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Kenji Kashiwabara, Masataka Nakamura

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# Brylawski's Decomposition of NBC Complexes of Abstract Convex Geometries and Their Associated Algebras

Kenji Kashiwabara<sup>1</sup> and Masataka Nakamura<sup>1</sup>

<sup>1</sup> Department of Systems Science, University of Tokyo, Komaba 3-8, Meguro, Tokyo, Japan

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We introduce a notion of a *broken circuit* and an *NBC complex* for an (abstract) convex geometry. Based on these definitions, we shall show the analogues of the Whitney-Rota's formula and Brylawski's decomposition theorem for broken circuit complexes on matroids for convex geometries. We also present an Orlik-Solomon type algebra on a convex geometry, and show the NBC generating theorem. This note is on the same line as the studies in [10, 11, 12].

**Keywords:** broken circuit, characteristic polynomial, NBC basis theorem

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## 1 Closure Systems, Matroids, and Convex Geometries

A collection  $K \subseteq 2^E$  of subsets of a finite set  $E$  is a *closure system* if

- (1)  $E \in K$ ,
- (2)  $X, Y \in K \implies X \cap Y \in K$ .

An element of  $K$  is called a *closed set*. A closure system determines a closure operator

$$\sigma(A) = \bigcap_{X \in K, A \subseteq X} X \quad (A \subseteq E). \quad (1.1)$$

An element in  $\bigcap \{X : X \in K\} = \sigma(\emptyset)$  is a *loop*, and  $K$  is *loop-free* if it has no loops.

A map  $Ex : 2^E \rightarrow 2^E$  defined by  $Ex(A) = \{e \in A : e \notin \sigma(A \setminus e)\}$  ( $A \subseteq E$ ) is an *extreme function*. We say that an element in  $Ex(A)$  is an *extreme element* of  $A$ , and we call an extreme element of the entire set  $E$  a *coloop*. A subset  $A \subseteq E$  is an *independent set* if  $Ex(A) = A$ . A set which is not independent is *dependent*, and a minimal dependent set is called a *circuit*. It is easy to see that any subset of an independent set is independent.

When a closure operator satisfies the Steinitz-McLane exchange property below,

$$\text{if } x, y \notin \sigma(A) \text{ and } y \in \sigma(A \cup x), \text{ then } x \in \sigma(A \cup y) \quad (x, y \in E, A \subseteq E), \quad (1.2)$$

then the corresponding closure system is the set of flats (closed sets) of a matroid  $M$  on  $E$ , and vice versa. The notions of an independent set and a circuit introduced above as a closure system agree with the ordinary definitions of matroid theory.

Let  $M$  be a matroid on  $E$ , and suppose we have a linear order  $\omega$  on  $E$ . When  $C$  is a circuit of  $M$  and  $e$  is the minimum element in  $C$  with respect to  $\omega$ , we call  $C \setminus e$  a *broken circuit*.

A subset of  $E$  is *nbc-independent* if it contains no broken circuits of  $M$ . Evidently an nbc-independent set is an independent set of  $M$ . The collection of nbc-independent sets forms a simplicial complex  $NBC(M, \omega)$ , which is called a *broken circuit complex* of  $M$  (with respect to  $\omega$ ).

When the closure operator satisfies the anti-exchange property below

$$\text{if } x, y \notin \sigma(A) \text{ and } y \in \sigma(A \cup x), \text{ then } x \notin \sigma(A \cup y) \quad (x, y \in E, A \subseteq E), \quad (1.3)$$

the closure system  $K$  is called an (*abstract*) *convex geometry*. Convex geometries arise from various combinatorial objects such as affine point configurations, chordal graphs, posets, semi-lattices, searches on a rooted graph, and so on. (See [4, 9].)

Since a convex geometry itself is a closure system, we have the corresponding definitions of an independent set and a circuit for a convex geometry. In a circuit of a convex geometry there exists uniquely an element that is not extreme. (In a circuit of a matroid there is no element that is extreme.) That is, a circuit  $C$  of a convex geometry contains a unique element  $e$  such that  $Ex(C) = C \setminus e$ . We say that  $e$  is the *root* of  $C$ , and  $X = C \setminus e$  is a *broken circuit* with respect to the root  $e$ . And  $(X, e)$  is a *rooted circuit*. Let us call a set *nbc-independent* if it contains no broken circuit. The collection of nbc-independent sets forms a simplicial complex, which is the *NBC complex* of  $K$  denoted by  $NBC(K)$ .

Note that to determine a broken circuit for a matroid it is required to assume a linear order on the underlying set, while there is no need to suppose such an order when we define a broken circuit for a convex geometry.

## 2 Whitney-Rota's Formula and Its Analogue

### 2.1 Matroid

The NBC complexes of matroids appear in the Whitney-Rota's formula. Let  $\mathcal{L}(M)$  be the lattice consisting of the closed sets (flats) of  $M$ . The characteristic polynomial  $p(M; \lambda)$  of  $M$  is defined by

$$p(\lambda; M) = \sum_{X \in \mathcal{L}(M)} \mu(\sigma(\emptyset), X) \lambda^{r(E) - r(X)}. \quad (2.1)$$

Then the Whitney-Rota's formula for matroids is described as

**Theorem 2.1 (Rota [14])** For an arbitrary linear order  $\omega$  on  $E$ , we have

$$p(\lambda; M) = \sum_{X \in NBC(M, \omega)} (-1)^{|X|} \lambda^{r(E) - r(X)}. \quad (2.2)$$

### 2.2 Convex Geometry

Let  $K$  be a loop-free convex geometry on a finite set  $E$ . The characteristic function of  $K$  is

$$p(\lambda; K) = \sum_{X \in K} \mu_K(\emptyset, X) \lambda^{|E| - |X|} \quad (2.3)$$

where  $\mu_K$  is the Möbius function of the lattice  $K$ . A set which is both closed and independent is a *free set*. The collection of the free sets constitutes a simplicial complex, called a free complex [3]. A free complex plays an important role in the counting formula of the interior points of an affine point configuration proved by Klain [8], and Edelman and Reiner [5]. A free complex of a convex geometry can be revealed to be equal to its NBC complex. That is,

**Theorem 2.2** A subset of  $E$  is a free set if and only if it is nbc-independent. Equivalently, the free complex of a convex geometry coincides with its NBC complex.

Edelman [3] explicitly determined the values of  $\mu_K$  as:

**Lemma 2.1 (Edelman [3])** For a closed set  $X \in K$ ,

$$\mu_K(\emptyset, X) = \begin{cases} (-1)^{|X|} & \text{if } X \text{ is free,} \\ 0 & \text{otherwise.} \end{cases} \quad (2.4)$$

Theorem 2.2 and Lemma 2.1 immediately give rise to the Whitney-Rota's formula for convex geometry:

**Theorem 2.3** For a convex geometry  $K$  and the characteristic polynomial (2.3), it holds that

$$p(\lambda; K) = \sum_{X \in NBC(K)} (-1)^{|X|} \lambda^{|E|-|X|}. \quad (2.5)$$

### 3 Brylawskis Decomposition and Its Analogue

#### 3.1 Matroid

Brylawski [2] showed a direct-sum decomposition theorem of NBC complex of a matroid below.

**Theorem 3.1 (Brylawski [2])** Let  $(M, \omega)$  be an ordered matroid, and  $x$  be the maximum element with respect to  $\omega$ . Then

$$NBC(M, \omega) = NBC(M \setminus x, \omega) \uplus (NBC(M/x, \omega) * x) \quad (3.1)$$

where  $NBC(M/x, \omega) * x = \{A \cup x : A \in NBC(M/x, \omega)\}$

#### 3.2 Convex Geometry

Let  $K$  be a convex geometry on  $E$ . For a coloop  $e$ ,  $K \setminus e = \{X : X \in K, e \notin X\}$  is a convex geometry on  $E \setminus e$ , which is a *deletion* of  $e$  from  $K$ . For any element  $e \in E$ ,  $K/e = \{X \setminus e : X \in K, e \in X\}$  is a convex geometry on  $E \setminus e$ , which is a *contraction* of  $e$  from  $K$ . We have Brylawski's decomposition theorem for convex geometries as

**Theorem 3.2** For a coloop  $x \in E$  of a convex geometry  $K$ , we have

$$NBC(K) = NBC(K \setminus x) \uplus (NBC(K/x) * x) \quad (3.2)$$

## 4 Orlik-Solomon Algebra and Its Analogues

### 4.1 Matroid

An NBC complex is known to provide a linear basis of the Orlik-Solomon algebra, which we shall describe below. Suppose  $E = \{e_1, \dots, e_n\}$ . Taking  $e_1, \dots, e_n$  as generators, we denote a graded external algebra over the free module  $\bigoplus_{e \in E} \mathbb{Z}e$  by  $\bigwedge E = \bigoplus_{i \in \mathbb{N}} \bigwedge^i E$ . A linear map  $\partial : \bigwedge E \rightarrow \bigwedge E$  is defined by

- (1)  $\partial_0 : \mathbb{Z} \rightarrow (0)$ ,                      (2)  $\partial_1 : \bigwedge^1 E \rightarrow \mathbb{Z}$  where  $\partial(e) = 1$     ( $e \in E$ ),  
 (3) for  $k = 2, \dots, n$ :

$$\partial_k : \bigwedge^k E \rightarrow \bigwedge^{k-1} E, \quad \partial_k(e_{i_1} \wedge \dots \wedge e_{i_k}) = \sum_{j=1}^k (-1)^{j-1} e_{i_1} \wedge \dots \wedge \widehat{e_{i_j}} \wedge \dots \wedge e_{i_k}$$

Although it is a little abuse of terminology, for the sake of simplicity, we associate a term  $e_X = x_1 \wedge \dots \wedge x_t$  in  $\bigwedge E$  with each subset  $X = \{x_1, \dots, x_t\} \subseteq E$ .

Suppose  $I(M)$  to be an ideal generated by  $\{\partial(e_C) : C \text{ is a circuit of } M \text{ with } |C| \geq 2\} \cup \{e : e \text{ is a loop of } M\}$ . Then the *Orlik-Solomon algebra* of  $M$  is defined as

$$OS(M) = \left( \bigwedge E \right) / I(M). \quad (4.1)$$

**Theorem 4.1 (NBC basis theorem for the Orlik-Solomon algebra [1], [13])** Let  $M$  be a matroid on  $E$ , and  $\omega$  be an arbitrary linear order on the underlying set. Then  $\{e_X : X \in NBC(M, \omega)\}$  is a linear basis of module  $OS(M)$ .

### 4.2 Convex Geometry

Suppose  $K$  to be a loop-free convex geometry on  $E = \{e_1, \dots, e_n\}$ . The graded external algebra  $\bigwedge E = \bigoplus_{i=0}^n \bigwedge^i E$  and a linear map  $\partial : \bigwedge E \rightarrow \bigwedge E$  are defined in the same way as before. And let  $I(K)$  be the ideal in  $\bigwedge E$  generated by  $\{\partial(e_C) : C \text{ is a circuit of } K\}$ , and let us define an *Orlik-Solomon type algebra of a convex geometry  $K$*  by

$$OS(K) = \left( \bigwedge E \right) / I(K). \quad (4.2)$$

It can be shown that  $\{e_X : X \in NBC(K)\}$  is a linear generating set of  $OS(K)$ . That is, although the NBC basis theorem (Theorem 4.1) does not hold for  $OS(K)$ , we have a weaker form, the NBC generating theorem, below.

**Theorem 4.2** An arbitrary element in  $OS(K)$  can be represented as a linear combination of the terms in  $\{e_X : X \in NBC(K)\}$ .

There is an alternative definition of an Orlik-Solomon type algebra so that the NBC basis theorem would be satisfied. Let  $J(K)$  be the ideal generated by  $\{e_X : X \text{ is a broken circuit of } K\}$ , and let us define an algebra

$$A(K) = \left( \bigwedge E \right) / J(K) \quad (4.3)$$

By definition  $\{e_X : X \in NBC(K)\}$  is necessarily a linear basis of module  $A(K)$ .

Hence the decomposition of Theorem 3.2 readily implies the short exact split sequence theorem for  $A(K)$ .

**Theorem 4.3** For a coloop  $x$  of a convex geometry  $K$ ,

$$0 \rightarrow A(K \setminus x) \xrightarrow{i_x} A(K) \xrightarrow{p_x} A(K/x) \rightarrow 0 \quad (4.4)$$

is an exact short split sequence.

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