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# Color critical hypergraphs and forbidden configurations

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The present paper connects sharpenings of Sauer’s bound on forbidden configurations with color critical hypergraphs. We define a matrix to be *simple* if it is a  $(0,1)$ -matrix with no repeated columns. Let  $F$  be a  $k \times l$   $(0,1)$ -matrix (the forbidden configuration). Assume  $A$  is an  $m \times n$  simple matrix which has no submatrix which is a row and column permutation of  $F$ . We define  $\text{forb}(m, F)$  as the best possible upper bound on  $n$ , for such a matrix  $A$ , which depends on  $m$  and  $F$ . It is known that  $\text{forb}(m, F) = O(m^k)$  for any  $F$ , and Sauer’s bound states that  $\text{forb}(m, F) = O(m^{k-1})$  for *simple*  $F$ . We give sufficient condition for non-simple  $F$  to have the same bound using linear algebra methods to prove a generalization of a result of Lovász on color critical hypergraphs.

**Keywords:** forbidden configuration, color critical hypergraph, linear algebra method

## 1 Introduction

A  $k$ -uniform hypergraph  $(V, \mathcal{E})$  is 3-color critical if it is not 2-colorable, but for all  $E \in \mathcal{E}$  the hypergraph  $(V, \mathcal{E} \setminus \{E\})$  is 2-colorable. Lovász [12] proved in 1976, that

$$|\mathcal{E}| \leq \binom{n}{k-1}$$

for a 3-color critical  $k$ -uniform hypergraph. Here we prove the following that can be considered as generalization of Lovász’ result.

**Theorem 1** *Let  $\mathcal{E} \subseteq \binom{[m]}{k}$  be a  $k$ -uniform set system on an underlying set  $X$  of  $m$  elements. Let us fix an ordering  $E_1, E_2, \dots, E_t$  of  $\mathcal{E}$  and a prescribed partition  $A_i \cup B_i = E_i$  ( $A_i \cap B_i = \emptyset$ ) for each member of  $\mathcal{E}$ . Assume that for all  $i = 1, 2, \dots, t$  there exists a partition  $C_i \cup D_i = X$  ( $C_i \cap D_i = \emptyset$ ), such that*

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$E_i \cap C_i = A_i$  and  $E_i \cap D_i = B_i$ , but  $E_j \cap C_i \neq A_j$  and  $E_j \cap D_i \neq B_j$  for all  $j < i$ . (That is, the  $i$ th partition cuts the  $i$ th set as it is prescribed, but does not cut any earlier set properly.) Then

$$t \leq \binom{m}{k-1} + \binom{m}{k-2} + \dots + \binom{m}{0}. \quad (1)$$

Theorem 1 was motivated by the following sharpening of Sauer's bound for forbidden configurations. Let  $F$  be a  $k \times l$  0-1 matrix, then  $\text{forb}(m, F)$  denotes maximum  $n$  such that there exists an  $m \times n$  simple matrix  $A$  such that no column and/or row permutation of  $F$  is a submatrix of  $A$ . Furthermore, let  $K_k$  denote the  $k \times 2^k$  simple 0-1 matrix consisting of all possible columns.

**Theorem 2** *Let  $F$  be contained in  $F_B = [K_k | t \cdot (K_k - B)]$  for an  $k \times (k+1)$  matrix  $B$  consisting of one column of each possible column sum. Then  $\text{forb}(m, F) = O(m^{k-1})$ .*

We explain the connection between Theorem 1 and Theorem 2.

The study of forbidden configurations is a problem in extremal set theory. The language we use here is matrix theory which conveniently encodes the problems. We define a *simple* matrix as a  $(0,1)$ -matrix with no repeated columns. Such a matrix can be thought of a set of subsets of  $\{1, 2, \dots, m\}$  with the columns encoding the subsets and the rows indexing the elements. Assume we are given a  $k \times l$   $(0,1)$ -matrix  $F$ . We say that a matrix  $A$  has no *configuration*  $F$  if no submatrix of  $A$  is a row and column permutation of  $F$  and so  $F$  is referred to as a *forbidden configuration* (sometimes called *trace*). A variety of combinatorial objects can be defined by forbidden configurations. For a simple  $m \times n$  matrix  $A$  which is assumed to have no configuration  $F$ , we seek an upper bound on  $n$  which will depend on  $m, F$ . We denote the best possible upper bound as  $\text{forb}(m, F)$ . Many results have been obtained about  $\text{forb}(m, F)$  including [2],[3],[5].

At this point all values known for  $\text{forb}(m, F)$  are of the form  $\Theta(m^e)$  for some integer  $e$ . We completed the classification for  $2 \times l$  matrices  $F$  in [2] and for  $3 \times l$  matrices  $F$  in [6]. We also put forward a conjecture on what properties of  $F$  drive the exponent  $e$ . Roughly speaking, we proposed a set of constructions and guessed that these constructions are sufficient to deduce the exponent  $e$  in the expression  $\Theta(m^e)$ .

We use the notation  $K_k$  to denote the  $k \times 2^k$  simple matrix of all possible columns on  $k$  rows. The basic result for  $\text{forb}(m, F)$  is as follows.

**Theorem 3** [Sauer [13], Perles and Shelah [14], Vapnik and Chervonenkis [15]] *We have that  $\text{forb}(m, K_k)$  is  $\Theta(m^{k-1})$ .*

In fact Theorem 3 is usually stated with  $\text{forb}(m, K_k) = \binom{m}{k-1} + \binom{m}{k-2} + \dots + \binom{m}{0}$  but the asymptotic growth of  $\Theta(m^{k-1})$  was what interested Vapnik and Chervonenkis.

One easy observation is that if we let  $A^c$  denote the 0-1-complement of  $A$  then  $\text{forb}(m, F^c) = \text{forb}(m, F)$ . Another observation is that if  $F'$  is a submatrix of  $F$ , then  $\text{forb}(m, F) \geq \text{forb}(m, F')$ . We let  $K_k^s$  denote the  $k \times \binom{k}{s}$  simple matrix of all possible columns of column sum  $s$ .

We use the notation  $[A|B]$  to denote the matrix obtained from concatenating the two matrices  $A$  and  $B$ . We use the notation  $k \cdot A$  to denote the matrix  $[A|A] \cdots [A]$  consisting of  $k$  copies of  $A$  concatenated together. We give precedence to the operation  $\cdot$  (multiplication) over concatenation so that for example  $[2 \cdot A|B]$  is the matrix consisting of the concatenation of  $B$  with the concatenation of two copies of  $A$ .

According to an earlier unpublished result of Füredi [10]  $\text{forb}(m, F) = O(m^k)$  for arbitrary  $k \times l$  configuration  $F$ . The goal of this paper is to give sufficient conditions that ensure  $\text{forb}(m, F) = O(m^{k-1})$ .

## 2 The boundary between $m^{k-1}$ and $m^k$

Theorem 3 implies that simple configurations all have  $\text{forb}(m, F) = O(m^{k-1})$ , thus we investigate  $f$ 's with multiple columns. First, we show that which configurations  $F$  have  $\text{forb}(m, F) = \Omega(m^k)$  using the direct product construction. Let  $A(k, 2)$  be defined as a minimal matrix with the property that any pair of rows has  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$  has both with 1's in some column and such that deleting a column of  $A(k, 2)$  would violate this property.

**Lemma 4** *Let  $F$  be a  $k \times l$  configuration.  $\text{forb}(m, F) = \Omega(m^k)$  if  $F$  contains  $2 \cdot K_k^l$  for  $2 \leq l \leq k - 2$  and  $l = 0, k$  or if  $F$  contains  $[2 \cdot K_k^1 | A(k, 2)]$ .*

**Proof:** We find that  $\text{forb}(m, F)$  is  $\Omega(m^k)$  if  $F$  contains  $2 \cdot K_k^l$  for  $0 \leq l \leq k$  and  $l \neq 1, k - 1$ . This follows since  $2 \cdot K_k^l$  is not contained in the  $k$ -fold product of  $l$   $K_{m/k}^1$ 's and  $k - l$   $K_{m/k}^{(m/k)-1}$ 's and so may deduce  $\text{forb}(m, 2 \cdot K_k^l)$  is  $\Omega(m^k)$ . To verify this for  $2 \leq l \leq k - 2$ , we note that any pair of rows of  $K_k^l$  has  $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$  and so if we have a submatrix that is a row and column permutation of  $K_k^l$ , we can only choose one row from either  $K_{m/k}^1$  or from  $K_{m/k}^{(m/k)-1}$ . The verification for  $K_k^0$  or  $K_k^k$  is easier.

For  $l = 1$  (the case  $l = k - 1$  is the  $(0,1)$ -complement) we can no longer assert that any pair of rows of  $K_k^l$  has  $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$  merely  $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$  and so can choose two rows from the copy of  $K_{m/k}^1$ , one row from each of  $k - 2$  of the  $K_{m/k}^{(m/k)-1}$  terms and generate a copy of  $2 \cdot K_k^1$ . (Theorem 5.1 of [6] shows that  $\text{forb}(m, K_k^1)$  is  $\Theta(m_{k-1})$ ). This is fixed by considering a minimal matrix  $A(k, 2)$  with the property that any pair of rows has  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$  has both with 1's in some column and such that deleting a column of  $A(k, 2)$  would violate this. As above, we have that if  $F$  contains  $[2 \cdot K_k^1 | A(k, 2)]$ , then  $\text{forb}(m, F)$  is  $\Omega(m^k)$ .  $\square$

Lemma 4 leaves two possibilities if we want  $\text{forb}(m, f)$  be bounded away from  $m^k$ . Either  $F$  is contained in a matrix  $F_B = [K_k | t \cdot (K_k - B)]$  for an  $k \times (k + 1)$  matrix  $B$  consisting of one column of each possible column sum or  $F$  is contained in a matrix  $[K_k^0 | t \cdot C]$  where  $C$  is a  $k$ -rowed simple matrix consisting of all columns which do not have 1's in both rows 1 and 2 and also with at least one 1. Note, that these are not mutually exclusive cases. Our main result Theorem 2 is that in the first case  $\text{forb}(m, F) = O(m^{k-1})$ .

**Proof of Theorem 2:** Let  $A$  be an  $m \times n$  simple 0-1 matrix, and  $B$  be a  $k \times (k + 1)$  matrix consisting of one column of each possible column sum. Suppose that  $A$  does not have  $F_B = [K_k | t \cdot (K_k - B)]$  as configuration. This implies that on a given  $k$ -tuple  $L$  of rows either  $K_k$  is missing, or if all possible columns of size  $k$  occur on  $L$ , then  $t \cdot (K_k - B)$  must be missing. This latter means, that for some  $0 \leq s \leq k$ , two columns of column sum  $s$  occur at most  $t - 1$  times on  $L$ , respectively. Let  $\mathcal{K}$  be the set of  $k$ -tuples of rows where the latter happens. Using Lemma 5 a set of columns of size  $O(m^{k-1})$  can be removed from  $A$  to obtain  $A'$ , so that for all  $L \in \mathcal{K}$  a column (in fact two) is missing on  $L$  in  $A'$ . However, this implies that  $K_k$  is not a configuration in  $A'$ , thus by Theorem 3  $A'$  has at most  $O(m^{k-1})$  columns.  $\square$

Let  $\mathcal{K}$  be a system of  $k$ -tuples of rows such that  $\forall K \in \mathcal{K}$  there are two  $(k \times 1)$  columns,  $\alpha_K \neq \beta_K$  specified. We say that a column  $x$  of  $A$  violates  $(K, \alpha_K)$ , if  $x|_K = \alpha_K$ , similarly,  $x$  violates  $(K, \beta_K)$ , if  $x|_K = \beta_K$ .

**Lemma 5** Assume, that for every  $K \in \mathcal{K}$  there are at most  $t - 1$  columns of  $A$  that violate  $(K, \alpha_K)$ , and at most  $t - 1$  columns of  $A$  violate  $(K, \beta_K)$ . Then there exists a subset  $X$  of columns of  $A$ , such that  $|X| = O(m^{k-1})$  and no column of  $A - X$  violates any of  $(K, \alpha_K)$  or  $(K, \beta_K)$ .

**Proof:** It can be assumed without loss of generality that for all  $K \in \mathcal{K}$   $\alpha_K = \alpha$  and  $\beta_K = \beta$  independent of  $K$ . Indeed, there are  $2^k \times 2^k$  possible  $\alpha_K, \beta_K$  pairs, that is a constant number of them. Thus,  $\mathcal{K}$  can be partitioned into a constant number of parts, so that in each part  $\alpha_K = \alpha$  and  $\beta_K = \beta$  holds. We apply induction on  $k$  using the simplification given above.  $k = 1$  is obvious.

Consider now  $k \times 1$  columns  $\alpha \neq \beta$ . Assume first, that  $\alpha \neq \bar{\beta}$ . That is, there is a coordinate where  $\alpha$  and  $\beta$  agree, say both have 1 as their  $\ell$ th coordinate. The case of a common 0 coordinate is similar. For the  $i$ th row of  $A$  we count how many columns have violation so that for some  $K \in \mathcal{K}$  the  $\ell$ th coordinate in  $K$  is exactly row  $i$ . Let  $\mathcal{K}_{i,\ell}$  be the set of these  $k$ -tuples from  $\mathcal{K}$ . Columns that have violation on  $k$ -tuples from  $\mathcal{K}_{i,\ell}$  have 1 in the  $i$ th row, let  $A_{i,1}$  denote matrix formed by the set of columns that have 1 in row  $i$ . If row  $i$  is removed from  $A_{i,1}$ , the remaining matrix  $A'_{i,1}$  is still simple. Let  $\mathcal{K}'_{i,\ell}$  denote the set of  $(k - 1)$ -tuples obtained from  $k$ -tuples of  $\mathcal{K}_{i,\ell}$  by removing their  $\ell$ th coordinate,  $i$ , furthermore let  $\alpha'$  ( $\beta'$ , respectively) denote the  $(k - 1) \times 1$  column obtained from  $\alpha$  ( $\beta$ ) by removing the  $\ell$ th coordinate, 1. Note, that  $\alpha' \neq \beta'$ . A column of  $A$  has a violation on  $K \in \mathcal{K}_{i,\ell}$  iff its counterpart in  $A'_{i,1}$  has a violation on the corresponding  $K' \in \mathcal{K}'_{i,\ell}$ . The number of those columns is at most  $cm^{k-2}$  by the inductive hypothesis. Since  $\mathcal{K} = \cup_{i=1}^m \mathcal{K}_{i,\ell}$ , we obtain that the number of columns of  $A$  having violation on some  $K \in \mathcal{K}$  is at most  $m \cdot cm^{k-2}$ .

Let us assume now, that  $\alpha = \bar{\beta}$ . A subset  $\mathcal{J} \subseteq \mathcal{K}$  is called *independent* if there exists an ordering  $J_1, J_2, \dots, J_g$  of the elements of  $\mathcal{J}$  such that for every  $J_i \in \mathcal{J}$  there exists an  $m \times 1$  0-1 column that violates  $J_i$  and does not violate any  $J_j \in \mathcal{J}$  for  $j < i$ . Let us call a *maximal* independent subset  $\mathcal{B}$  of  $\mathcal{K}$  a *basis* of  $\mathcal{K}$ . If a column of  $A$  has a violation on  $K \in \mathcal{K}$ , then it has a violation on some  $B \in \mathcal{B}$ , as well. Indeed, either  $K \in \mathcal{B}$  holds, or if  $K \notin \mathcal{B}$ , then by the maximality of  $\mathcal{B}$ ,  $K$  cannot be added to it as a  $|\mathcal{B}| + 1$ st element in the order, so the column having violation on  $K$  must have a violation on  $B \in \mathcal{B}$ , for some  $B$ . By Theorem 1 for a basis  $\mathcal{B}$  we have

$$|\mathcal{B}| \leq \binom{m}{k-1} + \binom{m}{k-2} + \dots + \binom{m}{0},$$

since a column violating a  $k$ -tuple  $B_i$  from  $\mathcal{B}$ , but none of  $B_j$  for  $j < i$ , gives an appropriate partition of the set of rows. Thus, there could be at most  $(2t - 2) \left[ \binom{m}{k-1} + \binom{m}{k-2} + \dots + \binom{m}{0} \right]$  columns violating some  $K \in \mathcal{K}$ .  $\square$

**Proof of Theorem 1:** We define a polynomial  $p_i(\underline{x}) \in \mathbb{R}[x_1, x_2, \dots, x_m]$  for each  $E_i$  as follows.

$$p_i(x_1, x_2, \dots, x_m) = \prod_{a \in A_i} (1 - x_a) \prod_{b \in B_i} x_b + (-1)^{k+1} \prod_{a \in A_i} x_a \prod_{b \in B_i} (1 - x_b) \quad (2)$$

Polynomials defined by (2) are multilinear of degree at most  $k - 1$ , since the product  $\prod_{e \in E_i} x_e$  cancels by the coefficient  $(-1)^{k+1}$ . Thus, they are from the space generated by monomials of type  $\prod_{j=1}^r x_{i_j}$ , for  $r = 0, 1, \dots, k - 1$ . The dimension of this space over  $\mathbb{R}$  is  $\binom{m}{k-1} + \binom{m}{k-2} + \dots + \binom{m}{0}$ .

We shall prove that polynomials  $p_1(\underline{x}), p_2(\underline{x}), \dots, p_t(\underline{x})$  are linearly independent over  $\mathbb{R}$ , which implies (1). Assume that

$$\sum_{i=1}^t \lambda_i p_i(\underline{x}) = 0 \quad (3)$$

is a linear combination of the  $p_i(\underline{x})$ 's that is the zero polynomial. Consider the partition  $C_t \cup D_t = X$ , and substitute  $x_c = 0$  if  $c \in C_t$  and  $x_d = 1$  if  $d \in D_t$  into (3). Then  $p_t(\underline{x}) = 1$ , but it is easy to see that  $p_k(\underline{x}) = 0$  for  $k < t$ . This implies that  $\lambda_t = 0$ . Now assume by induction on  $j$ , that  $\lambda_t = \lambda_{t-1} = \dots = \lambda_{t-j+1} = 0$ . Take the partition  $C_{t-j} \cup D_{t-j} = X$  and substitute into (3)  $x_c = 0$  if  $c \in C_{t-j}$  and  $x_d = 1$  if  $d \in D_{t-j}$ . Then, as before,  $p_{t-j}(\underline{x}) = 1$ , but  $p_k(\underline{x}) = 0$  for  $k < t-j$ . This implies  $\lambda_{t-j} = 0$ , as well. Thus, all coefficients in (3) must be 0, hence the polynomials are linearly independent.  $\square$

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