors).

- If the polyhedron  $P$  is *in general position* with respect to the cone  $K$  (meaning: all flats spanned by facets of  $K$  and  $P$  intersect transversally), then

$$M_n \approx n^{f/d}$$

where  $f$  is the dimension of  $max_K(P)$  [3].

The situations described above are in some sense the most degenerate and most generic ones, respectively. In the former case, all faces of  $P$  are parallel to some faces of  $K$ . Generically, a small perturbation would lead to a  $K, P$  being in general position. Intermediate situations are relevant, however: for example, if  $P$  is given as a set of solution of a system of linear inequalities  $P = \{Ax \leq b\}$ , the sparsity of the matrix  $A$  would lead to a problem intermediate between the those above.



**Figure 1:** Generic deformations of the polyhedron  $P$ : in a smooth 1-dimensional family, typical representative has polynomial growth ( $M_n \approx n^0$  on the left;  $M_n \approx n^{1/2}$  on the right), and has logarithmic terms for exceptional values of parameter ( $M_n \approx \log n$  for the middle display).

### 1.3 gap problem

It has been noticed [8] that there is a certain *gap* in the possible asymptotic behaviors of  $M_n$  as a function of  $n$ , at least in dimension 2: if the expected number grows faster than  $\log(n)$ , then it is asymptotically  $\Omega(\sqrt{n})$ . This, clearly, motivates the problem:

**Question 2** *What are the possible asymptotics of  $M_n$ , at least for convex polyhedral  $K, P$  (and uniform sample from  $P$ )?*

In other words, what else, beyond observed so far behaviors  $M_n \approx n^{f/d}$ ;  $M_n \approx \log^{d-1}(n)$  can happen within our polyhedral setup?

In this work we answer both Questions, 1 and 2.

#### 1.3.1 acknowledgment

I was told about the gap problem by Mordecai Golin during AofA'06; many thanks!

## 2 main result

Consider the set of points

$$\Delta = \{(m, c) \in \mathbb{Z}^2, 0 \leq c \leq m \leq d\}.$$

We will call a pair  $(r, \mu)$ ,  $r > 0$ ,  $\mu \in \mathbb{N}$  *admissible*, if the intersection of the ray

$$c = rm, m \geq 0$$

with the set  $\Delta$  contains at least at least  $\mu$  points (beyond the origin). In particular, in an admissible pair  $(r, \mu)$ ,  $r$  is rational.

**Theorem 1** *The expected number of maximal elements  $M_n = \mathbb{E}(\max_K(X))$  with respect to a convex closed polyhedral cone  $K$ , where  $X$  is the uniform sample from polyhedral  $P \subset V$  of size  $n$ , satisfies*

$$M_n \approx n^{1-r} \log^{\mu-1}(n), \tag{1}$$

for some admissible pair  $(r, \mu)$ . For any admissible  $(r, \mu)$ , there exists a pair  $(P, K)$  satisfying (1).

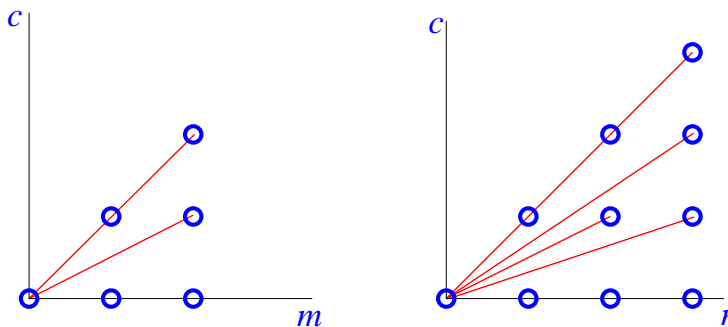


Figure 2: Asymptotics of  $M_n$  in dimensions 2,3.

### 2.1 examples

1. For  $d = 2$ , the Figure 2.1 shows all possible alternatives: the point  $(m, c) = (1, 1)$  corresponds to  $M_n \approx n^0$  (compare Fig. 1.2, left); the point  $(2, 2)$  correspond to  $M_n \approx \log n$  (Fig. 1.2, middle) and the point  $(2, 1)$  corresponds to  $M_n \approx n^{1/2}$ . In particular, one can see the “gap”.
2. For  $d = 3$ , asymptotic growth rates for  $M_n$  are
  - $M_n \approx n^{1-r}, r = 1/3, 1/2, 2/3$ ;
  - $M_n \approx \log^\nu(n), \nu = 0, 1, 2$ .
3. For illustrative purposes, some examples in dimension 8: The three segments illustrate the following

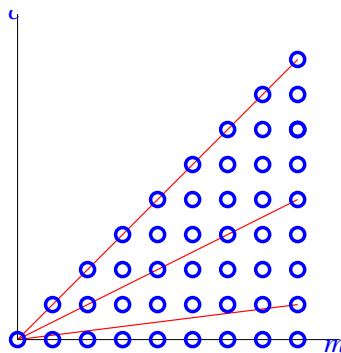


Figure 3: Asymptotics of  $M_n$  in dimension 8.

asymptotics (bottom-to-top):

- $M_n \approx n^{7/8}$ ;
- $M_n \approx n^{3/4} \log^3(n)$ ;

- $\log^7(n)$ .

**Remark:** In generic situation (when all faces of  $P$  are transversal to all faces of  $K$ ), all multiplicities  $\mu$  are equal to 0.

### 2.2 specific polytopes and cones

The general result of the previous section depends, of course, on the results describing the asymptotics of  $M_n$  for specific instances of  $(P, K)$ . To describe those, we will need to define *flags* and their *spectra*.

Consider the (ranked) *face poset* of the polytope  $P$ : its elements are faces of  $P$  of all dimensions ordered by inclusion (thus the maximal face is of dimension  $d$ , the polytope  $P$  itself, then the facets of  $P$  of dimension  $d - 1$  and so on. We will denote the set of faces of dimension  $l$  as  $\mathcal{P}_l$ .

Flags are the chains of *adjoining* proper faces of  $P$ :

$$\mathcal{F} = (f_1, f_2, \dots, f_l) : f_1 \subset f_2 \subset \dots \subset f_l, f_i \in \mathcal{P}_{d_i}; d_l < d.$$

A flag is *full* if it has length  $d$ , i.e. it includes facets of all dimensions between 0 and  $d - 1$ .

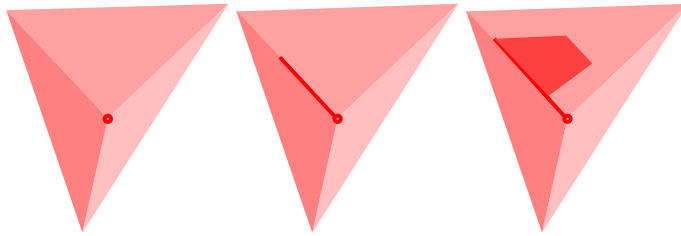


Figure 4: A flag.

We associate to each face  $f$  of  $P$  its *deficiency*  $\delta(f)$ : it is just the dimension of the intersection

$$(K + x) \cap P$$

for a (relative) interior point  $x \in f$ . One can easily check that the deficiency is well-defined, i.e. does not depend on particular point in the relative interior of a face.

Deficiencies in a flag do not decrease:

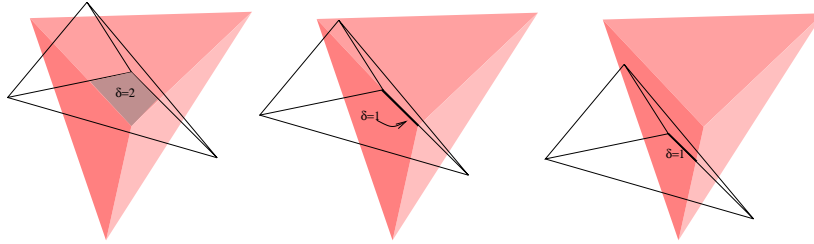
$$\delta(f_1) \leq \delta(f_2), f_1 \subset f_2.$$

**Definition:** The spectrum of a flag  $F = (f_1 \subset f_2 \subset \dots \subset f_l)$  is the multiset

$$\sigma(\sigma) = \left\{ \frac{c(f_1)}{m(f_1)}, \frac{c(f_2)}{m(f_2)}, \dots, \frac{c(f_l)}{m(f_l)} \right\}$$

(where elements are counted with multiplicities), where

$$c(f_i) = \text{codim}(f_i) = d - \dim(f_i)$$



**Figure 5:** Deficiencies of some of the facets for polytope on Figure 2.2; the deficiencies of the facets in the flag on Figure 2.2 are equal to  $(0, 1, 2)$ , and its spectrum is  $\{1^3\}$  (1 with multiplicity 3).

is the codimension of a face, and

$$m(f_i) = d - \delta(f_i)$$

is its *codeficiency*.

**Definition:** For a flag  $\mathcal{F}$  define its leading exponent to be

$$r(\mathcal{F}) = \min\{r : r \in \sigma(\mathcal{F})\},$$

the smallest number in the flag spectrum. The multiplicity  $\mu(\mathcal{F})$  is the multiplicity of  $r(\mathcal{F})$  in  $\sigma(\mathcal{F})$ .

The spectra of flags are crucial for us due to our second main result,

**Theorem 2** *The asymptotics of  $M_n$ , as  $n \rightarrow \infty$  is given by*

$$M_n \approx n^{1-r_*} \log^{\mu-1}(n), \tag{2}$$

where  $r_* = \min_{\mathcal{F}} r(\mathcal{F})$  is the smallest of the leading exponents of flags, and  $\mu$  is the largest multiplicity of  $r_*$  among all the flags.

**Remark:** It is immediate that extending a flag only increases its spectra, and thus can lead only to faster growing terms in (2). Hence to analyze the leading asymptotics of  $M_n$  it is enough to consider only full flags.

**Remark:** In fact, the entire asymptotic expansion of  $M_n$  is governed by the spectra of flags. Essentially, for each pair  $(c(\mathcal{F}), m(\mathcal{F}))$  such that  $r = c/m$  occurs in the flag spectrum with multiplicity  $\mu$  contributes to the asymptotic expansion of  $M_n$  an asymptotic series

$$\sum_{0 \leq l; 0 \leq \nu < \mu} a_{l,\nu} n^{1-(c+l)/m} \log^\nu(n).$$

We will not go into details, postponing a detailed exposition of this (and all the remaining) topics to a separate paper.

### 3 techniques

The techniques are a melange of Fubini theorem, an elementary version of *resolution of singularities* and some fairly standard results from the theory of generalized functions. We will not even attempt to present any details of the proofs (which, while not complicated, would require quite a bit of supporting machinery), but rather will sketch the major steps and outline main constructions.

### 3.1 Fubini

We will denote by  $\lambda$  the Lebesgue measure on  $V$ , and by  $|P| = \lambda(P)$  the volume of the polytope  $P$ .

For a point  $x \in P$ , denote by

$$u(x) = |P|^{-1} \lambda(K + x \cap P)$$

the probability that a random point from  $P$  dominates  $x$ .

**Lemma 1** *The function  $u$  is piece-wise polynomial in  $V$ : there exists a polyhedral subdivision of  $P$  (spline subdivision) such that the restriction of  $u$  to each of its polyhedra is polynomial.*

We will use the following formula which is a more or less straightforward corollary of Fubini and a formula for  $M_n$  implied by conditioning on the positions of a point in  $X$  and finding the probability that this point is maximal:

**Lemma 2**

$$M_n = n \int (1-t)^{n-1} dB(t), \quad (3)$$

where

$$B(t) = |P|^{-1} \lambda\{x \in P : u(x) \leq t\}, \quad (4)$$

is the probability that  $M$  evaluated at a random point in  $P$  is  $\leq t$ .

The advantage of using (3) is the decoupling of geometry: we can concentrate now on the properties of the function  $B(t)$ . Indeed, Karamata-type Tauberian theorems would translate the asymptotics of  $B$  near zero into the asymptotics of  $M_n, n \rightarrow \infty$ .

To analyze  $B$  near 0, we need to analyze  $u$  near the points where  $u$  vanishes. This can be done locally.

### 3.2 resolution of singularities

The domains where  $u$  is polynomial can adjoin the faces of  $P$  in rather intricate fashion. Hence a resolution of  $P$  would be helpful. In particular, it would be convenient to arrive at the domain with controlled singularities, specifically, with normal intersections (where each face of codimension  $k$  is adjoined to at most  $k$  faces of codimension 1).

To do so we resolve the singularities of  $P$  (i.e. its faces of codimension 2 and higher), in such a way that all facets (independently of their dimensions) would lift to faces of dimension  $(d-1)$  in the resolved polytope, the flags of length 2 (pairs of incident facets) would lift to faces of dimension  $(d-2)$  and so on, with full flags corresponding to the vertices in the resolution.

The easiest way to visualize such a resolution is as follows:

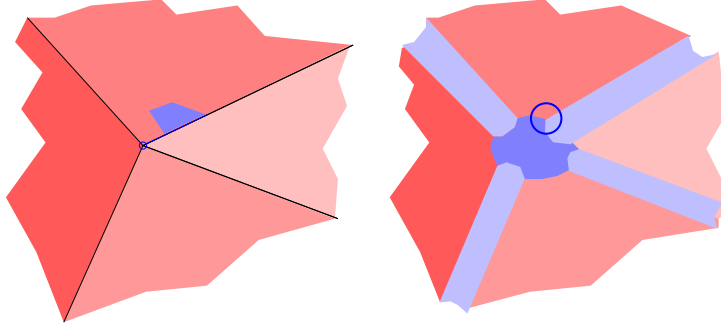
For each face  $f^l$  of dimension  $l < d$  of the polytope  $P$  consider the  $\epsilon^{l+1}$ -tube  $T(f)$  around this face (i.e. the set of all points in  $P$  at the distance at most  $\epsilon^{l+1}$  from the affine subspace (of dimension  $l$ ) spanned by  $f^l$ ). For small enough  $\epsilon$  these tubes intersect transversally, and only if the corresponding faces adjoin.

Let  $P_\epsilon$  be the complement to the union of these tubes,

$$P_\epsilon = P - \cup_{l < d} \cup_{f^l \subset P} T(f).$$

It is immediate that the facets of  $P_\epsilon$  are in one-to-one correspondence with flags of  $P$ .





**Figure 6:** Resolution of vertices: non-simple vertex becomes (topologically) simple after the resolution. Vertices of the resolved polytope correspond to the flags of the original polytope  $P$ : one such pair, flag  $\leftrightarrow$  vertex, is shown.

**Proposition 1** *The (topological) polyhedron  $P_\epsilon$  resolves  $P$ : there exists a (natural) mapping  $\pi : P_\epsilon \rightarrow P$  which is a diffeomorphism on the interior of  $P_\epsilon$  taking each codimension 1 facet of  $P_\epsilon$  to the corresponding facet of  $P$ . (More generally, the faces of  $P_\epsilon$  of codimension  $k$  correspond to flags in  $P$  of length  $k$ .) Further, the polyhedron  $P_\epsilon$  is a manifold with corners: locally, it is diffeomorphic to an open ball at the origin in  $\mathbb{R}^d$  intersected with  $s \leq d$  halfspaces  $x_1 \geq 0, \dots, x_s \geq 0$ .*

Also, the resolution moves the boundaries between the spline domains away from the vertices of  $P_\epsilon$ :

**Proposition 2** *The boundaries between polyhedra of the spline subdivision lift to subvarieties (with corners) of  $P_\epsilon$  which do not contain the vertices of  $P_\epsilon$ .*

### 3.3 Gelfand-Leray forms and elementary Laplace integrals

Using the proposition 1 we can localize the integral (4). Indeed, we can represent

$$B(t) = |P|^{-1} \int_P \mathbf{1}(u(x) < t) d\lambda \tag{5}$$

as

$$|P|^{-1} \int_{P_\epsilon} \mathbf{1}(\pi^* u(x) < t) d\rho \tag{6}$$

$$= |P|^{-1} \sum_{\mathcal{F}} \int_{P_\epsilon} \phi_{\mathcal{F}}(y) \mathbf{1}(\pi^* u < t) d\rho. \tag{7}$$

Here  $\rho = \pi^* \lambda$  is the pull-back of the Lebesgue measure (possible as  $\pi$  is a diffeomorphism onto the interior of  $P$ ), and

$$\sum_{\mathcal{F}} \phi_{\mathcal{F}} = 1, \phi_{\mathcal{F}} \geq 0$$

is the partition of unity such that  $\phi_{\mathcal{F}} = 0$  outside of the small vicinity of the stratum of  $P_\epsilon$  corresponding to flag  $\mathcal{F}$ .

The advantage of the decomposition (7) stems from the following

**Lemma 3** Let  $y$  be an interior point of a  $s$ -dimensional facet of  $P_\epsilon$  corresponding to a flag  $\mathcal{F} = (f_1 \subset f_2 \subset \dots \subset f_l)$ . In particular, one can choose a coordinate system centered at  $y$  such that near the origin, the polyhedron  $P_\epsilon$  is given by  $\{y_1 \geq 0, \dots, y_l \geq 0\}$ . Then

- the density of  $\rho$  behaves as

$$\frac{d\rho}{d\lambda}(y) = U_\rho(y) \prod_1^s y_i^{c_i-1},$$

and the lift of the function  $u$  behave

$$\pi^*u(y) = U_u(y) \prod_1^s y_i^{m_i}.$$

Here  $c_i = c(f_i)$  and  $m_i = m(f_i)$ , and  $U_\rho, U_u$  are nonvanishing functions,  $U_\rho$  smooth, and  $U_u$  continuous and smooth in vicinities of the vertices of  $P_\epsilon$ .

Now we could apply directly the results of [1], expressing the asymptotics of the “elementary Laplace integral”

$$\int_{y \geq 0} U(y) e^{-t \Pi_i y_i^{m_i}} y_1^{c_1-1} \dots y_s^{c_s-1} dy_1 \dots dy_s,$$

or, more directly, can analyze the poles of the Mellin transform

$$\int_y (\pi^*u(y))^z y_1^{c_1-1} \dots y_s^{c_s-1} dy_1 \dots dy_s$$

and apply the results of [10] relating them to the Laplace integrals. One can then see that the leading terms of the asymptotics come from vicinities of the vertices of  $P_\epsilon$ , leading to Theorem 2.

## 4 concluding remarks

- The main results of this note give an algorithm of computing the asymptotics of  $M_n$ , in the polyhedral setup. The method requires, on its face, to enumerate and to analyze all the flags of a polyhedron, the number of which grows superexponentially with the dimension. There are obvious shortcuts, and a more efficient way to find the growth rates of  $M_n$  might be quite feasible.
- What happens if the cone  $K$  is not polyhedral but rather semi-algebraic? A lot of the elements of the proofs survive; but some auxiliary results (especially Lemma 3) would need some rethinking.

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