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# Fast separation in a graph with an excluded minor<sup>†</sup>

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Let  $G$  be an  $n$ -vertex  $m$ -edge graph with weighted vertices. A pair of vertex sets  $A, B \subseteq V(G)$  is a  $\frac{2}{3}$ -separation of order  $|A \cap B|$  if  $A \cup B = V(G)$ , there is no edge between  $A \setminus B$  and  $B \setminus A$ , and both  $A \setminus B$  and  $B \setminus A$  have weight at most  $\frac{2}{3}$  the total weight of  $G$ . Let  $\ell \in \mathbb{Z}^+$  be fixed. Alon, Seymour and Thomas [*J. Amer. Math. Soc.* 1990] presented an algorithm that in  $\mathcal{O}(n^{1/2}m)$  time, either outputs a  $K_\ell$ -minor of  $G$ , or a separation of  $G$  of order  $\mathcal{O}(n^{1/2})$ . Whether there is a  $\mathcal{O}(n + m)$  time algorithm for this theorem was left as open problem. In this paper, we obtain a  $\mathcal{O}(n + m)$  time algorithm at the expense of  $\mathcal{O}(n^{2/3})$  separator. Moreover, our algorithm exhibits a tradeoff between running time and the order of the separator. In particular, for any given  $\epsilon \in [0, \frac{1}{2}]$ , our algorithm either outputs a  $K_\ell$ -minor of  $G$ , or a separation of  $G$  with order  $\mathcal{O}(n^{(2-\epsilon)/3})$  in  $\mathcal{O}(n^{1+\epsilon} + m)$  time.

**Keywords:** graph algorithm, separator, minor

## 1 Introduction

We consider graphs  $G$  that are simple, finite, and undirected. Let  $V(G)$  and  $E(G)$  denote the vertex and edge sets of  $G$ . Let  $|G| := |V(G)|$  and  $\|G\| := |E(G)|$ . A *separation* of  $G$  is a pair  $\{A, B\}$  of vertex sets  $A, B \subseteq V(G)$  such that  $A \cup B = V(G)$ , and there is no edge with one endpoint in  $A \setminus B$  and the other endpoint in  $B \setminus A$ . The *order* of  $\{A, B\}$  is  $|A \cap B|$ . The set  $A \cap B$  is called a *separator* of  $G$ . A *weighting* of  $G$  is a function  $w : V(G) \rightarrow \mathbb{R}^+$ . Let  $w(S) := \sum_{v \in S} w(v)$  for all  $S \subseteq V(G)$ , and  $w(G) := w(V(G))$ . We say  $(G, w)$  is a *weighted graph*. A separation  $\{A, B\}$  of a weighted graph  $(G, w)$  is an  $\alpha$ -separation if  $w(A \setminus B) \leq \alpha \cdot w(G)$  and  $w(B \setminus A) \leq \alpha \cdot w(G)$ .

A ‘separator theorem’ is of the format: for some  $0 < \alpha < 1$  and  $0 < \epsilon \leq 1$ , every graph  $G$  from a certain family has an  $\alpha$ -separation of order  $\mathcal{O}(|G|^{1-\epsilon})$ . Applications of separator theorems are numerous, and include VLSI circuit layout, approximation algorithms using the divide-and-conquer paradigm, solving sparse systems of linear equations, pebbling games, and graph drawing. See the recent monograph by Rosenberg and Heath [9] for more details.

A seminal theorem due to Lipton and Tarjan [5] states that every weighted planar graph  $G$  has a  $\frac{2}{3}$ -separation of order  $\mathcal{O}(|G|^{1/2})$  that can be computed in  $\mathcal{O}(|G| + \|G\|)$  time. This result was generalised

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for graphs with an excluded minor by Alon *et al.* [1] (see [2, 3, 7] for related results). A graph  $H$  is a *minor* of a graph  $G$  if  $H$  can be obtained from a subgraph of  $G$  by contracting edges, in which case we say that  $G$  has an  $H$ -*minor*. The Kuratowski-Wagner Theorem states that a graph is planar if and only if it has no  $K_5$ -minor and no  $K_{3,3}$ -minor. An  $H$ -*model* in  $G$  is a set of disjoint connected subgraphs  $\{X_v : v \in V(H)\}$  indexed by the vertices of  $H$ , such that for every edge  $vw \in E(H)$ , there is an edge  $xy \in E(G)$  with  $x \in X_v$  and  $y \in X_w$ . Clearly  $G$  has an  $H$ -minor if and only if  $G$  has an  $H$ -model. We choose to work with  $H$ -models rather than  $H$ -minors.

**Theorem 1 (Alon *et al.* [1])** *There is an algorithm with running time  $\mathcal{O}((\ell \cdot |G|)^{1/2} \cdot (|G| + \|G\|))$  that, given  $\ell \in \mathbb{Z}^+$  and a weighted graph  $(G, w)$ , either outputs:*

- (a) a  $K_\ell$ -model of  $G$ , or
- (b) a  $\frac{2}{3}$ -separation of  $(G, w)$  of order at most  $\ell^{3/2} \cdot |G|^{1/2}$ .

Suppose that  $\ell$  is fixed. It follows from a result of Mader [6] (see Theorem 3) that Theorem 1 can be implemented in  $\mathcal{O}(|G|^{3/2} + \|G\|)$  time. Alon *et al.* [1] left as an open problem whether linear time is possible. The main result of this paper is the following partial answer to this question. We obtain a linear running time at the expense of a slightly larger separator (and a larger dependence on  $\ell$ ). Moreover, our algorithm exhibits a tradeoff between running time (ranging from  $\mathcal{O}(n)$  to  $\mathcal{O}(n^{3/2})$ ) and the order of the separator (ranging from  $\mathcal{O}(n^{2/3})$  to  $\mathcal{O}(n^{1/2})$ ).

**Theorem 2** *There is an algorithm with running time  $\mathcal{O}(2^{(3\ell^2+7\ell-3)/2} \cdot |G|^{1+\epsilon} + \ell \cdot \|G\|)$  that, given  $\epsilon \in [0, \frac{1}{2}]$ ,  $\ell \in \mathbb{Z}^+$ , and a weighted graph  $(G, w)$ , either outputs:*

- (a) a  $K_\ell$ -model of  $G$ , or
- (b) a  $\frac{2}{3}$ -separation of  $(G, w)$  of order at most  $2^{(\ell^2+3\ell+1)/2} \cdot |G|^{(2-\epsilon)/3}$ .

Note that for applications to divide-and-conquer algorithms a separation of order  $\mathcal{O}(|G|^{1-\epsilon})$ , for some constant  $\epsilon > 0$ , is all that is needed.

The idea behind the proof of Theorem 2 is simple. We now outline the proof for fixed  $\ell$  and with  $\epsilon = 0$ . Suppose that in  $\mathcal{O}(|G| + \|G\|)$  time, we can find a partition of  $V(G)$  into  $|G|^{2/3}$  connected subgraphs  $\{S_1, S_2, \dots, S_{|G|^{2/3}}\}$ , each containing  $\mathcal{O}(|G|^{1/3})$  vertices. Let  $H$  be the weighted graph obtained from  $G$  by contracting each  $S_i$  to a vertex  $v_i$  with weight  $w(v_i) = w(S_i)$ . Then apply Theorem 1 to  $H$  to either obtain a  $K_\ell$ -model in  $H$  which defines a  $K_\ell$ -model in  $G$ , or a  $\frac{2}{3}$ -separation  $\{A, B\}$  of  $H$  with order  $\mathcal{O}(|H|^{1/2}) = \mathcal{O}(|G|^{1/3})$ , in which case  $\{\bigcup\{S_i : v_i \in A\}, \bigcup\{S_i : v_i \in B\}\}$  is a  $\frac{2}{3}$ -separation of  $G$  with order  $\mathcal{O}(|G|^{2/3})$ . The running time is  $\mathcal{O}(|H|^{3/2} + \|H\|) \subseteq \mathcal{O}(|G| + \|G\|)$ . The proof of Theorem 2 is actually a little different from this outline. In particular, the subgraphs  $S_i$  will not necessarily be connected, but we will still be able to convert the output from Theorem 1 applied to  $H$  to the desired output for  $G$ . By relaxing the connectivity condition, we are able to prove that an appropriate partition exists.

We will use the following notation for a graph  $G$ . For  $x \in V(G)$ , let  $N(x) := \{y \in V(G) : xy \in E(G)\}$ . For a subgraph  $X$  of  $G$ , let  $N(X) := \bigcup\{N(x) \setminus X : x \in X\}$ . Where there is no confusion, a set of vertices  $S \subseteq V(G)$  will also refer to the subgraph of  $G$  induced by  $S$ .

## 2 Mader's Theorem

This section contains a number of easily proved results—see the full version of the paper for details. We start with an algorithmic version of a theorem of Mader [6] (cf. [8]).

**Theorem 3** Given a graph  $G$  with  $\|G\| \geq 2^{\ell-3} \cdot |G|$  (for some  $\ell \in \mathbb{Z}^+$ ), a  $K_\ell$ -model of  $G$  can be computed in  $\mathcal{O}(\ell(|G| + \|G\|))$  time.

Note that if we ignore the running time, Theorem 3 is far from best possible. Kostochka [4] and Thomason [10] independently proved that if  $\|G\| \in \Omega(\ell\sqrt{\log \ell} \cdot |G|)$  then  $G$  has a  $K_\ell$ -model. Theorem 3 implies the following slightly faster version of Theorem 1 (for fixed  $\ell$ )

**Theorem 4** There is an algorithm with running time  $\mathcal{O}(2^{2\ell} \cdot |G|^{3/2} + \ell \cdot \|G\|)$  that, given  $\ell \in \mathbb{Z}^+$  and a weighted graph  $(G, w)$ , either outputs:

- (a) a  $K_\ell$ -model of  $G$ , or
- (b) a  $\frac{2}{3}$ -separation of  $(G, w)$  of order at most  $\ell^{3/2} \cdot |G|^{1/2}$ .

A  $k$ -clique of  $G$  is a (not necessarily maximal) set of  $k$  pairwise adjacent vertices of  $G$ . If every subgraph of  $G$  has a vertex of degree at most  $d$ , then  $G$  is  $d$ -degenerate. For example, Theorem 3 implies that a graph with no  $K_\ell$ -minor is  $2^{\ell-2}$ -degenerate. It is easily proved that a  $d$ -degenerate graph  $G$  with no  $k$ -clique has less than  $d^{k-1} \cdot |G|$  cliques. Hence a graph  $G$  with no  $K_\ell$ -minor has less than  $2^{(\ell-2)(\ell-1)} \cdot |G|$  cliques. For an algorithm, we have the following result.

**Lemma 1** Given a graph  $G$  with no  $k$ -clique and at least  $2^{(\ell-2)(k-1)} \cdot |G|$  cliques (for some  $\ell \in \mathbb{Z}^+$ ), a  $K_\ell$ -minor of  $G$  can be computed in  $\mathcal{O}(\ell(|G| + \|G\|))$  time.

### 3 Proof of Theorem 2

Let  $G$  be a graph. Let  $\mathcal{A}$  be a partition of  $V(G)$ . Let  $H$  be the graph obtained from  $G$  by collapsing each part  $S \in \mathcal{A}$  to a single vertex  $v$ , and replacing parallel edges by a single edge. Denote  $H_v := S$ . We say  $\{H_v : v \in V(H)\}$  is an  $H$ -partition of  $G$ . Furthermore,  $\{H_v : v \in V(H)\}$  is a connected  $H$ -partition of  $G$  if  $vw \in E(H)$  if and only if there is an edge of  $G$  between every component of  $H_v$  and every component of  $H_w$ . We prove the following lemma.

**Lemma 2** There is an algorithm with running time  $\mathcal{O}(2^{2\ell} \cdot |G| + \|G\|)$  that, given  $\ell, k \in \mathbb{Z}^+$  and a graph  $G$ , outputs a connected  $H$ -partition of  $G$  such that either:

- (a)  $H$  has a  $K_\ell$ -model (which is also output), or
- (b)  $|H| \leq 2^{\ell^2+\ell-1} \cdot |G| \cdot k^{-1}$ , and  $|H_x| < 2k$  for all  $x \in V(H)$ .

**Proof of Theorem 2 assuming Lemma 2:** Apply Lemma 2 with  $k = \lfloor |G|^{(1-2\epsilon)/3} \rfloor$ . First suppose that Lemma 2 outputs a  $K_\ell$ -model  $\{S_1, S_2, \dots, S_\ell\}$  of  $H$ . Thus each  $S_i$  is a connected subgraph of  $H$ . Choose a connected component  $Z_v$  of  $H_v$  for each  $v \in V(H)$ . Let  $T_i := \bigcup \{Z_v : v \in S_i\}$ . Then  $\{T_1, T_2, \dots, T_\ell\}$  is a  $K_\ell$ -model of  $G$ .

Otherwise  $|H| \leq 2^{\ell^2+\ell-1} \cdot |G|^{2(1+\epsilon)/3}$ , and  $|H_x| < 2|G|^{(1-2\epsilon)/3}$  for all  $x \in V(H)$ . Let  $w(v) := w(H_v)$  for all  $v \in V(H)$ . Apply Theorem 4 to  $(H, w)$ . The running time is

$$\mathcal{O}(2^{2\ell} \cdot |H|^{3/2} + \ell \cdot \|H\|) \subseteq \mathcal{O}(2^{2\ell} \cdot (2^{\ell^2+\ell-1} \cdot |G|^{2(1+\epsilon)/3})^{3/2} + \ell \cdot \|G\|) \subseteq \mathcal{O}(2^{(3\ell^2+7\ell-3)/2} \cdot |G|^{1+\epsilon} + \ell \cdot \|G\|) .$$

We either obtain a  $K_\ell$ -model of  $H$ , or a  $\frac{2}{3}$ -separation of  $H$  with order at most  $\ell^{3/2} \cdot |H|^{1/2}$ . In the first case,  $G$  has a  $K_\ell$ -model as proved above.

Now suppose that we obtain a  $\frac{2}{3}$ -separation  $\{A, B\}$  of  $(H, w)$  with order

$$|A \cap B| \leq \ell^{3/2} \cdot |H|^{1/2} \leq \ell^{3/2} \cdot (2^{\ell^2+\ell-1} |G|^{2(1+\epsilon)/3})^{1/2} \leq 2^{(\ell^2+3\ell-1)/2} \cdot |G|^{(1+\epsilon)/3} .$$

Let  $X := \bigcup\{H_v : v \in A\}$  and  $Y := \bigcup\{H_v : v \in B\}$ . Then  $\{X, Y\}$  is a separation of  $G$  with order

$$|X \cap Y| = |\bigcup\{H_v : v \in A \cap B\}| \leq 2^{(\ell^2 + 3\ell - 1)/2} \cdot |G|^{(1+\epsilon)/3} \cdot 2|G|^{(1-2\epsilon)/3} \leq 2^{(\ell^2 + 3\ell + 1)/2} \cdot |G|^{(2-\epsilon)/3}.$$

We have  $w(X \setminus Y) = w(A \setminus B) \leq \frac{2}{3}w(H) = \frac{2}{3}w(G)$ . Similarly  $w(B \setminus A) \leq \frac{2}{3}w(G)$ .  $\square$

**Proof of Lemma 2:**

**Step 1:** Using a breadth-first search algorithm, compute a maximal set  $\mathcal{A}$  of connected subgraphs of  $G$  such that  $|S| = k$  for all  $S \in \mathcal{A}$ . Let  $\mathcal{B}$  be the set of connected components of  $G \setminus \bigcup\{S \in \mathcal{A}\}$ . Then  $\mathcal{A} \cup \mathcal{B}$  is a partition of  $V(G)$ , and there is no edge of  $G$  between distinct  $T_1, T_2 \in \mathcal{B}$ . Note that  $|T| < k$  for all  $T \in \mathcal{B}$ , as otherwise  $T$  would contain a connected subgraph  $X$  with  $|X| = k$ , which could be added to  $\mathcal{A}$ .

**Step 2:** Let  $H$  be the graph obtained from  $G$  by contracting each set  $S \in \mathcal{A} \cup \mathcal{B}$  into a single vertex  $v$  with  $H_v := S$ , and replacing parallel edges by a single edge. Since each  $S \in \mathcal{A} \cup \mathcal{B}$  is connected in  $G$ ,  $\{H_v : v \in V(H)\}$  is a connected  $H$ -partition of  $G$ . Let  $A := \{v \in V(H) : H_v \in \mathcal{A}\}$  and  $B := \{v \in V(H) : H_v \in \mathcal{B}\}$ . A vertex  $v$  of  $H$  is *big* if  $|H_v| \geq k$ . A vertex  $v$  of  $H$  is *small* if  $|H_v| < k$ . Observe that every vertex in  $A$  is big,  $B$  is an independent set of  $H$ , and every vertex in  $B$  is small. Partition  $B = C \cup D \cup E$ , where  $C := \{v \in B : \deg_H(v) \geq 2^{\ell-2}\}$ ,  $D := \{v \in B : \ell - 1 \leq \deg_H(v) < 2^{\ell-2}\}$ , and  $E := \{v \in B : \deg_H(v) \leq \ell - 2\}$ .

**Step 3:** Suppose that  $|C| \geq |A|$ . Then the subgraph  $C \cup A$  of  $H$  has at least  $2^{\ell-2} \cdot |C|$  edges and at most  $2|C|$  vertices. By Theorem 3, a  $K_\ell$ -model of  $C \cup A$  can be computed in  $\mathcal{O}(\ell \cdot |G|)$  time. We now assume that  $|C| < |A|$ .

**Step 4:** For each vertex  $v \in D \cup E$ , if there is a pair  $x, y \in A$  of distinct neighbours of  $v$ , such that  $\{x, y\}$  has not been assigned any vertex in  $D \cup E$ , then assign  $v$  to  $\{x, y\}$ . This step can be implemented in  $\mathcal{O}(2^{2\ell} \cdot |G|)$  time, since each vertex in  $D \cup E$  has degree at most  $2^{\ell-2}$ .

Suppose that there is a vertex  $v \in D$  that is not assigned. Let the neighbourhood of  $v$  be  $\{x_1, x_2, \dots, x_d\}$ . Then  $d \geq \ell - 1$ . Thus for all  $1 \leq i < j \leq d$ , there is a distinct vertex  $v_{i,j}$  that is assigned to the pair  $\{x_i, x_j\}$ , and  $v_{i,j}$  is adjacent to both  $x_i$  and  $x_j$ . In the graph obtained from  $H$  by contracting each edge  $x_i v_{i,j}$ , the subgraph  $\{x_1, x_2, \dots, x_d, v\}$  is a clique on at least  $\ell$  vertices. Thus  $H$  has a  $K_\ell$ -model. We now assume that every vertex in  $D$  is assigned.

Let  $E^*$  be the set of assigned vertices in  $E$ . Consider the graph obtained from  $A \cup D \cup E^*$  by contracting the edge  $vx$  for each  $v \in D \cup E^*$  assigned to the pair  $\{x, y\}$ . This graph has  $|A|$  vertices and at least  $|D| + |E^*|$  edges. Thus if  $|D| + |E^*| \geq 2^{\ell-3} \cdot |A|$ , then by Theorem 3,  $H$  has a  $K_\ell$ -model that can be computed in  $\mathcal{O}(\ell \cdot |G|)$  time. We now assume that  $|D| + |E^*| < 2^{\ell-3} \cdot |A|$ .

**Step 5:** Partition  $E \setminus E^* = \bigcup\{P_1, P_2, \dots, P_s\}$  such that for all  $u, v \in E \setminus E^*$ , we have  $N(u) = N(v)$  if and only if both  $u, v \in P_i$  for some  $1 \leq i \leq s$ . For all  $1 \leq i \leq s$ , partition  $P_i = \bigcup(P_{i,1}, P_{i,2}, \dots, P_{i,t_i})$  such that for all  $1 \leq j \leq t_i - 1$ ,  $k \leq |\bigcup\{H_v : v \in P_{i,j}\}| < 2k$ , and  $|\bigcup\{H_v : v \in P_{i,t_i}\}| < k$ . This is possible since  $|H_v| < k$  for all  $v \in P_i$ . Collapse each set  $P_{i,j}$  into a single vertex  $p_{i,j}$  in  $H$ , whose associated subgraph in  $G$  is  $H_{p_{i,j}} := \bigcup\{H_v : v \in P_{i,j}\}$ . Since the vertices in  $P_{i,j}$  have the same neighbourhood,  $\{H_v : v \in V(H)\}$  remains a connected partition of  $G$ . Let  $E_{\text{big}} = \{p_{i,j} : 1 \leq i \leq s, 1 \leq j \leq t_i - 1\}$  and  $E_{\text{small}} = \{p_{i,t_i} : 1 \leq i \leq s\}$ . Then every vertex in  $E_{\text{big}}$  is big and every vertex in  $E_{\text{small}}$  is small.

Suppose that  $|E_{\text{small}}| \geq 2^{\ell^2} \cdot |A|$ . Let  $X$  be the graph obtained from  $A$  by adding a clique on  $N(v)$  for each vertex  $v \in E_{\text{small}}$ . Since distinct vertices in  $E_{\text{small}}$  have distinct neighbourhoods, this process adds at least  $|E_{\text{small}}| \geq 2^{\ell^2} \cdot |A|$  cliques. Thus by Lemma 1, a  $K_\ell$ -model of  $X$  can be computed in  $\mathcal{O}(|G|)$  time. For every edge  $x_i x_j$  in this  $K_\ell$ -model that is in  $X$  but not in  $A$ , we have  $x_i, x_j \in N(v)$  for some  $v \in E_{\text{small}}$ . Since  $v$  is not assigned, there is a vertex  $u \in D \cup E^*$  assigned to  $\{x_i, x_j\}$ , and  $u$  is adjacent to both  $x_i$  and  $x_j$ . Since  $u$  is not in the  $K_\ell$ -model, we can include  $u$  in the connected subgraph of the  $K_\ell$ -model that contains  $x_i$  or  $x_j$ , and we obtain a  $K_\ell$ -model in  $A \cup D \cup E^*$  (in particular, without the edge  $x_i x_j$ ). Now assume that  $|E_{\text{small}}| < 2^{\ell^2} \cdot |A|$ .

**Step 6:** We have now partitioned  $V(H)$  into sets  $A \cup E_{\text{big}}$  of big vertices, and sets  $C \cup D \cup E^* \cup E_{\text{small}}$  of small vertices. We have proved that  $|C| < |A|$ ,  $|D| + |E^*| < 2^{\ell-3} \cdot |A|$ , and  $|E_{\text{small}}| < 2^{\ell^2} \cdot |A|$ . Thus the number of small vertices is less than  $(1 + 2^{\ell-3} + 2^{\ell^2} + 1) \cdot |A| \leq 2^{\ell^2 + \ell - 2} \cdot |A|$ . By definition, the number of big vertices in  $H$  is at most  $|G| \cdot k^{-1}$ . In particular,  $|A| \leq |G| \cdot k^{-1}$ . Thus  $|H| \leq 2^{\ell^2 + \ell - 1} \cdot |G| \cdot k^{-1}$ . Moreover, every  $|H_v| < 2k$  for every vertex  $v \in V(H)$ .  $\square$

## References

- [1] NOGA ALON, PAUL SEYMOUR, AND ROBIN THOMAS. A separator theorem for nonplanar graphs. *J. Amer. Math. Soc.*, 3(4):801–808, 1990.
- [2] ERIC D. DEMAINE AND MOHAMMADTAGHI HAJIAGHAYI. Graphs excluding a fixed minor have grids as large as treewidth, with combinatorial and algorithmic applications through bidimensionality. In *Proc. 16th Annual ACM-SIAM Symp. on Discrete Algorithms (SODA '05)*. ACM, 2005.
- [3] MARTIN GROHE. Local tree-width, excluded minors, and approximation algorithms. *Combinatorica*, 23(4):613–632, 2003.
- [4] ALEXANDR V. KOSTOCHKA. Lower bound of the Hadwiger number of graphs by their average degree. *Combinatorica*, 4(4):307–316, 1984.
- [5] RICHARD J. LIPTON AND ROBERT E. TARJAN. A separator theorem for planar graphs. *SIAM J. Appl. Math.*, 36(2):177–189, 1979.
- [6] WOLFGANG MADER. Homomorphieeigenschaften und mittlere Kantendichte von Graphen. *Math. Ann.*, 174:265–268, 1967.
- [7] SERGE PLOTKIN, SATISH RAO, AND WARREN D. SMITH. Shallow excluded minors and improved graph decompositions. In *Proc. 5th Annual ACM-SIAM Symp. on Discrete Algorithms (SODA '94)*, pp. 462–470. ACM, 1994.
- [8] NEIL ROBERTSON AND PAUL D. SEYMOUR. Graph minors. XIII. The disjoint paths problem. *J. Combin. Theory Ser. B*, 63(1):65–110, 1995.
- [9] ARNOLD L. ROSENBERG AND LENWOOD S. HEATH. *Graph separators, with applications*. Frontiers of Computer Science. Kluwer, 2001.
- [10] ANDREW THOMASON. An extremal function for contractions of graphs. *Math. Proc. Cambridge Philos. Soc.*, 95(2):261–265, 1984.

