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► **To cite this version:**

Elena Dyakonova. Survival probability of a critical multi-type branching process in random environment. Fourth Colloquium on Mathematics and Computer Science Algorithms, Trees, Combinatorics and Probabilities, 2006, Nancy, France. pp.375-380, 10.46298/dmtcs.3477 . hal-01184679

**HAL Id: hal-01184679**

**<https://inria.hal.science/hal-01184679>**

Submitted on 17 Aug 2015

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# Survival probability of a critical multi-type branching process in random environment

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We study a multi-type branching process in i.i.d. random environment. Assuming that the associated random walk satisfies the Doney-Spitzer condition, we find the asymptotics of the survival probability at time  $n$  as  $n \rightarrow \infty$ .

**Keywords:** branching processes in random environment, Doney-Spitzer condition, survival probability

## Introduction

Branching processes in random environment constitute an important part of the theory of branching processes (see, for example, (1), (2), (4)-(7), (9)-(14)). A branching process in random environment was first considered by Smith and Wilkinson (10). The subsequent papers (2), (7), (11) investigated single- and multi-type Galton-Watson processes in random environment. The asymptotics of the survival probability of the critical branching processes in a random environment generated by a sequence of independent identically distributed random variables under the condition  $\mathbf{E}X^2 < \infty$  for the increment  $X$  of the associated random walk was found in (6), (9) for single-type processes, and in (4) for multi-type processes. Recent papers (1), (5), (12)-(14) study the survival probability for an extended class of the critical single-type branching processes in random environment where the case  $\mathbf{E}X^2 = \infty$  is not excluded and, moreover,  $\mathbf{E}X$  may not exist. The present paper investigates an extended class of multi-type critical branching processes in random environment whose associated random walks satisfy the Doney-Spitzer condition. In particular, we generalize some results established in (1) and (4) concerning the asymptotic behavior of survival probability.

Let  $Z(n) = (Z_1(n), \dots, Z_p(n))$ ,  $n = 0, 1, \dots$ , be a  $p$ -type Galton-Watson branching process in a random environment. This process can be described as follows.

Let  $\mathbf{N}_0 = \{0, 1, 2, \dots\}$  and  $\mathbf{N}_0^p$  be the set of all vectors  $t = (t_1, \dots, t_p)$  with non-negative integer coordinates. Denote by  $(\Delta_1, \mathcal{B}(\Delta_1))$  a set of probability measures on  $\mathbf{N}_0^p$  with  $\sigma$ -algebra  $\mathcal{B}(\Delta_1)$  of Borel sets endowed with the metric of total variation, and by  $(\Delta, \mathcal{B}(\Delta))$  the  $p$ -times product of the space  $(\Delta_1, \mathcal{B}(\Delta_1))$  on itself. Let  $\mathbf{F} = (\mathbf{F}^{(1)}, \dots, \mathbf{F}^{(p)})$  be a random variable (random measure) taking values in  $(\Delta, \mathcal{B}(\Delta))$ . An infinite sequence  $\Pi = (\mathbf{F}_0, \mathbf{F}_1, \mathbf{F}_2, \dots)$  of independent identically distributed copies of  $\mathbf{F}$  is said to form a random environment and we will say that  $\mathbf{F}$  generates  $\Pi$ . A sequence of random  $p$ -dimensional vectors  $Z(0), Z(1), Z(2), \dots$  with non-negative integer coordinates is called a  $p$ -type branching process in random environment  $\Pi$ , if  $Z(0)$  is independent of  $\Pi$  and for all  $n \geq 0, z = (z_1, \dots, z_p) \in \mathbf{N}_0^p$  and  $f_0, f_1, \dots \in \Delta$

$$\begin{aligned} \mathcal{L}(Z(n+1) \mid Z(n) = (z_1, \dots, z_p), \Pi = (f_1, f_2, \dots)) \\ &= \mathcal{L}(Z(n+1) \mid Z(n) = (z_1, \dots, z_p), \mathbf{F}_n = f_n) \\ &= \mathcal{L}((\xi_{n,1}^{(1)} + \dots + \xi_{n,z_1}^{(1)}) + (\xi_{n,1}^{(2)} + \dots + \xi_{n,z_2}^{(2)}) + \dots + (\xi_{n,1}^{(p)} + \dots + \xi_{n,z_p}^{(p)})), \end{aligned} \quad (1)$$

where  $f_n = (f_n^{(1)}, f_n^{(2)}, \dots, f_n^{(p)}) \in \Delta$ ,  $\xi_{n,i}^{(i)}, \xi_{n,2}^{(i)}, \dots, \xi_{n,z_i}^{(i)}, i = 1, \dots, p$ , are independent  $p$ -dimensional random vectors, and for each  $i = 1, \dots, p$  the random vectors  $\xi_{n,1}^{(i)}, \xi_{n,2}^{(i)}, \dots, \xi_{n,z_i}^{(i)}$  are identically distributed according to the measure  $f_n^{(i)}$ . Relation (1) defines a branching Galton-Watson process  $Z(n)$  in random environment which describes the evolution of a particle population  $Z(n) = (Z_1(n), \dots, Z_p(n))$ ,  $n = 0, 1, \dots$ , where  $Z_i(n), i = 1, \dots, p$ , is the number of type  $i$  particles in the  $n$ -th generation.

<sup>†</sup>Supported by RFBR grant 05-01-00035, grant Scientific School-4129.2006.1 and by the program "Contemporary Problems of Theoretical Mathematics" of the Russian Academy of Sciences.

This population evolves as follows. If  $\mathbf{F}_n = f_n$  then each of the  $Z_i(n)$  particles of type  $i$  existing at the time  $n$ , produces offspring in accordance with the  $p$ -dimensional probability measure  $f_n^{(i)}$  independently of the reproduction of other particles. Thus, the  $i$ -th component of the vector  $Z(n+1) = (Z_1(n+1), \dots, Z_p(n+1))$  is equal to the number of type  $i$  particles among all direct descendants of the particles of the  $n$ -th generation. The distribution of  $Z(0)$  will be specified later.

### The main results

Let  $J^p$  be the set of all column vectors  $s = (s_1, \dots, s_p)^T, 0 \leq s_i \leq 1, i = 1, \dots, p$ . For  $s \in J^p$  and  $t \in \mathbf{N}_0^p$  set  $s^t = \prod_{i=1}^p s_i^{t_i}$ . Taking into account existence of a one-to-one correspondence between probability measures and generating functions we associate with  $\mathbf{F} = (\mathbf{F}^{(1)}, \dots, \mathbf{F}^{(p)})$  generating  $\Pi$  a random  $p$ -dimensional column vector  $F(s) = (F^{(1)}(s), \dots, F^{(p)}(s))^T, s \in J^p$ , whose components are  $p$ -dimensional (random) generating functions  $F^{(i)}(s)$  corresponding to  $\mathbf{F}^{(i)}, 1 \leq i \leq p$ :

$$F^{(i)}(s) = \sum_{t \in \mathbf{N}_0^p} \mathbf{F}^{(i)}(\{t\})s^t, s \in J^p.$$

In a similar way we associate with the component  $\mathbf{F}_n = (\mathbf{F}_n^{(1)}, \dots, \mathbf{F}_n^{(p)}), n \geq 0$ , of the random environment  $\Pi = (\mathbf{F}_0, \mathbf{F}_1, \mathbf{F}_2, \dots)$  a random vector  $F_n(s) = (F_n^{(1)}(s), \dots, F_n^{(p)}(s))^T, s \in J^p$ , the components of which are multidimensional (random) generating functions  $F_n^{(i)}(s)$ , corresponding to  $\mathbf{F}_n^{(i)}, 1 \leq i \leq p$ ,

$$F_n^{(i)}(s) = \sum_{t \in \mathbf{N}_0^p} \mathbf{F}_n^{(i)}(\{t\})s^t.$$

Let  $e_j, j = 1, \dots, p$ , be the  $p$ -dimensional row vector whose  $j$ -th component is equal to 1 and the others are zeros,  $\bar{0} = (0, \dots, 0)$  be the  $p$ -dimensional row vector all whose components are zeros, and let  $\bar{1} = (1, \dots, 1)^T$  be the  $p$ -dimensional column vector all whose components are equal to 1. For  $x = (x_1, \dots, x_p)$  and  $y = (y_1, \dots, y_p)^T$  we set  $|x| = \sum_{i=1}^p |x_i|, |y| = \sum_{i=1}^p |y_i|, (x, y) = \sum_{i=1}^p x_i y_i$ . Let  $A = \|A(i, j)\|_{i,j=1}^p$  be an arbitrary positive  $p \times p$  matrix. Denote by  $\rho(A)$  the Perron root of  $A$  and by  $u(A) = (u_1(A), \dots, u_p(A))^T$  and  $v(A) = (v_1(A), \dots, v_p(A))$  the right and left eigenvectors of  $A$  corresponding to the eigenvalue  $\rho(A)$  and such that

$$|v(A)| = 1, (v(A), u(A)) = 1.$$

For vector-valued generating functions  $F(s)$  and  $F_n(s)$  we introduce the mean matrices

$$M = M(\mathbf{F}) = \|M(i, j)\|_{i,j=1}^p = \left\| \frac{\partial F^{(i)}(\bar{1})}{\partial s_j} \right\|_{i,j=1}^p$$

and

$$M_n = M_n(\mathbf{F}_n) = \|M_n(i, j)\|_{i,j=1}^p = \left\| \frac{\partial F_n^{(i)}(\bar{1})}{\partial s_j} \right\|_{i,j=1}^p.$$

Let  $\mathcal{C}_\alpha, 0 < \alpha < 1$ , be the class of all matrices  $A = \|A(i, j)\|_{i,j=1}^p$  such that

$$\alpha \leq \frac{A(i_1, j_1)}{A(i_2, j_2)} \leq \alpha^{-1}, 1 \leq i_1, i_2, j_1, j_2 \leq p.$$

One of our basic hypotheses is the following condition.

**Assumption A0.** There exist a number  $0 < \alpha < 1$  and a positive row vector  $v = (v_1, \dots, v_p), |v| = 1$ , such that, with probability 1

$$M = M(\mathbf{F}) \in \mathcal{C}_\alpha,$$

and

$$vM = \rho(M)v. \tag{2}$$

Set  $\rho = \rho(M), \rho_n = \rho(M_n), n \geq 0$ . It is not difficult to see that in our settings  $X := \ln \rho, X_i := \ln \rho_{i-1}, i \geq 1$ , are independent and identically distributed random variables. Our next hypothesis imposes a restriction on the so-called associated random walk  $S = (S_0, S_1, \dots)$ , where

$$S_n = X_1 + \dots + X_n, n \geq 1, S_0 = 0.$$

**Assumption A1.** There exists a number  $0 < a < 1$  such that

$$\mathbf{P}(S_n > 0) \rightarrow a, n \rightarrow \infty. \tag{3}$$

Extending the known classification of single-type branching processes in random environment (see (1), (12)), we call a  $p$ -type branching process  $Z(n), n \geq 0$ , in random environment  $\Pi$  critical if its associated random walk is of the oscillating type, i.e.,  $\limsup_{n \rightarrow \infty} S_n = +\infty$  a.s. and  $\liminf_{n \rightarrow \infty} S_n = -\infty$  a.s. It is known that any random walk satisfying Assumption A1 oscillates. From now on we consider only critical  $p$ -type branching processes in random environment.

Let  $0 =: \gamma_0 < \gamma_1 < \dots$  be the strict descending ladder epochs of  $S$ . Put

$$V(x) := \sum_{i=0}^{\infty} \mathbf{P}(S_{\gamma_i} \geq -x), x \geq 0; V(x) = 0, x < 0.$$

Since  $S$  is oscillating, the following relation holds (3):

$$\mathbf{E}V(x + X) = V(x), x \geq 0. \tag{4}$$

For  $d \in \mathbf{N}_0$  set

$$O_d = \{t = (t_1, \dots, t_p) \in \mathbf{N}_0^p \mid t_i < d, i = 1, \dots, p\}, U_d = \mathbf{N}_0^p \setminus O_d.$$

Introduce the random variable

$$\kappa(d) = \sum_{t \in U_d} \sum_{i=1}^p v_i \sum_{j,k=1}^p \mathbf{F}^{(i)}(\{t\}) t_j t_k / \rho^2, d \in \mathbf{N}_0,$$

where  $v = (v_1, \dots, v_p)$  is from (2). Our next condition is connected with the random variable  $\kappa(d)$ , which is a generalization of the standardized truncated second moment of the reproduction law to the multi-type case.

**Assumption A2.** There exist  $\varepsilon > 0$  and  $d \in \mathbf{N}_0$  such that

$$\mathbf{E}(\ln^+ \kappa(d))^{1/a+\varepsilon} < \infty, \mathbf{E} \left( V(X)(\ln^+ \kappa(d))^{1/a+\varepsilon} \right) < \infty.$$

Let  $T = \min\{n \geq 0 : Z(n) = \bar{0}\}$  be the extinction moment for  $Z(n)$ . Introduce the random variables

$$Q^{(i)}(n) = \mathbf{P}(T > n \mid Z(0) = e_i, \Pi), Q(n) = (Q^{(1)}(n), \dots, Q^{(p)}(n)),$$

and let

$$q_i(k) = \mathbf{P}(T > k \mid Z(0) = e_i) = \mathbf{E}Q^{(i)}(k).$$

Note that under Assumptions A0 and A1  $Q^{(i)}(n) \rightarrow 0$   $\mathbf{P}$ -a.s. as  $n \rightarrow \infty$  for all  $1 \leq i \leq p$ , since  $\mathbf{P}$ -a.s.

$$(v, Q(n)) \leq \min_{0 \leq k \leq n-1} |vM_0 \cdots M_k| \leq \exp\left\{ \min_{0 \leq k \leq n-1} S_k \right\} \rightarrow 0$$

as  $n \rightarrow \infty$ . Denote by  $u(n) = (u_1(n), \dots, u_p(n))^T := u(M_0 \cdots M_n), n \geq 0$ , the right eigenvector of the product  $M_0 \cdots M_n$ , corresponding to the Perron root  $\rho(M_0 \cdots M_n) = \rho_0 \cdots \rho_n$ . To investigate the asymptotic behavior of  $q_i(n)$  and  $Q^{(i)}(n)$  as  $n \rightarrow \infty$  we need the following statement describing the behavior of  $u(n)$ .

**Theorem 1** *If Assumption A0 is valid, then there exist a random vector  $u = (u_1, \dots, u_p)^T$  and a function  $g(n) \geq 0, g(n) \rightarrow 0, n \rightarrow \infty$ , such that with probability 1*

$$|u_i(n) - u_i| \leq g(n), i = 1, \dots, p.$$

In addition,

$$(v, u) = 1, \alpha \leq u_i \leq 1/v^*,$$

where  $v^* = \min(v_1, \dots, v_p)$  and  $v = (v_1, \dots, v_p)$  is from (2).

The following statement describes the behavior of  $Q(n)$  as  $n \rightarrow \infty$ .

**Theorem 2** Assume Assumptions A0 and A1. Then  $\mathbf{P}$ -a.s., as  $n \rightarrow \infty$ ,

$$\frac{Q_i(n)}{(v, Q(n))} \rightarrow u_i, \quad i = 1, \dots, p,$$

where  $u = (u_1, \dots, u_p)$  is from Theorem 1.

Now we are ready to formulate the main result of the paper.

**Theorem 3** Assume Assumptions A0, A1, and A2. Then, as  $n \rightarrow \infty$ ,

$$q_i(n) \sim c_i n^{-(1-a)} l(n), \quad c_i > 0, \quad i = 1, \dots, p,$$

where  $l(n)$  is a function slowly varying at infinity.

Note that under our approach one of the key facts to prove Theorems 1, 2, 3 is convergence in distribution, as  $n \rightarrow \infty$ , of the products  $\prod_{i=0}^n M_i \rho_i^{-1}$  of random matrices to a limit matrix whose distribution is not concentrated at zero matrix. It is known (8) that for  $p = 2$  the products  $\prod_{i=0}^n M_i \rho_i^{-1}$  of the positive bounded independent identically distributed  $2 \times 2$  matrices  $A_i = M_i \rho_i^{-1}$  converges in distribution, as  $n \rightarrow \infty$ , to a limit matrix whose distribution is not concentrated at zero matrix if and only if all the matrices  $A_i$  have a common positive right or left eigenvector. Hence, for the 2-type process  $Z(n)$  our assumption on existence of a common positive left eigenvector of the matrices  $M$  is essential indeed.

Observe also that Assumption A1 covers non-degenerate random walks with zero mean and finite variance of there increments, as well as all non-degenerate symmetric random walks. In these cases  $a = 1/2$ . Another example when Assumption A1 is valid gives the random walk, whose increments have distribution belonging to the domain of attraction of a stable law.

In conclusion we give an example where Assumption A2 is fulfilled (given that Assumption A0 is valid as well). Clearly, if the measure  $\mathbf{F}$  generating our random environment has a bounded support, i.e., if there exists a  $p$ -dimensional cube  $B = [0, b]^p, b > 0$ , such that  $\mathbf{P}(\mathbf{F}(B) = 1) = 1$ , then Assumption A2 holds since  $\kappa(d) = 0$   $\mathbf{P}$ -a.s. for  $d > b$ .

One can show that if  $\mathbf{F}$  satisfies Assumption A0 and

$$F(s) = \bar{1} - \frac{M(\bar{1} - s)}{1 + \gamma(\bar{1} - s)}, \quad s \in J^p,$$

where the  $p$ -dimensional random row vector  $\gamma$  with positive components and the random matrix  $M$  are such that the components of the vector  $y = (M\bar{1})/|\gamma|$  are uniformly bounded from below then Assumption A2 holds true.

Note that if the distribution of  $X = \ln \rho$  has a regular varying tail then Assumptions A1 and A2 can be replaced by the following hypotheses (see (13) or (1)):

**Assumption A1'**. There exist constants  $c_n, n \geq 0$ , such that as  $n \rightarrow \infty$  the scaled sums  $c_n S_n$  converge weakly to a stable distribution  $\mu$  with parameter  $\beta \in (0, 2]$ . The limit law  $\mu$  is not one-side, i.e.,  $0 < \mu(\mathbb{R}^+) < 1$ .

**Assumption A2'**. There exist  $\varepsilon > 0$  and  $d \in \mathbb{N}_0$  such that

$$\mathbf{E}(\ln^+ \kappa(d))^{\beta+\varepsilon} < \infty,$$

where  $\beta$  is from Assumption A1'.

Note that Assumption A1' implies the validity of Assumption A1 with  $a = \mu(\mathbb{R}^+)$ , and Assumption A2 is stronger than Assumption A2' since  $a\beta \leq 1$ .

**Corollary 1** Assume Assumptions A0 and A1'. Then the statement of Theorem 2 remains true.

Using the proof of Theorem 3 and results of paper (1), one can obtain also the following statement.

**Theorem 4** Assume Assumptions A0, A1', and A2'. Then the statement of Theorem 3 remains true.

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