



**HAL**  
open science

## Spanning trees of finite Sierpiński graphs

Elmar Teufl, Stephan Wagner

► **To cite this version:**

Elmar Teufl, Stephan Wagner. Spanning trees of finite Sierpiński graphs. Fourth Colloquium on Mathematics and Computer Science Algorithms, Trees, Combinatorics and Probabilities, 2006, Nancy, France. pp.411-414, 10.46298/dmtcs.3494 . hal-01184698

**HAL Id: hal-01184698**

**<https://inria.hal.science/hal-01184698v1>**

Submitted on 17 Aug 2015

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# The number of spanning trees of finite Sierpiński graphs

Elmar Teufl<sup>1†</sup> and Stephan Wagner<sup>2‡</sup>

<sup>1</sup>Fakultät für Mathematik, Universität Bielefeld, P.O.Box 100131, 33501 Bielefeld, Germany

<sup>2</sup>Institut für Mathematik, Technische Universität Graz, Steyrergasse 30, 8010 Graz, Austria

We show that the number of spanning trees in the finite Sierpiński graph of level  $n$  is given by

$$\sqrt[4]{\frac{3}{20}} \left(\frac{5}{3}\right)^{-n/2} (\sqrt[4]{540})^{3^n}.$$

The proof proceeds in two steps: First, we show that the number of spanning trees and two further quantities satisfy a 3-dimensional polynomial recursion using the self-similar structure. Secondly, it turns out, that the dynamical behavior of the recursion is given by a 2-dimensional polynomial map, whose iterates can be computed explicitly.

**Keywords:** combinatorial enumeration, spanning trees, finite Sierpiński graphs

## 1 Introduction

The enumeration of spanning trees in a finite graph ranges among the classical tasks of combinatorics and has been studied for more than 150 years. The number of spanning trees in a graph  $X$  is often called the *complexity* and denoted by  $\tau(X)$ . Let us recall the famous Matrix-Tree Theorem of Kirchhoff [Kirchhoff(1847)]: The complexity  $\tau(X)$  of a graph  $X$  is equal to any cofactor of the Laplace matrix of  $X$ , which is the degree matrix of  $X$  minus the adjacency matrix of  $X$ . As a consequence the number of spanning trees in the complete graph with  $n$  vertices is given by  $n^{n-2}$ .

In the following the complexity of finite Sierpiński graphs is computed. These graphs are discrete analoga of the well-known Sierpiński gasket (see [Sierpinski(1915)]) and can be defined as follows: Denote by

$$\mathbf{e}_1 = (1, 0, 0), \quad \mathbf{e}_2 = (0, 1, 0), \quad \text{and} \quad \mathbf{e}_3 = (0, 0, 1)$$

the canonical basis vectors of  $\mathbb{R}^3$ . For  $n = 0$  the Sierpiński graph  $X_0$  is given by  $VX_0 = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$  and  $EX_0 = \{\{\mathbf{e}_1, \mathbf{e}_2\}, \{\mathbf{e}_2, \mathbf{e}_3\}, \{\mathbf{e}_3, \mathbf{e}_1\}\}$ . For  $n > 0$  the Sierpiński graph  $X_n$  is defined iteratively by

$$VX_n = (2^{n-1}\mathbf{e}_1 + VX_{n-1}) \cup (2^{n-1}\mathbf{e}_2 + VX_{n-1}) \cup (2^{n-1}\mathbf{e}_3 + VX_{n-1})$$

and

$$EX_n = (2^{n-1}\mathbf{e}_1 + EX_{n-1}) \uplus (2^{n-1}\mathbf{e}_2 + EX_{n-1}) \uplus (2^{n-1}\mathbf{e}_3 + EX_{n-1}).$$

The graph  $X_n$  is called *Sierpiński graph of level  $n$* ; see Figure 1 for  $X_0$ ,  $X_1$ , and  $X_2$ . A simple computation shows that  $|VX_n| = \frac{3}{2}(3^n + 1)$  and  $|EX_n| = 3^{n+1}$ .

## 2 Main result

The complexity  $\tau(X_n)$  of the Sierpiński graph of level  $n$  is given by

$$\tau(X_n) = \sqrt[4]{\frac{3}{20}} \left(\frac{5}{3}\right)^{-n/2} (\sqrt[4]{540})^{3^n}.$$

The first numbers of this sequence are 3, 54, 524880, 803355125990400000, ...

<sup>†</sup>Supported by the Marie Curie Fellowship MEIF-CT-2005-011218

<sup>‡</sup>Supported by the project S9611 of the Austrian Science Fund FWF

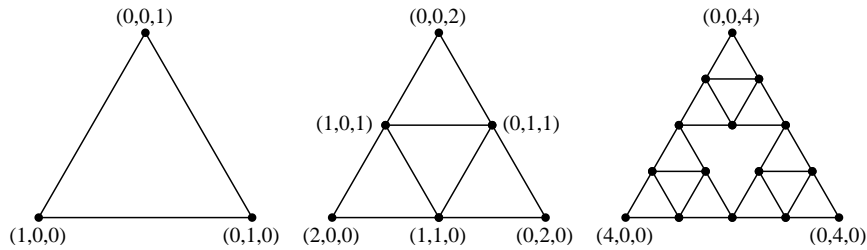


Fig. 1: The Sierpiński graphs  $X_0$ ,  $X_1$ , and  $X_2$ .

Since every spanning tree is a subset of the edge set  $EX_n$ , it is natural to rewrite this formula in terms of  $|EX_n|$ : This yields

$$\tau(X_n) = \sqrt[4]{\frac{5}{12}} |EX_n|^{(1-2/d_s)/2} (\sqrt[12]{540})^{|EX_n|}.$$

Here  $d_s$  is the spectral dimension of the Sierpiński gasket, which is given by

$$d_s = 2 \frac{\log 3}{\log 5}.$$

The spectral dimension was first introduced using the integrated density of states of the Laplacian on the infinite Sierpiński graph. Later on, the exponent  $d_s$  was studied from several points of view, see for example [Barlow(1998), Kigami(2001)] and the references therein. Furthermore, note that the fraction  $\frac{5}{3}$  is the so-called resistance scaling factor of the Sierpiński gasket.

We conjecture that the formula

$$\tau(X_n) = C |EX_n|^{\beta(1-2/d_s)} \alpha^{|EX_n|}$$

holds for a large class of sequences  $(X_n)_{n \geq 0}$  of finite self-similar graphs, where  $C > 0$ ,  $\beta \geq 0$  and  $\alpha \in (1, 2)$  are some constants; see [Teufl and Wagner(2006)].

### 3 Proof

The proof is based on the work in the paper [Teufl and Wagner(2006)]. For  $n \geq 0$  and  $k \in \{1, 2, 3\}$  denote by  $\mathcal{F}_k(n)$  the set of spanning forests in  $X_n$  with  $k$  components, so that each component contains at least one vertex of  $B_n = 2^n \{e_1, e_2, e_3\}$ , and set  $\mathcal{F}(n) = \mathcal{F}_1(n) \uplus \mathcal{F}_2(n) \uplus \mathcal{F}_3(n)$ . For  $n \geq 0$  and  $i \in \{1, 2, 3\}$  let  $X_n^i$  be the subgraph of  $X_n$ , which is induced by  $2^{n-1}e_i + VX_{n-1}$ , and note that  $X_n^i$  is isomorphic to  $X_{n-1}$ . The restriction of a spanning forest in  $\mathcal{F}_k(n)$  to  $X_n^i$  yields a spanning forest in  $\mathcal{F}_j(n-1)$  for some  $j \in \{1, 2, 3\}$ . Hence each spanning forest in  $\mathcal{F}(n)$  can be decomposed into three forests in  $\mathcal{F}(n-1)$ . For example, each spanning tree in  $\mathcal{F}_1(n)$  is composed of two trees in  $\mathcal{F}_1(n-1)$  and one forest in  $\mathcal{F}_2(n-1)$ , see Figure 2.

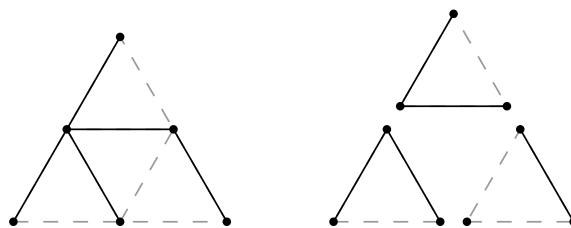


Fig. 2: The decomposition of a spanning tree of  $X_1$ .

Denote by  $c(n)$  the 3-dimensional counting vector

$$c(n) = (c_1(n), c_2(n), c_3(n)) = (|\mathcal{F}_1(n)|, \frac{1}{3}|\mathcal{F}_2(n)|, |\mathcal{F}_3(n)|).$$

Note that there are three possibilities to arrange the vertices of  $B_n$  in two components of a spanning forest in  $\mathcal{F}_2(n)$ , and the factor  $\frac{1}{3}$  takes care of that. Furthermore, the complexity  $\tau(X_n)$  is given by  $\tau(X_n) = c_1(n)$ .

The decomposition of forests in  $\mathcal{F}(n)$  into three forests in  $\mathcal{F}(n-1)$  implies that  $c(n)$  satisfies the polynomial recursion

$$c(n) = Q(c(n-1))$$

for  $n > 0$ , where  $Q$  is given by

$$Q : \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} 6x_1^2x_2 \\ 7x_1x_2^2 + x_1^2x_3 \\ 14x_2^3 + 12x_1x_2x_3 \end{pmatrix}$$

The initial vector is given by  $c(0) = (3, 1, 1)$ .

Now define the map  $v : \mathbb{R}_+^2 \rightarrow \mathbb{R}^3$  by

$$v(a, b) = (3a, b, b^2/a),$$

then  $c(0) = v(1, 1)$  and  $c(1) = v(18, 30)$ . Finally, define  $T : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+^2$  by

$$T(a, b) = (18a^2b, 30ab^2),$$

then the function  $Q$  maps the set  $v(\mathbb{R}_+^2)$  into itself and  $Q \circ v = v \circ T$ .

Thus the vector  $c(n)$  is given by

$$c(n) = v(T^n(1, 1)),$$

where  $T^n$  is the  $n$ -fold iterate of  $T$ . The  $n$ -fold iterate of  $T$  can be computed explicitly:

$$T^n(a, b) = \begin{pmatrix} \left(\frac{5}{3}\right)^{-n/2} \left(\sqrt[4]{540}\right)^{3^n-1} a^{(3^n+1)/2} b^{(3^n-1)/2} \\ \left(\frac{5}{3}\right)^{n/2} \left(\sqrt[4]{540}\right)^{3^n-1} a^{(3^n-1)/2} b^{(3^n+1)/2} \end{pmatrix}$$

for all  $n \geq 0$ . This implies the statement.

## 4 Conclusion

The ideas of this proof can be formalized and generalized to a large class of fractal-like graphs. In addition, it is also possible to count different combinatorial objects in those graphs; for example matchings (independent edge subsets), connected subsets, subtrees, colorings, factors, and vertex or edge coverings. Using a decomposition argument it is possible to obtain a multi-dimensional polynomial recursion. Of course, one cannot expect to get explicit formulas in general. However, the asymptotic behavior can be deduced under suitable conditions. See [Teufl and Wagner(2006)] for more information.

An interesting reformulation of the Matrix-Tree Theorem states, that the complexity of a graph  $X$  is given by the product of all non-zero eigenvalues of the Laplace matrix of  $X$  taking the multiplicities into account divided by the number of vertices. Note that the Dirichlet spectrum of the Laplace matrix of finite Sierpiński graphs  $X_n$  with respect to the boundary  $B_n$  is well understood, see [Shima(1991)]. However, it seems to be much more difficult to understand the normal spectrum.

## References

- [Barlow(1998)] M. T. Barlow. Diffusions on fractals. In *Lectures on probability theory and statistics (Saint-Flour, 1995)*, pages 1–121. Springer Verlag, Berlin, 1998.
- [Kigami(2001)] J. Kigami. *Analysis on fractals*, volume 143 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2001.
- [Kirchhoff(1847)] G. Kirchhoff. Über die Auflösung der Gleichungen, auf welche man bei der Untersuchung der linearen Verteilung galvanischer Ströme geführt wird. *Ann. Phys. Chem.*, 72:497–508, 1847.
- [Shima(1991)] T. Shima. On eigenvalue problems for the random walks on the Sierpiński pre-gaskets. *Japan J. Indust. Appl. Math.*, 8(1):127–141, 1991.
- [Sierpinski(1915)] W. Sierpinski. Sur une courbe dont tout point est un point de ramification. *Compt. Rendus Acad. Sci. Paris*, 160:302–305, 1915.
- [Teufl and Wagner(2006)] E. Teufl and S. Wagner. Enumeration problems for classes of self-similar graphs. preprint, 2006.

