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# Constrained exchangeable partitions

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For a class of random partitions of an infinite set a de Finetti-type representation is derived, and in one special case a central limit theorem for the number of blocks is shown.

**Keywords:** exchangeability, paintbox, stick-breaking

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## 1 Introduction

Under a *partition* of the set  $\mathbb{N}$  we shall mean a sequence  $(b_1, b_2, \dots)$  of subsets of  $\mathbb{N}$  such that (i) the sets  $b_j$  are disjoint, (ii)  $\cup_j b_j = \mathbb{N}$ , (iii) if  $b_k = \emptyset$  then also  $b_{k+1} = \emptyset$  and (iv) if  $b_{k+1} \neq \emptyset$  then  $\min b_k < \min b_{k+1}$ . Condition (iv) says that the sequence of minimal elements of the blocks is increasing. One can think of partition as a mapping which sends a generic element  $j \in \mathbb{N}$  to one of the infinitely many blocks, in such a way that conditions (iii) and (iv) are fulfilled.

A random partition  $\Pi = (B_k)$  of  $\mathbb{N}$  (so, with random blocks  $B_k$ ) is a random variable with values in the set of partitions of  $\mathbb{N}$ . This concept can be made precise by means of a projective limit construction and the measure extension theorem. To this end, one identifies  $\Pi$  with consistent partitions  $\Pi_n := \Pi|_{[n]}$  ( $n = 1, 2, \dots$ ) of finite sets  $[n] := \{1, \dots, n\}$ . Note that the restriction  $\Pi_n$ , which is obtained by removing all elements not in  $[n]$ , still has the blocks in the order of increase of their least elements.

There is a well developed theory of exchangeable partitions [1; 13; 17]. Recall that  $\Pi = (B_j)$  is *exchangeable* if the law of  $\Pi$  is invariant under all bijections  $\sigma : \mathbb{N} \rightarrow \mathbb{N}$ . Partitions with weaker symmetry properties have also been studied. Pitman [16] introduced *partially exchangeable* random partitions of  $\mathbb{N}$  with the property that the law of  $\Pi$  is invariant under all bijections  $\sigma : \mathbb{N} \rightarrow \mathbb{N}$  that preserve the order of blocks, meaning that the sequence of the least elements of the sets  $\sigma(B_1), \sigma(B_2), \dots$  is also increasing. Pitman [16] derived a de Finetti-type representation for partially exchangeable partitions and established a criterion for their exchangeability. Kerov [14] studied a closely related structure of virtual permutations of  $\mathbb{N}$ , which may be seen as partially exchangeable partitions with some total ordering of elements within each of the blocks. Kallenberg [13] characterised *spreadable* partitions whose law is invariant under increasing injections  $\sigma : \mathbb{N} \rightarrow \mathbb{N}$ .

In this note we consider constrained random partitions of  $\mathbb{N}$  which satisfy the condition that, for a fixed integer sequence  $\rho = (\rho_1, \rho_2, \dots)$  with  $\rho_k \geq 1$ , each block  $B_k$  contains  $\rho_k$  least elements of  $\cup_{j \geq k} B_j$ , for every  $k$  with  $B_k \neq \emptyset$ . It is easy to check that this condition holds if and only if the sequence comprised of  $\rho_1$  least elements of  $B_1$ , followed by  $\rho_2$  least elements of  $B_2$  and so on, is itself an increasing sequence. We shall focus on the constrained partitions with the following symmetry property.

**Definition 1** For a given sequence  $\rho$ , we call  $\Pi$  *constrained exchangeable* if  $\Pi$  is a constrained partition with respect to  $\rho$  and the law of  $\Pi$  is invariant under all bijections  $\sigma : \mathbb{N} \rightarrow \mathbb{N}$  that preserve this property.

Since the law of  $\Pi$  is uniquely determined by the laws of finite restrictions  $\Pi_n$ , the constrained exchangeability of  $\Pi$  amounts to the analogous property of  $\Pi_n$ 's for each  $n = 1, 2, \dots$ . To gain a feeling of the property, the reader is suggested to check that for  $\rho = (1, 2, 1, \dots)$  the partition  $\Pi_8$  assumes the values  $(\{1, 3, 5\}, \{2, 4, 6\}, \{7, 8\})$  and  $(\{1, 2, 3\}, \{4, 5, 8\}, \{6, 7\})$  with the same probability.

Every partition of  $\mathbb{N}$  is constrained with respect to  $\rho = (1, 1, \dots)$ , and every constrained exchangeable partition with this  $\rho$  is partially exchangeable in the sense of Pitman [16]. In principle, any constrained exchangeable partition may be reduced to some Pitman's partially exchangeable partition by isolating  $\rho_k - 1$  least elements of  $B_k$  in  $\rho_k - 1$  singleton blocks, for each  $\rho_k > 1$ , but this viewpoint will not be adopted here.

For general  $\rho \neq (1, 1, \dots)$  the constrained exchangeable partitions which are also exchangeable are rather uninteresting, since they cannot have infinitely many blocks:

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**Proposition 2** *Let  $\Pi$  be a constrained partition with respect to some  $\rho$  which has  $\rho_k > 1$  for some  $k$ . If  $\Pi$  is exchangeable then  $\Pi$  has at most  $k$  nonempty blocks.*

**Proof:** Suppose  $B_k \neq \emptyset$ , then by Kingman’s representation of exchangeable partitions [17] the set  $\cup_{j \geq k} B_j$  contains infinitely many elements. For the same reason  $\#B_k \geq 2$  implies that  $B_k$  is an infinite set, and that partition  $\Pi'$  obtained by restricting  $\Pi$  to  $\cup_{j \geq k} B_j$  and re-labelling the elements of  $\cup_{j \geq k} B_j$  by  $\mathbb{N}$  in increasing order is an exchangeable partition of  $\mathbb{N}$ . But then with probability one  $\Pi'$  is the trivial single-block partition, because elements 1 and 2 are always in the same block.  $\square$

In many contexts where random partitions appear, exchangeability is an obvious kind of symmetry. Constrained exchangeability may appear when some initial elements of the blocks play a special role of ‘establishing’ the block. To illustrate, consider the following situation. Suppose there is a sequence of independent random points sampled from some distribution on  $\mathbb{R}^d$ . Define  $D_1$  as the convex hull of the first  $\rho_1$  points,  $D_2$  as the convex hull of the first  $\rho_2$  points not in  $D_1$ ,  $D_3$  as the convex hull of the first  $\rho_3$  points not in  $D_1 \cup D_2$ , etc. Divide  $\mathbb{R}^d$  in disjoint nonempty subsets  $G_1 = D_1, G_2 = D_2 \setminus D_1, G_3 = D_3 \setminus (D_1 \cup D_2), \dots$ . A constrained exchangeable partition  $\Pi$  of  $\mathbb{N}$  is defined then by assigning to block  $B_k$  the indices of  $\rho_k$  initial points that determine  $D_k$  and the indices of all further points that hit  $G_k$ .

Of course, there is nothing special in the convex hulls construction, and any other way of ‘spanning’ a spatial domain  $D_k$  on  $\rho_k$  sample points and then ‘peeling’ the space in  $G_k$ ’s will also result in a constrained exchangeable partition. An example of this kind related to multidimensional records will be given.

In what follows we extend Pitman’s [16] sequential realisation of partitions via frequencies of blocks, to cover arbitrary constrained exchangeable partitions. Generalising a result on exchangeable partitions [6] we shall also derive a central limit theorem for the number of blocks of finite partitions  $\Pi_n = \Pi|_{[n]}$  in one important case of partitions induced by a ‘stick-breaking’ scheme.

## 2 Constrained sampling

We fix throughout a sequence of positive integers  $\rho$ . Recall that a *composition* is a finite sequence of positive integers called parts, e.g.  $(3, 1, 2)$  is a composition of  $6 = 3 + 1 + 2$  with three parts. We say that a composition  $\lambda = (\lambda_1, \dots, \lambda_\ell)$  is a *constrained composition of  $n$*  if  $\lambda_j \geq \rho_j$  for  $j = 1, \dots, \ell - 1$  and  $|\lambda| := \sum \lambda_j = n$ .

For each  $\lambda$  a constrained composition of  $n$ , the following random algorithm, which may be called *constrained sampling*, yields another constrained composition  $\mu$  of  $n - 1$ . Imagine a row of boxes labeled  $1, \dots, \ell$  and occupied by  $\lambda_1, \dots, \lambda_\ell$  white balls. Let  $\Lambda_j := \lambda_j + \dots + \lambda_\ell, j \leq \ell$ . At the first step,  $\rho_1$  balls in box 1 are re-painted black and then a white ball is drawn uniformly at random from all  $\Lambda_1 - \rho_1$  white balls. If the ball drawn was in box 1, the ball is deleted and the new composition is  $\mu = (\lambda_1 - 1, \lambda_2, \dots, \lambda_\ell)$ , and if the ball drawn was in some other box, it is returned to the box and the process continues, so that at the second step  $\rho_2$  balls in box 2 are re-painted black and a white ball is drawn uniformly at random from boxes  $2, \dots, \ell$ . If the second ball drawn was in box 2, the ball is deleted and the new composition is  $\mu = (\lambda_1, \lambda_2 - 1, \lambda_3, \dots, \lambda_\ell)$ , and so on. If the procedure does not terminate in  $\ell - 1$  steps, then a ball is deleted from the last box and the new composition is  $\mu = (\lambda_1, \dots, \lambda_{\ell-1}, \lambda_\ell - 1)$ . By this description, for  $j < \ell$  the transition probability from  $\lambda$  to  $\mu = (\dots, \lambda_j - 1, \dots)$  is

$$\frac{\Lambda_2}{(\Lambda_1 - \rho_1)} \cdots \frac{\Lambda_j}{(\Lambda_{j-1} - \rho_{j-1})} \frac{(\lambda_j - \rho_j)}{(\Lambda_j - \rho_j)},$$

while the transition probability from  $\lambda$  to  $\mu = (\dots, \lambda_\ell - 1)$  is

$$\frac{\Lambda_2}{(\Lambda_1 - \rho_1)} \cdots \frac{\Lambda_\ell}{(\Lambda_{\ell-1} - \rho_{\ell-1})}.$$

A random sequence  $\mathcal{C} = (\mathcal{C}_n)$  of constrained compositions of integers  $n = 1, 2, \dots$  is called *consistent* if  $\mathcal{C}_{n-1}$  has the same law as the composition derived from  $\mathcal{C}_n$  by the above constrained sampling procedure, for each  $n > 1$ . Every consistent sequence  $(\mathcal{C}_n)$  is an inverse Markov chain with some co-transition probabilities depending only on  $\rho$ . By analogy with [5] a consistent sequence  $\mathcal{C}$  will be called a *constrained composition structure*.

If the constraints are determined by  $\rho = (1, 1, \dots)$ , the constrained sampling amounts to a co-transition rule related to the partially exchangeable partitions in [16]. The unconstrained sampling (corresponding to  $\rho = (0, 0, \dots)$ ) leads to composition structures studied in [5; 9; 10].

### 3 Basic representation

For  $\Pi$  a constrained partition of  $\mathbb{N}$  with blocks  $B_1, B_2, \dots$  we define, for each  $n = 1, 2, \dots$ , a composition  $\mathcal{C}_n$  of  $n$  as the finite sequence of positive values in  $\#(B_1 \cap [n]), \#(B_2 \cap [n]), \dots$ . We call this composition the *shape* of  $\Pi_n$  and write  $\mathcal{C}_n = \text{shape}(\Pi_n)$ .

The number of constrained partitions of  $[n]$  with shape  $\lambda$  is equal to

$$d(\lambda) := \prod_{j=1}^{\ell-1} \binom{\Lambda_j - \rho_j}{\lambda_j - \rho_j}. \quad (1)$$

Similarly, the number of partitions of  $[n]$  with shape  $\lambda = (\lambda_1, \dots, \lambda_\ell)$  and whose restriction on  $[n']$  (for  $n' < n$ ) has shape  $\mu = (\mu_1, \dots, \mu_k)$  is equal to

$$d(\lambda, \mu) := \left[ \prod_{j=1}^{\ell-1} \binom{M_j - \Lambda_j}{\mu_j - \lambda_j} \right] \binom{M_\ell - \Lambda_\ell - (\rho_\ell - \lambda_\ell)_+}{\mu_\ell - \rho_\ell \vee \lambda_\ell} \left[ \prod_{j=\ell+1}^{k-1} \binom{M_j - \rho_j}{\mu_j - \rho_j} \right], \quad (2)$$

where  $M_j = \mu_j + \dots + \mu_k, j \leq k$ .

Introduce a function of compositions

$$p(\lambda) := \mathbb{P}(\text{shape}(\Pi_n) = \lambda).$$

It is easy to check that the consistency of  $\Pi_n$ 's with respect to restrictions implies that the  $\mathcal{C}_n$ 's are consistent in the sense of constrained sampling, therefore appealing to Kolmogorov's measure extension theorem we have:

**Proposition 3** *The formula*

$$\mathbb{P}(\Pi_n = \cdot) = p(\text{shape}(\cdot))/d(\text{shape}(\cdot))$$

*establishes a canonical homeomorphism between the distributions of constrained exchangeable partitions of  $\mathbb{N}$  and constrained composition structures. Conditionally given  $\mathcal{C}_n = \text{shape}(\Pi_n) = \lambda$  the distribution of  $\Pi_n$  is uniform on the set of constrained partitions of  $[n]$  with shape  $\lambda$ .*

The following basic construction modifies the one exploited in [14; 16]. Let  $(P_1, P_2, \dots)$  be an arbitrary sequence of random variables satisfying  $P_k \geq 0$  and  $\sum_k P_k \leq 1$ . A constrained exchangeable partition  $\Pi$  directed by  $(P_k)$  is defined as follows. Conditionally given  $(P_k)$  the partition is obtained by successive extension of  $\Pi_n$  to  $\Pi_{n+1}$ , for each  $n = 1, 2, \dots$ , according to the rules: given  $\Pi_n$  with  $\text{shape}(\Pi_n) = (\lambda_1, \dots, \lambda_\ell)$ , the element  $n + 1$

- (i) joins the block  $B_j, j < \ell$ , with probability  $P_j$ ,
- (ii) if  $\lambda_\ell < \rho_\ell$  joins the block  $B_\ell$  with probability  $1 - \sum_{j=1}^{\ell-1} P_j$ ,
- (iii) and if  $\lambda_\ell \geq \rho_\ell$  joins the block  $B_\ell$  with probability  $P_\ell$  or starts the new block  $B_{\ell+1}$  with probability  $1 - \sum_{j=1}^{\ell} P_j$ .

Explicitly, for the function  $p$  of compositions we have the formula

$$p(\lambda) = d(\lambda) \mathbb{E} \left\{ \left[ \prod_{j=1}^{\ell-1} \left( 1 - \sum_{i=1}^{j-1} P_i \right)^{\rho_j} P_j^{\lambda_j - \rho_j} \right] \left( 1 - \sum_{i=1}^{\ell-1} P_i \right)^{\rho_\ell \wedge \lambda_\ell} P_\ell^{(\lambda_\ell - \rho_\ell)_+} \right\}. \quad (3)$$

The next de Finetti-type result states that the construction covers all possible constrained exchangeable partitions. The proof is only sketched, since it follows the same lines as in [14; 16].

**Proposition 4** *For  $\Pi$  a constrained exchangeable partition, the normalised shapes  $\text{shape}(\Pi_n)/n$  (considered as sequences padded by infinitely many zeroes) converge in the product topology with probability one to some random limit  $(P_1, P_2, \dots)$  satisfying  $P_j \geq 0$  and  $\sum_j P_j \leq 1$ . Conditionally given  $(P_k)$  the partition  $\Pi$  is recovered according to the above rules (i)-(iii).*

**Proof:** The key point is to show the existence of frequencies. This can be concluded from de Finetti's theorem for 0 – 1 exchangeable sequences by noting that the indicators  $1(m \text{ belongs to block } B_k)$  for  $m > n$  are conditionally exchangeable given that the block  $B_k$  has at least  $\rho_k$  representatives in  $[n]$ . Alternatively, one can use, as in [14], more direct Martin boundary arguments which exploit the explicit formulas (1) and (2) to show that the pointwise limit of ratios  $d(\cdot, \mu)/d(\mu)$ , as  $m = |\mu| \rightarrow \infty$ , exists if and only if  $\mu_j/m$  converge for every  $j$ .  $\square$

### 4 The formation sequence

Nacu [15] established that the law of a partially exchangeable partition is uniquely determined by the law of the increasing sequence of the least elements of blocks. We show that a similar result holds for every constrained exchangeable partition  $\Pi$  with general  $\rho$ .

We define the *formation sequence* to be the sequence obtained by selecting the  $\rho_k$ th least element of the block  $B_k$  of  $\Pi$ , for  $k = 1, 2, \dots$ . For a composition  $\lambda$  let  $q(\lambda)$  be the probability that the formation sequence starts with elements  $\lambda_1, \lambda_1 + \lambda_2, \dots, \lambda_1 + \dots + \lambda_\ell$ . Let  $(P_j)$  be the frequencies as in Section 3, and introduce the variables

$$H_k = 1 - \sum_{j=1}^k P_j, \text{ so } P_k = H_{k-1} - H_k,$$

where we set  $H_0 = 1$ . Then

$$q(\lambda_1, \dots, \lambda_\ell) = \mathbb{E} \left[ \prod_{j=1}^{\ell-1} \binom{\lambda_{j+1} - 1}{\rho_{j+1} - 1} H_j^{\rho_{j+1}} (1 - H_j)^{\lambda_{j+1} - \rho_{j+1}} \right]. \tag{4}$$

Comparing this with (3) written in the same variables (where  $H_0 = 1$ ) we obtain for constrained compositions

$$p(\lambda) = d(\lambda) \mathbb{E} \left\{ \left[ \prod_{j=1}^{\ell-1} H_{j-1}^{\rho_j} (H_{j-1} - H_j)^{\lambda_j - \rho_j} \right] H_{\ell-1}^{\rho_\ell \wedge \lambda_\ell} (H_{\ell-1} - H_\ell)^{(\lambda_\ell - \rho_\ell)_+} \right\}, \tag{5}$$

which leads to the following conclusion:

**Proposition 5** *There is an invertible linear transition from  $p$  to  $q$ . Hence each of these two functions on compositions uniquely determines the law of  $\Pi$ .*

**Proof:** The substantial part of the claim is showing that we can compute  $p$  from  $q$ . To that end, start by observing that  $p$  is uniquely determined by the values on compositions of the type  $(\lambda_1, \dots, \lambda_{\ell-1}, \rho_\ell)$ . To see that this follows from the consistency for various  $n$ , argue by induction in  $m = 0, \dots, \rho_\ell$  for compositions  $(\dots, \rho_\ell - m)$ . Now, for such compositions whose last part meets the constraint exactly, (5) and (4) involve the same factors of the type  $H_j^{\rho_j}$ , hence  $p$  can be reduced to  $q$  by expanding each factor  $(H_{j-1} - H_j)^k = ((1 - H_j) - (1 - H_{j-1}))^k$  using the binomial formula.  $\square$

### 5 The paintbox

Paintbox representations based on the uniform sampling from  $[0, 1]$  are often used to model exchangeable structures and their relatives [5; 8; 9; 17]. We shall design a version that is appropriate for constrained exchangeable partitions.

Let  $1 = H_0 \geq H_1 \geq H_2 \geq \dots \geq 0$  be an arbitrary nonincreasing random sequence. Let  $(U_n)$  be a sequence of independent  $[0, 1]$ -uniform random points, independent of  $(H_k)$ . We define a new sequence  $(U_n) |_\rho(H_k)$  with some of  $U_n$ 's replaced by  $H_k$ 's, as follows. Replace  $U_1, \dots, U_{\rho_1}$ , by  $H_1$ . Then replace the first  $\rho_2$  entries which belong to  $U_{\rho_1}, U_{\rho_1+1}, \dots$  and hit  $[0, H_1[$  by  $H_2$ . Inductively, when  $H_1, \dots, H_k$  get used, respectively,  $\rho_1, \dots, \rho_k$  times, keep on screening uniforms until replacing the first  $\rho_{k+1}$  points hitting  $[0, H_k[$  by  $H_{k+1}$ . Eventually all  $H_k$ 's will enter the resulting sequence.

The construction has an interpretation in terms of the classical theory of records (see [7; 14] for a special case).

**Proposition 6** *Conditionally given  $(H_k)$ , the sequence  $(U_n) |_\rho(H_k)$  has the same distribution as  $(U_n)$  conditioned on the event that the sequence of lower records in  $(U_n)$  is  $(H_k)$ , with the record value  $H_k$  repeated  $\rho_k$  times.*

In this framework, we define a partition  $\Pi$  by assigning to block  $B_k$  all integers which label the entries of  $(U_n) |_\rho(H_k)$  falling in  $[H_k, H_{k-1}[$ . Given  $(H_k)$ , the chance for  $U_n$  to hit  $[H_j, H_{j-1}[$  is  $P_j = H_{j-1} - H_j$ , therefore the construction is equivalent to that defined by the rules (i)-(iii) above.

## 6 Stick-breaking partitions

Explicit evaluation of the function  $p$  is possible when the frequencies involve a kind of independence. To this end, it is convenient to introduce yet another set of variables  $(W_k)$  (sometimes called *residual fractions*) which satisfy  $H_k = W_1 \cdots W_k, W_k \in [0, 1]$ . In these variables (5) becomes

$$p(\lambda) = d(\lambda) \mathbb{E} \left\{ \left[ \prod_{j=1}^{\ell-1} W_j^{\Lambda_j - \lambda_j} (1 - W_j)^{\lambda_j - \rho_j} \right] (1 - W_\ell)^{(\lambda_\ell - \rho_\ell)_+} \right\} \tag{6}$$

where  $\Lambda_j = \lambda_j + \dots + \lambda_\ell$ . As in [16], if the  $W_k$ 's are independent, (6) assumes the product form

$$p(\lambda) = \prod_{k=1}^{\ell} q_k(\Lambda_k : \lambda_k) \tag{7}$$

with the *decrement matrices*

$$q_k(n : m) = \binom{n - \rho_k}{m - \rho_k} \mathbb{E} [(1 - W_k)^{m - \rho_k} W_k^{n - m}], \quad 1 \leq m \leq n, \tag{8}$$

and the convention  $\binom{-i}{-j} = 1 (i = j)$  for negative arguments of the binomial coefficients. In fact, (7) forces representation (8) (this fact is implicit in [14; 16] in the case of partially exchangeable partitions):

**Proposition 7** *A constrained composition structure  $(\mathcal{C}_n)$  satisfies (7) with some decrement matrices  $q_k, k = 1, 2, \dots$ , if and only if there exist independent  $[0, 1]$ -valued random variables  $(W_k)$  such that (8) holds.*

**Proof:** We argue the ‘only if’ part. For  $n < \rho_1$  we have  $q_1(n : n) = 1$  by definition. For  $n \geq \rho_1$  the constrained sampling consistency yields

$$q_1(n : m) = \frac{m + 1 - \rho_1}{n + 1 - \rho_1} q_1(n + 1 : m + 1) + \frac{n + 1 - m}{n + 1 - \rho_1} q_1(n + 1 : m)$$

which is the familiar Pascal-triangle recursion in the variables  $n - \rho_1, m - \rho_1$ , therefore the integral representation (8) follows as a known consequence of the Hausdorff moments problem. The case  $k > 1$  is completely analogous. The independence of the  $W_k$ 's is obvious from (7).  $\square$

We note in passing that the product formula (7) with a single decrement matrix leads, in a related setting of regenerative composition structures, to a nonlinear recursion and a very different conclusion [9]. See [11] for product formulas of another kind in the exchangeable case.

Suppose now that  $W_k$ 's are independent and have beta( $a_k, b_k$ ) distributions, whose density is

$$(1 - s)^{a_k - 1} s^{b_k - 1} / \mathbb{B}(a_k, b_k).$$

The rows of the decrement matrices are then Pólya-Eggenberger distributions

$$q_k(n : m) = \binom{n - \rho_k}{m - \rho_k} \frac{(a_k)_{m - \rho_k} (b_k)_{n - m}}{(a_k + b_k)_{n - \rho_k}}, \quad m = 1, \dots, n.$$

For instance, taking positive integer  $a_k, b_k$  and  $\rho_k = a_k + b_k - 1$ , a partition  $\Pi$  is constructed as follows: replace  $U_1, \dots, U_{\rho_1}$  by the value  $H_1$  equal to the  $b_1$ th minimal order statistic of these points, then replace the first  $\rho_2$  uniforms that hit  $[0, H_1[$  by the value  $H_2$  equal to the  $b_2$ th minimal order statistic of these hits, etc, thus defining a partition via  $(U_n) |_{\rho}(H_k)$ . A distinguished class of structures of this kind is the Ewens-Pitman two-parameter family of exchangeable partitions [12; 16; 17] with  $\rho = (1, 1, \dots), a_k = \theta + k\alpha$  and  $b_k = 1 - \alpha$  (for suitable  $\alpha$  and  $\theta$ ); the product formula simplifies in this case due to a major telescoping of factors.

## 7 Counting the blocks

Let  $K_n$  be the number of blocks of  $\Pi_n$ , which in the  $(U_j) |_{\rho}(H_k)$ -representation coincides with the number of intervals  $]H_1, H_0], ]H_2, H_1], \dots$  discovered by the  $n$  first terms of the sequence. Conditionally given  $(H_k), K_n$  is the number of certain independent geometric summands which sum to no more than  $n$ . In

particular, the difference between the  $k$ th and the  $(k + 1)$ st entries of the formation sequence follows the negative binomial distribution with parameters  $\rho_k, H_k$ .

We shall proceed by assuming a ‘stick-breaking’ scheme  $H_k = W_1 \cdots W_k$ ,  $k = 1, 2, \dots$ , with independent identically distributed  $W_k$ ’s. We assume further that the logarithmic moments  $\mu = \mathbb{E}[-\log W_1]$ ,  $\sigma^2 = \text{Var}[-\log W_1]$  are both finite. The idea is to derive a CLT for  $K_n$  from the standard CLT for renewal processes [4]. Similar technique was used in [6; 7], but in the new situation we need to also limit the growth of  $\rho_k$  as  $k \rightarrow \infty$ .

We will show that  $K_n$  is asymptotic to  $J_n := \max\{k : H_k > 1/n\}$ . The last quantity is indeed asymptotically Gaussian with the mean  $(\log n)/\mu$  and the variance  $(\log n)/(\sigma^2\mu^{-3})$  because  $J_n$  is just the number of renewal epochs within  $[0, \log n]$  of the renewal process with steps  $-\log W_k$ . In loose terms, we will exploit a ‘cut-off phenomenon’: typically, only a few points out of  $n$  uniforms fall below  $1/n$ , while for  $k < J_n$  essentially all intervals get hit, with exponentially growing occupancy numbers when scanned backwards in  $k$  from  $k = J_n$ .

**Proposition 8** *Suppose  $\Pi$  is directed by  $H_k = W_1 \cdots W_k$ , where for  $k = 1, 2, \dots$  the  $W_k$ ’s are i.i.d. with finite logarithmic moments  $\mu = \mathbb{E}[-\log W_1]$ ,  $\sigma^2 = \text{Var}[-\log W_1]$ . If*

$$\log \left[ \sum_{j=1}^k \rho_j \right] = o(k), \text{ as } k \rightarrow \infty, \tag{9}$$

*then the strong law of large numbers holds, i.e.  $K_n \sim \mu^{-1} \log n$  a.s.. Moreover, the random variable  $(K_n - \mathbb{E}[K_n])/\sqrt{\text{Var}[K_n]}$  converges in law to the standard Gaussian distribution, whereas the moments satisfy*

$$\mathbb{E}[K_n] \sim \frac{\log n}{\mu}, \quad \text{Var}[K_n] \sim \frac{\log n}{\sigma^2\mu^{-3}}. \tag{10}$$

**Proof:** By the construction of  $(U_j) |_{\rho(H_k)}$ , we have a dichotomy:  $U_n \in ]H_k, H_{k-1}]$  implies that either  $U_n$  will enter the transformed sequence or will get replaced by some  $H_i \geq H_k$ . Let  $U_{1n} < \dots < U_{nn}$  be the order statistics of  $U_1, \dots, U_n$ . It follows that

- (i) if  $U_{jn} > H_k$  then  $K_n \leq j + k$ ,
- (ii) if  $U_{mn} < H_k$  for  $m = \sum_{i=1}^k \rho_k$  then  $K_n \geq k$ .

Define  $\xi_n$  by  $U_{\xi_n, n} < 1/n < U_{\xi_n+1, n}$  and recall that  $J_n$  was defined by  $H_{J_n+1} \leq 1/n < H_{J_n}$ , thus  $\xi_n$  is the number of uniforms to the left of  $1/n$ , and  $J_n$  follows the CLT. Clearly,  $J_n$  and  $\xi_n$  are independent and  $\xi_n$  is binomial( $n, 1/n$ ). By (i), we have  $K_n \leq J_n + \xi_n$  where  $\xi_n$  is approximately Poisson(1), which yields the desired upper bound.

The lower bound is more delicate. Introduce

$$\psi_n = c \sum_{j=1}^{\lfloor \log n \rfloor} \rho_j$$

where  $c$  should be selected sufficiently large. Then by the assumption (9)  $\log \psi_n = o(\log n)$ , which is enough to assure that the number, say  $L_n$ , of the  $H_k$ ’s larger than  $\psi_n/n$  is still asymptotic to  $J_n$ . Because  $L_n$  is close to Gaussian with moments as in (10), an easy large deviation estimate implies that the inequality  $L_n < (c/2) \log n$  holds with probability at least  $1 - n^{-2}$ . On the other hand, the number of uniforms smaller  $\phi_n/n$  is also close to Gaussian with both central moments about  $\phi_n$ , hence, in view of  $\phi_n > c \log n$ , a similar estimate shows that this number is at least  $(c/2) \log n$  with probability at least  $1 - n^{-2}$ . By application of (ii) with  $k = L_n$  we see that the lower bound  $K_n > L_n$  holds up to an event of probability  $O(n^{-2})$ . This completes the proof of the CLT. Finally, since both  $J_n$  and  $L_n$  are asymptotic to  $\mu^{-1} \log n$  almost surely, the Borel-Cantelli lemma implies that the same is valid for  $K_n$ .  $\square$

## 8 A continuous time process

The sequential construction of  $\Pi$  from the frequencies  $(P_k)$  can be embedded in continuous time by letting the elements  $1, 2, \dots$  arrive at epochs of a rate-1 Poisson process on  $\mathbb{R}_+$ . Let  $R_t$  be the total frequency of

the blocks which are not represented by the elements arrived before  $t$ , then  $R = (R_t)$  is a nonincreasing pure-jump process with piecewise-constant paths and  $R_0 = 1$ .

Suppose as in Section 7 that  $W_k$ 's are independent and identically distributed. The process  $R$  is then easy to describe: if after the  $(k - 1)$ st jump the process  $R$  is in state  $s$  then the time in this state has distribution  $\text{gamma}(\rho_k, s)$ , and thereafter the state is changed to  $sW_k$ . The sojourns in consecutive states  $1, W_1, W_1W_2, \dots$  are independent. The instance  $\rho = (1, 1, \dots)$  corresponds to a known self-similar Markov process which appears as a ‘tagged particle’ process in random fragmentation models [1]. For general  $\rho$  the process is no longer Markovian, as one needs to also include the time spent in the current state to summarise the history.

A minor adjustment of Proposition 8 to the continuous-time setting allows to conclude that under the same assumptions the number of jumps during the time  $[0, T]$  is approximately Gaussian, as  $T \rightarrow \infty$ . In fact, the process  $R$  is well defined for arbitrary positive values  $\rho_k$  ( $k = 1, 2, \dots$ ), in which case an analogous CLT is readily acquired by interpolation from the case of integer  $\rho_k$ 's.

## 9 Example: the chain records

Next is an example of Pitman’s partially exchangeable partitions, so the constraint is  $\rho = (1, 1, \dots)$ . Consider a Borel space  $\mathcal{Z}$  endowed with a distribution  $\mu$  and some measurable strict partial order  $\prec$ . For a sample  $V_1, V_2, \dots$  from  $(\mathcal{Z}, \mu)$ , we say that a *chain record* occurs at index  $j$  if either  $j = 1$ , or  $j > 1$  and  $V_j$  is  $\prec$ -smaller than the last chain record in the sequence  $V_1, \dots, V_{j-1}$ . The instance of  $\mathbb{R}^d$  with the natural coordinate-wise partial order was discussed in [7].

Let  $R_k, k = 1, 2, \dots$ , be the sample values when the chain records occur; the sequence  $(R_k)$  is a ‘greedy’ falling chain of the partially ordered sample  $(V_j)$ . Introducing the lower sets  $L_v := \{u \in \mathcal{Z} : u \prec v\}$ , we define  $D_k := L_{R_k}, G_k := D_k \setminus D_{k-1}$  (where  $D_0 := \emptyset$ ), and we define a constrained exchangeable partition  $\Pi = (B_k)$  as in Section 1. The frequencies of  $B_k$ 's are  $P_k = \mu(L_{G_k})$ , and we have  $H_k = \mu(L_{R_k})$ , as is easily seen.

To guarantee a ‘stick-breaking’ form of  $(H_k)$ , as in Section 6, we need to assume a self-similarity property of the sampling space. We may call  $(\mathcal{Z}, \mu, \prec)$  *regenerative* if (i)  $\mu(L_v) > 0$  for  $\mu$ -almost all points  $v \in \mathcal{Z}$ , and (ii) the lower section  $L_v$  with conditional measure  $\mu(\cdot)/\mu(L_v)$  is isomorphic, as a partially ordered probability space, to the whole space  $(\mathcal{Z}, \mu, \prec)$ . Since all  $L_v$ 's are in this sense the same, the  $H_k$ 's undergo stick-breaking with i.i.d. residual fractions whose distribution is the same as that of  $L_{V_1}$ . Under the hypothesis of Proposition 8, the number of chain records among the first  $n$  sample points is approximately Gaussian, since this number coincides with the number of blocks of  $\Pi_n$ . A class of regenerative spaces is comprised of the Bollobás-Brightwell box-spaces [3], which have all intervals  $\{u : v \prec u \prec w\}$  for  $v \prec w$  isomorphic to the whole space (and not only lower sections).

Further examples of regenerative spaces appear, in a disguise, in the context of multidimensional data structures like quad-trees or simplex-trees [2]. More generally, constrained exchangeability appears in connection with data structures which allow multiple key storage at a node of the search tree.

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