



S-constrained random matrices

Sylvain Gravier, Bernard Ycart

► To cite this version:

Sylvain Gravier, Bernard Ycart. *S*-constrained random matrices. Fourth Colloquium on Mathematics and Computer Science Algorithms, Trees, Combinatorics and Probabilities, Sep 2006, Nancy, France. pp.357-364, 10.46298/dmtcs.3480 . hal-01184682

HAL Id: hal-01184682

<https://inria.hal.science/hal-01184682>

Submitted on 17 Aug 2015

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

S-constrained random matrices

Sylvain Gravier^{1†} and Bernard Ycart^{2‡}

¹CNRS - UJF, ERTé "Maths à Modeler", Groupe de Recherche GéoD - Laboratoire Leibniz, 46, avenue Félix Viallet, 38031 Grenoble Cedex (France)

²LMC, CNRS UMR 5523, BP 53, 38041 Grenoble cedex 9, FRANCE

Let S be a set of d -dimensional row vectors with entries in a q -ary alphabet. A matrix M with entries in the same q -ary alphabet is S -constrained if every set of d columns of M contains as a submatrix a copy of the vectors in S , up to permutation. For a given set S of d -dimensional vectors, we compute the asymptotic probability for a random matrix M to be S -constrained, as the numbers of rows and columns both tend to infinity. If n is the number of columns and $m = m_n$ the number of rows, then the threshold is at $m_n = \alpha_d \log(n)$, where α_d only depends on the dimension d of vectors and not on the particular set S . Applications to superimposed codes, shattering classes of functions, and Sidon families of sets are proposed. For $d = 2$, an explicit construction of a S -constrained matrix is given.

Keywords: random matrix, Poisson approximation, superimposed code, shattering, VC dimensions, Sidon families

1 Introduction

We propose S -constrained matrices as a unifying framework for two seemingly remote notions, one in the field of cryptography (superimposed codes), the other one in information theory (shattering classes of functions).

Definition 1 Let q , d and s be three fixed integers. Let $S = \{\eta_1, \dots, \eta_s\}$ be a set of d -dimensional row vectors, with entries in the q -ary alphabet $\{0, \dots, q-1\}$.

Let $m \geq d$ and $n \geq s$ be two integers. Let $M = (M_{i,j})$ be a $m \times n$ matrix with entries in the same q -ary alphabet.

The matrix M is said to be S -constrained if for every subset \mathbf{j} of $[n] = \{1, \dots, n\}$, with $|\mathbf{j}| = d$ elements, there exist s indices i_1, \dots, i_s in $[m]$ such that for all $h = 1, \dots, s$ and for all $j \in \mathbf{j}$,

$$M_{i_h,j} = \eta_h(j) .$$

In other words, every subset of d columns of the matrix M must contain as a submatrix a copy of all row vectors in S , up to permutation.

The matrices considered here can be interpreted either as q -ary codes (see Cohen and Schaathun's review (5)) or as classes of functions from $[n]$ into $\{0, \dots, q-1\}$ (see for instance Anthony and Bartlett (2)). In the first interpretation, the *columns* of the matrix are the words of the code, in the second one, the *rows* are the functions of the class.

To motivate Definition 1, recall ((16) p. 276) that a binary code is called (w, r) -superimposed if for every subsets of words W , R (sets of columns of the matrix) with respective cardinalities w, r , there exists a position (row index) on which every word of W is 1 and every word of R is 0. Let S be the set of all binary vectors with w ones and r zeros. The code is (w, r) -superimposed if and only if the corresponding matrix is S -constrained.

Consider now the interpretation of a q -ary matrix as a class of functions (rows of the matrix). A set \mathbf{j} of d coordinates (column indices), is *shattered* by the class, if the restrictions of the functions to those coordinates contain all possible q^d functions. The *Vapnik-Chervonenkis dimension* of the class is the size of the largest shattered set, its *testing dimension* is the maximal d such that all sets of size d are shattered (see (3)). Let S be the set of all q^d vectors of size d . The testing dimension of the class is $\geq d$ if and only if the corresponding matrix is S -constrained.

[†]Email: Sylvain.Gravier@imag.fr

[‡]Email: Bernard.Ycart@imag.fr

The aim of this paper is to obtain asymptotic bounds on the size of S -constrained matrices, as the number of columns n tends to infinity, and the number of rows $m = m_n$ increases as a function of n . The problem of finding bounds on matrices satisfying certain constraints, has given rise to an extensive literature: see *e.g.* Kim *et al.* (16) for (w, r) -superimposed codes, Cohen and Schaathun (5; 6) for separating codes, Li *et al.* (17) for hashing codes. The dual problem of finding matrices *not containing* any copy of S has also been considered by a number of authors, following Sauer (20): see Steele (21) for matrices not shattering any sets of size d , and more recently Anstee *et al.* (1) on matrices with forbidden configurations.

We are mainly concerned with probabilistic bounds, derived from examining the threshold of the desired property for random matrices: see (6; 16; 17) for comparisons between probabilistic and deterministic bounds. See also (22; 23), for probabilistic bounds on classes of binary functions under different random models.

Our main result gives the asymptotic probability for a random matrix to be S -constrained. By *random matrix*, we mean a matrix whose entries are mutually independent and uniformly distributed on the alphabet $\{0, \dots, q-1\}$. Its distribution is denoted by \mathbb{P} .

Theorem 1 *Let S be a set of s q -ary vectors of size d . Denote by α_d the following (positive) real.*

$$\alpha_d = -\frac{d}{\log(1 - q^{-d})} . \quad (1)$$

Assume $m = m_n$ is such that:

$$\lim_{n \rightarrow \infty} m_n - \alpha_d \log n = c , \quad (2)$$

where c is a real constant. Let M be q -ary random matrix with m_n rows and n columns. Then:

$$\lim_{n \rightarrow \infty} \mathbb{P}(M \text{ is } S\text{-constrained}) = \exp\left(-\frac{s}{d!}(1 - q^{-d})^c\right) . \quad (3)$$

As a consequence of Theorem 1, $m_n = \alpha_d \log n$ is a probabilistic bound for S -constrained matrices: if the number of rows is such that $m_n - \alpha_d \log n$ tends to $-\infty$, then with high probability (*w.h.p.*), a random matrix is not S -constrained. If $m_n - \alpha_d \log n$ tends to $+\infty$, it is S -constrained *w.h.p.* Observe that the threshold $\alpha_d \log n$ only depends on the dimension of the vectors in S , and not on any particular set S . Thus if $m_n - \alpha_d \log n$ tends to $+\infty$ a random matrix, interpreted as a code, is (w, r) -superimposed for any w, r such that $w + r = d$. Interpreted as a class of functions, it has testing dimension at least d . We shall see in Section 3 another application to Sidon families of sets.

The question naturally arises of whether $m_n = \alpha_d \log n$ is optimal. We will answer it by the negative in section 4. In the case $d = 2$, we construct a S -constrained matrix having $m_n \leq \beta_2 \log n$ rows, with

$$\beta_2 = \frac{q^2}{\log 2} < \alpha_2 = -\frac{2}{\log(1 - q^{-2})} .$$

The problem of finding bounds for the size of S -constrained matrices has been investigated in the different contexts of combinatorics, cryptography and information theory. Deterministic and probabilistic bounds for q -ary superimposed codes are given by (16). D'yachkov and Rykov (8) used a probabilistic approach to construct superimposed codes in the binary case. Cohen and Schaathun (5) propose a very thorough review on bounds for separating codes, completed by (6). Other results concerning Sidon families, cover-free families and superimposed codes can be found in (10; 8; 13; 19; 12; 9; 16).

The rest of the paper is organised as follows. Section 2 is dedicated to the proof of Theorem 1, based on a classical Poisson approximation technique. Section 3 details the application of S -constrained binary matrices to Sidon families of sets. In Section 4, we construct an explicit S -constrained matrix in the case $d = 2$.

2 Proof of Theorem 1

Before proving Theorem 1, we will comment a few of its consequences.

The convergence in (3) expresses a rather sharp concentration result. Consider a random matrix with $m_n = \alpha \log n$ rows, where $\alpha \neq \alpha_d$ for all d , and let d be the largest integer such that $\alpha > \alpha_d$. Then if S is any set of d -dimensional vectors, the matrix will be S -constrained *w.h.p.* But for any set of $(d+1)$ -dimensional vectors, it will not be S -constrained *w.h.p.*

For $m_n = \alpha_d \log n + c$, the probability to be S -constrained converges to a non trivial value, that does depend on the set S , but only as a decreasing function of its cardinality s : the more constraints are imposed,

the less likely it is to satisfy them all. Table 1 below gives numerical values of the asymptotic probability (3), for $q = 2$, $c = 5$, $d = 1, \dots, 10$, and for the two extreme values of $s = 1$ and $s = 2^d$. It turns out that there is relatively little difference between satisfying all 2^d possible constraints, or just a single one.

d	1	2	3	4	5	6	7	8	9	10
$s = 1$	0.9692	0.8881	0.9181	0.9703	0.9929	0.9987	0.9998	1.0000	1.0000	1.0000
$s = 2^d$	0.9394	0.6221	0.5047	0.6171	0.7965	0.9211	0.9759	0.9938	0.9986	0.9997

Tab. 1: Asymptotic probability for a random matrix to be *S*-constrained for a set of d -dimensional vectors, of size $s = 1$ or $s = 2^d$, for $q = 2$, $c = 5$ and $d = 1, \dots, 10$.

Theorem 1 provides an existence result for *S*-constrained matrices: if $m_n = \alpha_d \log n + c$, then for n large enough, the probability for a random matrix to be *S*-constrained is strictly positive, hence there must exist *S*-constrained matrices of size $m_n \times n$. But it also provides an algorithmic way to construct such a matrix. Assume the parameters are such that the probability for a random matrix to be *S*-constrained is $1/2$, then drawing random matrices until one which is *S*-constrained is found, will output the desired matrix after 2 random drawings on average.

Theorem 1 is a Poisson approximation result (see Barbour et al. (4) for a general reference). The technique of proof, based on the Stein-Chen method, is quite standard: we shall use the results stated in Janson (14).

Proof: Let $\mathbf{j} \subset [n]$ be a set of d column indices. For any d -dimensional q -ary row vector η , we denote by $C_{\mathbf{j},\eta}$ the set of all $m \times n$ q -ary matrices such that there exists a row of M whose restriction to \mathbf{j} is η . We denote by $C_{\mathbf{j}}$ the intersection of the $C_{\mathbf{j},\eta}$'s over all vectors $\eta \in S$

$$C_{\mathbf{j}} = \bigcap_{\eta \in S} C_{\mathbf{j},\eta}.$$

The set $C_{\mathbf{j}}$ is made of those matrices whose columns indexed by \mathbf{j} contain a copy of all vectors in S . The set of *S*-constrained matrices is the intersection over all possible subsets of d indices \mathbf{j} , of the events $C_{\mathbf{j}}$.

$$C = \bigcap_{|\mathbf{j}|=d} C_{\mathbf{j}}.$$

Our aim is to compute the probability of C under the uniform distribution on all $m \times n$ q -ary matrices, denoted by \mathbb{P} . We begin with the probability of $C_{\mathbf{j}}$, for $\mathbf{j} \subset [n]$ such that $|\mathbf{j}| = d$. We denote by \overline{B} the complement of an event B . Recall that $C_{\mathbf{j}} = \bigcap C_{\mathbf{j},\eta}$, hence $\overline{C_{\mathbf{j}}} = \bigcup \overline{C_{\mathbf{j},\eta}}$, where the union extends over all elements η of S . If $h \leq s$ and η_1, \dots, η_h are distinct elements of S , then the probability that among m random rows none of them coincides with one of the η_i 's on \mathbf{j} is:

$$\mathbb{P}(\overline{C_{\mathbf{j},\eta_1}} \cap \dots \cap \overline{C_{\mathbf{j},\eta_h}}) = (1 - hq^{-d})^m. \quad (4)$$

By the formula giving the probability of a union (formula (1.5) p. 99 of Feller (11)), one gets

$$\mathbb{P}(\overline{C_{\mathbf{j}}}) = \sum_{h=1}^s (-1)^{h-1} \binom{s}{h} (1 - hq^{-d})^m.$$

If $m = m_n$ tends to infinity, this sum tends to 0 and the first term dominates.

$$\mathbb{P}(\overline{C_{\mathbf{j}}}) = s(1 - q^{-d})^{m_n} (1 + o(1)).$$

Recall that $\alpha_d = -d/\log(1 - q^{-d})$. For $m_n = \alpha_d \log n + c + o(1)$,

$$\mathbb{P}(\overline{C_{\mathbf{j}}}) = s(1 - q^{-d})^c n^{-d} (1 + o(1)). \quad (5)$$

Let us now introduce the integer-valued random variable X which counts the number of failed events among the $C_{\mathbf{j}}$'s.

$$X = \sum_{|\mathbf{j}|=d} \mathbb{I}_{\overline{C_{\mathbf{j}}}},$$

where \mathbb{I}_B denotes the indicator function of an event B . Obviously, a random matrix is S -constrained, iff $X = 0$. The expectation of X is

$$\mathbb{E}(X) = \binom{n}{d} \mathbb{P}(\overline{C_j}) . \quad (6)$$

As n tends to infinity,

$$\binom{n}{d} = \frac{n^d}{d!} (1 + o(1)) .$$

Therefore, for $m_n = \alpha_d \log n + c + o(1)$,

$$\lim_{n \rightarrow \infty} \mathbb{E}(X) = \frac{s}{d!} (1 - q^{-d})^c . \quad (7)$$

Comparing (7) with (3), there remains to prove that

$$\lim_{n \rightarrow \infty} \mathbb{P}(X = 0) = \lim_{n \rightarrow \infty} \exp(-\mathbb{E}(X)) ,$$

We will prove that a Poisson approximation holds for X . The family of indicators $(\mathbb{I}_{\overline{C_j}})$ is *dissociated*, in the sense of Janson (14) p. 10: the two sets of random variables $\{\mathbb{I}_{\overline{C_j}}, j \in J_1\}$ and $\{\mathbb{I}_{\overline{C_{j'}}}, j' \in J_2\}$ are independent whenever every $j \in J_1$ is disjoint from every $j' \in J_2$. Denote by Γ the set of all $j \subset [n]$ with $|j| = d$. For $j \in \Gamma$, denote by Γ_j the set of all j' such that $j \cap j' \neq \emptyset$. By Theorem 4 p. 10 of (14), the total variation distance between the distribution of X and the Poisson distribution with parameter $\mathbb{E}(X)$ is bounded above by

$$(1 \wedge \mathbb{E}(X))^{-1} \left(\sum_{j \in \Gamma} \sum_{j' \in \Gamma_j} \mathbb{P}(\overline{C_j}) \mathbb{P}(\overline{C_{j'}}) + \sum_{j \in \Gamma} \sum_{j' \in \Gamma_j \setminus \{j\}} \mathbb{P}(\overline{C_j} \cap \overline{C_{j'}}) \right) \quad (8)$$

The result will follow by proving that each of the two sums in (8) converges to zero. The first sum has $O(n^{2d-1})$ terms, each of order $O(n^{-2d})$, by (5). We decompose the second sum according to the number of elements in $j \cap j'$ as follows:

$$\sum_{j \in \Gamma} \sum_{j' \in \Gamma_j \setminus \{j\}} \mathbb{P}(\overline{C_j} \cap \overline{C_{j'}}) = \sum_{h=1}^{d-1} \Delta_h ,$$

where

$$\Delta_h = \sum_{|j \cap j'|=h} \mathbb{P}(\overline{C_j} \cap \overline{C_{j'}}) .$$

Clearly, there are $O(n^{2d-h})$ terms in Δ_h . We have

$$\overline{C_j} \cap \overline{C_{j'}} = \left(\bigcup_{\eta \in S} \overline{C_{j,\eta}} \right) \cap \left(\bigcup_{\zeta \in S} \overline{C_{j',\zeta}} \right) = \bigcup_{\eta, \zeta} (\overline{C_{j,\eta}} \cap \overline{C_{j',\zeta}}) .$$

The probability of any of these intersections is:

$$\mathbf{P}(\overline{C_{j,\eta}} \cap \overline{C_{j',\zeta}}) = \begin{cases} (1 - 2q^{-d} + q^{-2d+h})^m & \text{if } \eta \equiv \zeta \text{ on } j \cap j' \\ (1 - 2q^{-d})^m & \text{otherwise.} \end{cases}$$

Hence

$$\Delta_h \leq a n^{2d-h} (1 - 2q^{-d} + q^{-2d+h})^m$$

for some positive a , not depending on n and m . For $m = m_n = \alpha_d \log n + c + o(1)$, there exists a positive constant b such that

$$\Delta_h \leq b n^{2d-h+\alpha_d \log(1-2q^{-d}+q^{-2d+h})} .$$

That Δ_h tends to zero follows from having a negative exponent, *i.e.*,

$$2d - h + \alpha_d \log(1 - 2q^{-d} + q^{-2d+h}) < 0 \quad (9)$$

for $d \geq 2$ and $h = 1, \dots, d-1$. Indeed, the left hand side of (9) vanishes both for $h = 0$ and $h = d$. As a function of h , its second derivative is positive on $[0, d]$ hence it is strictly convex. Therefore it is strictly negative for all $h = 1, \dots, d-1$. Hence Δ_h tends to zero with increasing n , which concludes the proof. \square

3 Sidon families

Let $\mathcal{R} = \{R_1, \dots, R_n\}$ be a family of subsets of $[m]$. To each of them, one can associate a m -dimensional binary column vector, whose i -th entry is 1 if $i \in R_j$ and 0 else. This defines the *incidence matrix* of the family, denoted by $M = (M_{i,j})$.

$$M_{i,j} = \mathbb{I}_{R_j}(i) .$$

Superimposed codes translate into *cover-free families*: \mathcal{R} is a (w, r) -cover-free family, if no intersection of w members of the family is covered by a union of r others (see (18; 9; 7; 16)). As already observed, the incidence matrix of a (w, r) -cover-free family is S -superimposed, where S is the set of vectors having w ones and r zeros. The notion we are studying in this section does not translate as straightforwardly.

Definition 2 A set family \mathcal{R} is a k -Sidon family if all the k -term unions are distinct :

$$\bigcup_{h=1}^k R_{j_h} \neq \bigcup_{h=1}^k R_{j'_h} ,$$

whenever $\{j_1, \dots, j_k\} \neq \{j'_1, \dots, j'_k\}$.

The notion was defined in (15) where the authors used the terminology ‘ UD_k code’ . One can also find in the literature the term ‘ k -superimposed family’ in (8). We adopt here the terminology of (13).

Contrarily to cover-free families, there is no set of vectors S such that the family is k -Sidon if and only if its incidence matrix is S -constrained. Proposition 1 gives a necessary and a sufficient condition.

Proposition 1 Let k be a positive integer. Let S_0 be the singleton containing only the null vector with k entries. Let S_1 be the set of all vectors of length $(k + 1)$ having a single one and k zeros.

1. If a family of sets is k -Sidon, then in its incidence matrix, at most one set of k columns does not contain S_0 .
2. If the incidence matrix is S_1 -constrained, then the family of sets is k -Sidon.

Proof: Consider first the necessary condition. A set of k columns does not contain the null vector, if and only if the union of the k sets is $[m]$. If the family is k -Sidon, this can happen at most once.

Let us now turn to the sufficient condition. Consider two sets of indices j_1, \dots, j_k and j'_1, \dots, j'_k . With no loss of generality, assume that $j_1 \neq j'_1$. Since the incidence matrix is S_1 -constrained, there exists an index $i \in [m]$ such that $M_{i,j_1} = 1$ and $M_{i,j'_1} = \dots = M_{i,j'_k} = 0$: the element i belongs to R_{j_1} , but not to $R_{j'_1} \cup \dots \cup R_{j'_k}$. Hence the family is k -Sidon. \square

Consider a random binary matrix in the sense of Theorem 1, with m_n rows and n columns. It is the incidence matrix of a family of n sets, provided its columns are distinct. This holds *w.h.p.* if m_n is large compared to $(2/\log 2) \log n$. As a consequence of point 2 of Proposition 1 and Theorem 1, if $m_n - \alpha_{k+1} \log n$ tends to $+\infty$, then the family is k -Sidon *w.h.p.* From point 1 of Proposition 1, for a k -Sidon family, the variable X counting the number of failures of the property of having a copy of S_0 is either 0 or 1. It follows from the Poisson approximation of Theorem 1, that the threshold for ‘ $X \leq 1$ ’ is the same as for ‘ $X = 0$ ’, *i.e.* $\alpha_k \log n$: if $m_n - \alpha_k \log n$ tends to $-\infty$, then the family will not be k -Sidon *w.h.p.*

4 Explicit construction

Given a set $S = \{\eta_1, \dots, \eta_s\}$ of s 2-dimensional vectors with entries in a q -ary alphabet, we will construct a $m \times n$ S -constrained matrix M with $m \leq \frac{q^2}{\log 2} \log n$. Notice that the size of this matrix is an improvement of the probabilistic bound obtained in Theorem 1. Indeed, the ratio

$$\frac{q^2/\log 2}{\alpha_2} = \frac{q^2/\log 2}{-2/\log(1 - q^{-2})}$$

is smaller than 1 for all q and tends to $1/(2 \log 2) \simeq 0.72$ as q increases.

Let k be a positive integer. We start with a pattern matrix P having $k + 1$ rows and 2^k columns. Its first row is null. Its rows with indices $2, \dots, k + 1$ are formed by all 2^k binary column vectors, ranked in alphabetical order. Here is the matrix P for $k = 3$.

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

We denote by \overline{P} the matrix $1 - P$.

Let S_1 be the set of all vectors (a, b) in S such that (b, a) also belongs to S , and $S_2 = S \setminus S_1$. Let s_1 and s_2 denote the respective cardinalities of S_1 and S_2 . We will use as an example

$$S = \{(0, 2), (2, 0), (2, 1)\}, \quad S_1 = \{(0, 2), (2, 0)\}, \quad S_2 = \{(2, 1)\}.$$

Given a set X of two dimensional vectors, we denote by $A(X)$ (resp.: $B(X)$) the column vector of their first (resp.: second) coordinates. In our example:

$$A(S_1) = \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \quad B(S_1) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \quad A(S_2) = (2), \quad B(S_2) = (1).$$

Let U be the $(s_1 + s_2)(k + 1) \times 2^{k+1}$ matrix obtained from the pattern matrix P by replacing all 0's (resp. all 1's) by the $(s_1 + s_2) \times 2$ block

$$\begin{pmatrix} A(S_1) & B(S_1) \\ A(S_2) & B(S_2) \end{pmatrix} \text{ (resp.: } \begin{pmatrix} B(S_1) & A(S_1) \\ B(S_2) & A(S_2) \end{pmatrix} \text{)}.$$

In our example:

$$U = \begin{pmatrix} \begin{array}{cc|cc|cc|cc|cc|cc|cc} 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 \\ 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 \\ 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 \end{array} \\ \hline \begin{array}{cc|cc|cc|cc|cc|cc|cc} 0 & 2 & 0 & 2 & 0 & 2 & 2 & 0 & 2 & 0 & 2 & 2 & 0 & 2 \\ 2 & 0 & 2 & 0 & 2 & 0 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 \\ 2 & 1 & 2 & 1 & 2 & 1 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 \end{array} \\ \hline \begin{array}{cc|cc|cc|cc|cc|cc|cc} 0 & 2 & 0 & 2 & 2 & 0 & 0 & 2 & 0 & 2 & 2 & 0 & 0 & 2 \\ 2 & 0 & 2 & 0 & 0 & 2 & 2 & 0 & 2 & 0 & 0 & 2 & 0 & 2 \\ 2 & 1 & 2 & 1 & 1 & 2 & 2 & 1 & 2 & 1 & 1 & 2 & 1 & 2 \end{array} \\ \hline \begin{array}{cc|cc|cc|cc|cc|cc|cc} 0 & 2 & 2 & 0 & 0 & 2 & 0 & 2 & 2 & 0 & 0 & 2 & 0 & 2 \\ 2 & 0 & 0 & 2 & 2 & 0 & 2 & 0 & 0 & 2 & 2 & 0 & 0 & 2 \\ 2 & 1 & 1 & 2 & 2 & 1 & 1 & 2 & 2 & 1 & 1 & 2 & 1 & 2 \end{array} \end{pmatrix}$$

Similarly, if S_2 is nonempty, we define V as the $s_2(k + 1) \times 2^{k+1}$ matrix obtained by replacing in \overline{P} all 0's (respectively 1's) by the $s_2 \times 2$ block:

$$(A(S_2), B(S_2)) \quad \text{(resp.: } (B(S_2), A(S_2)))$$

In our example:

$$V = \begin{pmatrix} \begin{array}{cc|cc|cc|cc|cc|cc|cc} 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 \\ 1 & 2 & 1 & 2 & 1 & 2 & 2 & 1 & 2 & 1 & 2 & 2 & 1 & 2 \\ 1 & 2 & 1 & 2 & 2 & 1 & 2 & 1 & 2 & 2 & 1 & 2 & 2 & 1 \\ 1 & 2 & 2 & 1 & 1 & 2 & 2 & 1 & 2 & 2 & 1 & 2 & 2 & 1 \end{array} \end{pmatrix}$$

If S_2 is empty, let $M_S = U$. If S_2 is not empty, let M_S be the matrix $\begin{pmatrix} U \\ V \end{pmatrix}$. The matrix M_S has $(s_1 + 2s_2)(k + 1)$ rows and 2^{k+1} columns. In any case, $(s_1 + 2s_2) \leq q^2$. Hence if $n = 2^{k+1}$ is the number of columns, the number of rows is no larger than $(q^2 / \log 2) \log n$. Observe moreover that contrarily to α_2 , the bound now depends on S : if S is a singleton, the matrix M_S only has $(2 / \log 2) \log n$ rows.

Theorem 2 *The matrix M_S is S -constrained.*

Proof: We must prove that any two columns contain a copy of all row vectors of S . Let j_1 and j_2 be two column indices. We will prove that there exist $s_1 + s_2$ row indices $i_1, \dots, i_{s_1}, i'_1, \dots, i'_{s_2}$ such that the submatrix of M_S indexed by $\{i_1, \dots, i_{s_1}, i'_1, \dots, i'_{s_2}\} \times \{j_1, j_2\}$ is either

$$\begin{pmatrix} A(S_1) & B(S_1) \\ A(S_2) & B(S_2) \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} B(S_1) & A(S_1) \\ A(S_2) & B(S_2) \end{pmatrix}.$$

Since, up to permutation of rows, $(A(S_1), B(S_1))$ is the same as $(B(S_1), A(S_1))$, this will be enough to prove that M_S is S -constrained.

Assume first that j_1 and j_2 have distinct parities. Then the submatrix indexed by $\{1, \dots, s_1\} \times \{j_1, j_2\}$ is either $(A(S_1), B(S_1))$ or $(A(S_1), B(S_1))$. If j_1 is odd then the submatrix indexed by $\{s_1 + 1, \dots, s_1 + s_2\} \times \{j_1, j_2\}$ is $(A(S_2), B(S_2))$. If j_1 is even, then the submatrix indexed by $\{s(k + 1) + 1, \dots, s(k + 1) + s_2\} \times \{j_1, j_2\}$ is $(A(S_2), B(S_2))$.

Assume now that j_1 and j_2 have the same parity. Let $j'_1 = \lceil j_1/2 \rceil$ and $j'_2 = \lceil j_2/2 \rceil$. By definition of P , there exists an index i , $1 < i \leq k + 1$, such that its coefficients of order (i, j'_1) and (i, j'_2) are distinct. So, the submatrix of M_S indexed by $\{(i - 1)s + 1, \dots, (i - 1)s + s_1\} \times \{j_1, j_2\}$ is either $(A(S_1), B(S_1))$ or $(B(S_1), A(S_1))$. Moreover, if the coefficient of order (i, j'_1) of P is 0, then the submatrix of M_S indexed by $\{(i - 1)s + s_1 + 1, \dots, (i - 1)s + s_1 + s_2\} \times \{j_1, j_2\}$ is $(A(S_2), B(S_2))$. Otherwise, the coefficient of order $(i + k + 1, j'_1)$ of P is 0, and the submatrix of M_S indexed by $\{(k + 1)s + (i - 1)s_2 + 1, \dots, (k + 1)s + is_2\} \times \{j_1, j_2\}$ is $(A(S_2), B(S_2))$. \square

5 Concluding remarks

The bounds deduced from Theorem 1 are probabilistic. Possibly, one could obtain a deterministic algorithm to construct such S -constrained matrices using a derandomization procedure which would lead to an algorithm similar to that of Hwang and Sós (13). Nevertheless, it would be even more interesting to get an ‘explicit’ construction for all d as we did in Section 4 for $d = 2$. Indeed, such a construction gives more information on the structure of extremal matrices.

Hwang and Sós (13) used the concept of *part-intersecting family*. Given a set S of cardinality N , a t -part-intersecting family is a family \mathcal{F} of subsets of S such that for all $A, B \in \mathcal{F}$

$$|A \cap B| < \frac{1}{t} \min_{A, B \in \mathcal{F}} \{|A|, |B|\}.$$

To understand how this problem relates with ours, consider the simpler constraint that every intersection $A \cap B$ has cardinality 2. For the incidence matrix, this means that any two columns must contain two copies of the row vector $(1, 1)$. The technique that was used in the proof of Theorem 1 relied on the fact that the elements of S were distinct, and it cannot be directly adapted to counting copies of a given row vector. So a natural generalisation of our setting would be to consider matrices such that any set of d columns contains, up to permutation, a copy of some fixed matrix, which could have distinct rows or not.

Another interesting extension would be to consider matrices such that any set of d columns contains a copy of at least one set of vectors among different sets S_1, \dots, S_k . This is related to separating codes, in a similar way as S -constrained matrices are related to superimposed codes.

References

- [1] R. Anstee, B. Fleming, Z. Füredi, and A. Sali. Color critical hypergraphs and forbidden configurations. In *Proc. EuroComb2005*, pages 117–122. DMTCS, 2005.
- [2] M. Anthony and P.L. Bartlett. *Neural Network Learning: Theoretical Foundations*. Cambridge University Press, 1999.
- [3] M. Anthony, G. Brightwell, D. Cohen, and J. Shawe-Taylor. On exact specification by examples. In *Proc. 5th Annual ACM Workshop on Comput. Learning Theory*, pages 311–318. ACM Press, New York, 1992.
- [4] A.D. Barbour, L. Holst, and S. Janson. *Poisson approximation*. Clarendon Press, Oxford, 1992.
- [5] G.D. Cohen and H.G. Schaathun. Asymptotic overview on separating codes. Technical Report 248, Dep. Informatics, Univ. Bergen, 2003.
- [6] G.D. Cohen and H.G. Schaathun. Upper bounds on separating codes. *IEEE Trans. Inf. Theory*, 50(6):1291–1295, 2004.
- [7] A. De Bonis and U. Vaccaro. Constructions of generalized superimposed codes with applications to group testing and conflict resolution in multiple access channels. *Theor. Comput. Sci.*, 306(1-3):223–243, 2003.
- [8] A.G. D’yachkov and V.V. Rykov. Bounds on the length of disjunctive codes. *Prob. Info. Transmission*, 18:166–171, 1983.

- [9] K. Engel. Interval packing and covering in the Boolean lattice. *Comb. Probab. Comput.*, 5(4):373–384, 1996.
- [10] P. Erdős, P. Frankl, and Z. Füredi. Families of finite sets in which no set is covered by the union of two others. *J. Combinatorial Theory*, 33:158–166, 1982.
- [11] W. Feller. *An introduction to probability theory and its applications*, volume I. Wiley, London, 3rd edition, 1968.
- [12] Z. Füredi. On r -cover free families. *J. Combinatorial Theory*, 73(1):172–173, 1996.
- [13] F.K. Hwang and V.T. Sós. Non-adaptive hypergeometric group testing. *Stud. Sci. Math. Hung.*, 22(1-4):257–263, 1987.
- [14] S. Janson. Coupling and Poisson approximation. *Acta Appl. Math.*, 34:7–15, 1994.
- [15] W.H. Kautz and R.C. Singleton. Nonrandom binary superimposed codes. *IEEE Trans. Inf. Theory*, 10:363–377, 1964.
- [16] H.K. Kim, V. Lebedev, and D.Y. Oh. Some new results on superimposed codes. *J. Combinatorial Designs*, 13(4):276–285, 2005.
- [17] P.C. Li, G.H.J. van Rees, and R. Wei. Constructions of 2-cover-free families and related separating hash families. Submitted, 2005.
- [18] C.J. Mitchell and F.C. Piper. Key storage in secure networks. *Disc. Appl. Math.*, 21(3):215–228, 1988.
- [19] M. Ruszinkó. On the upper bound of the size of the r -cover-free families. *J. Combinatorial Theory*, 66(2):302–310, 1994.
- [20] N. Sauer. On the density of families of sets. *J. Combinatorial Theory (A)*, 13:145–147, 1972.
- [21] J.M. Steele. Existence of submatrices with all possible columns. *J. Combinatorial Theory*, 24(1):84–88, 1978.
- [22] B. Ycart and J. Ratsaby. The VC dimension of k -uniform random hypergraphs. *Rand. Struct. Algo.*, to appear, 2006.
- [23] B. Ycart and J. Ratsaby. VC-dimensions of random function classes. Submitted, 2006.