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# Pairwise Intersections and Forbidden Configurations

R. P. Anstee<sup>1 †</sup> and Peter Keevash<sup>2 ‡</sup>

<sup>1</sup>Mathematics Department, The University of British Columbia, Vancouver, B.C., Canada V6T 1Z2

<sup>2</sup>Department of Mathematics, California Institute of Technology Pasadena, California, USA

Let  $f_m(a, b, c, d)$  denote the maximum size of a family  $\mathcal{F}$  of subsets of an  $m$ -element set for which there is no pair of subsets  $A, B \in \mathcal{F}$  with

$$|A \cap B| \geq a, \quad |\bar{A} \cap B| \geq b, \quad |A \cap \bar{B}| \geq c, \quad \text{and} \quad |\bar{A} \cap \bar{B}| \geq d.$$

By symmetry we can assume  $a \geq d$  and  $b \geq c$ . We show that  $f_m(a, b, c, d)$  is  $\Theta(m^{a+b-1})$  if either  $b > c$  or  $a, b \geq 1$ . We also show that  $f_m(0, b, b, 0)$  is  $\Theta(m^b)$  and  $f_m(a, 0, 0, d)$  is  $\Theta(m^a)$ . This can be viewed as a result concerning forbidden configurations and is further evidence for a conjecture of Anstee and Sali. Our key tool is a strong stability version of the Complete Intersection Theorem of Ahlswede and Khachatrian, which is of independent interest.

**Keywords:** forbidden configurations, extremal set theory, intersecting set systems, uniform set systems, (0,1)-matrices

Let  $f_m(a, b, c, d)$  denote the maximum size of a family  $\mathcal{F}$  of subsets of an  $m$ -element set for which there is no pair of subsets  $A, B \in \mathcal{F}$  with

$$|A \cap B| \geq a, \quad |\bar{A} \cap B| \geq b, \quad |A \cap \bar{B}| \geq c, \quad \text{and} \quad |\bar{A} \cap \bar{B}| \geq d.$$

By symmetry we can assume  $a \geq d$  and  $b \geq c$ .

**Theorem 1** *Suppose  $a \geq d$  and  $b \geq c$ . Then  $f_m(a, b, c, d)$  is  $\Theta(m^{a+b-1})$  if either  $b > c$  or  $a, b \geq 1$ . Also  $f_m(a, 0, 0, d)$  is  $\Theta(m^a)$  and  $f_m(0, b, b, 0)$  is  $\Theta(m^b)$ .*

Some motivation for studying this function comes from the forbidden configuration problem for matrices popularised by the first author. We can identify a family  $\mathcal{A} = \{A_1, \dots, A_n\}$  of subsets of  $[m]$  with an  $m \times n$  (0, 1)-matrix  $A$  determined by incidence, i.e.  $A_{ij}$  is 1 if  $i \in A_j$ , otherwise 0. Such a matrix is *simple*, by which we mean it has no repeated columns. Let  $F$  be a (0, 1)-matrix (not necessarily simple). We define  $\text{forb}(m, F)$  to be the largest  $n$  for which there is a simple  $m \times n$  (0, 1)-matrix  $A$  that does not contain an  $F$  configuration, i.e. a submatrix which is a row and column permutation of  $F$ . If we interpret

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$A, F$  as incidence matrices of systems  $\mathcal{A}, \mathcal{F}$  (the latter possibly having sets with multiplicity) then  $A$  has an  $F$  configuration exactly when  $\mathcal{A}$  has  $\mathcal{F}$  as a *trace*, i.e.  $\mathcal{F} \subset \{A \cap X : A \in \mathcal{A}\}$  for some  $X \subset [m]$ .

The first forbidden configuration result was obtained independently by Sauer [6], Perles, Shelah [7], Vapnik and Chervonenkis [8]. When  $F$  is the  $k \times 2^k$   $(0, 1)$ -matrix with all possible distinct columns they showed that  $\text{forb}(m, F) = \sum_{i=0}^{k-1} \binom{m}{i}$ . For a general  $k$ -row matrix  $F$ , Füredi obtained an  $O(m^k)$  upper bound on  $\text{forb}(m, F)$ , but it seems hard to determine the order of magnitude of  $\text{forb}(m, F)$  for each  $F$ . This was achieved when  $F$  has 2 rows by Anstee, Griggs and Sali [2] and for 3 rows by Anstee and Sali [3], but is open in general.

It is not hard to see that if  $F$  consists of a single column with  $s$  0's and  $t$  1's then  $\text{forb}(m, F)$  is  $\Theta(m^{\max\{s-1, t-1\}})$ . In this paper we solve the problem when  $F$  has two columns. Let  $F_{abcd}$  be the  $(a+b+c+d) \times 2$   $(0, 1)$ -matrix which has  $a$  rows of [11],  $b$  rows of [10],  $c$  rows of [01],  $d$  rows of [00]. Then  $\text{forb}(m, F_{abcd}) = f_m(a, b, c, d)$  as defined above.

In [3] a conjecture was made for the asymptotic behaviour of  $\text{forb}(m, F)$  as a function of  $m$  and  $F$ . In particular, a restricted set of constructions of simple matrices were described in [3] that were conjectured to predict the asymptotics of  $\text{forb}(m, F)$ . These were used in this paper to predict the asymptotics in Theorem 1 as well as to provide construction. This is further evidence for the conjecture in [3].

Our key tool is a strong stability version of the Complete Intersection Theorem of Ahlswede and Khachatrian [1], which is of independent interest. Strong stability results have been employed with success by the second author, for example in [4],[5]. First we recall some notation. Let numbers  $k, r_1, r_2$  be given and suppose  $G$  and  $H$  are disjoint sets with  $|G| = k - r_1 + r_2$ . We define  $\mathcal{I}_{r_1, r_2}^k$  on the pair  $(H, G)$  to be the family consisting of all sets of size  $k$  in  $G \cup H$  that intersect  $G$  in at least  $k - r_1 = |G| - r_2$  points. Note that any two sets in  $\mathcal{I}_{r_1, r_2}^k$  have at least  $|G| - 2r_2 = k - r_1 - r_2$  points in common, i.e.  $\mathcal{I}_{r_1, r_2}^k$  is  $(k - r)$ -intersecting, where  $r = r_1 + r_2$ .

We also define  $\mathcal{F}_{r_1, r_2}^k$  on the pair  $(H, G)$  to be the family consisting of all sets of size  $k$  in  $G \cup H$  that intersect  $G$  in exactly  $k - r_1 = |G| - r_2$  points. Clearly this is a subsystem of  $\mathcal{I}_{r_1, r_2}^k$  and  $|\mathcal{I}_{r_1, r_2}^k \setminus \mathcal{F}_{r_1, r_2}^k|$  is of a lower order of magnitude than  $|\mathcal{I}_{r_1, r_2}^k|$  and  $|\mathcal{F}_{r_1, r_2}^k|$ . In particular, if the systems are defined on the ground set  $[m]$  with  $k = \Theta(m)$  then  $|\mathcal{I}_{r_1, r_2}^k|$  and  $|\mathcal{F}_{r_1, r_2}^k|$  are  $\Theta(m^r)$ , whereas  $|\mathcal{I}_{r_1, r_2}^k \setminus \mathcal{F}_{r_1, r_2}^k| < m^{r-2}$ . The Complete Intersection Theorem, conjectured by Frankl, and proved by Ahlswede and Khachatrian [1], is that any  $k$ -uniform,  $(k - r)$ -intersecting family of maximum size on a given ground set is isomorphic to  $\mathcal{I}_{r-p, p}^k$ , for some  $0 \leq p \leq r$ , which depends on the size of the ground set. We prove the following result.

**Theorem 2** *Suppose  $\mathcal{A}$  is a  $k$ -uniform  $(k - r)$ -intersecting set system on  $[m]$  of size at least  $(5r)^{5r} m^{r-1}$ . Then  $\mathcal{A} \subset \mathcal{I}_{r-p, p}^k$  for some  $0 \leq p \leq r$ .*

We use this theorem in our proofs of the upper bounds in Theorem 1 in cases where  $\mathcal{A}$  is a  $k$ -uniform  $(k - r)$ -intersecting set system satisfying some additional properties. If  $|\mathcal{A}|$  is small, we can ignore it for the purposes of upper bounds. If  $|\mathcal{A}|$  is large enough to matter for the upper bounds, we can use the fact that  $\mathcal{A} \subset \mathcal{I}_{r-p, p}^k$  to deduce structure in  $\mathcal{A}$  (e.g. the partition  $G, H$  above) which we can exploit in our proofs.

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