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Density of universal classes of series-parallel graphs

Jaroslav Nešetřil^{1†} and Yared Nigussie^{1‡}

¹ Department of Applied Mathematics
Institute for Theoretical Computer Science(ITI)
Charles University
Malostranské nám.25
11800 Praha 1 Czech Republic

A class of graphs \mathcal{C} ordered by the homomorphism relation is *universal* if every countable partial order can be embedded in \mathcal{C} . It was shown in [1] that the class \mathcal{C}_k of k -colorable graphs, for any fixed $k \geq 3$, induces a universal partial order. In [4], a surprisingly small subclass of \mathcal{C}_3 which is a proper subclass of K_4 -minor-free graphs (\mathcal{G}/K_4) is shown to be universal. In another direction, a density result was given in [9], that for each rational number $a/b \in [2, 8/3] \cup \{3\}$, there is a K_4 -minor-free graph with circular chromatic number equal to a/b . In this note we show for each rational number a/b within this interval the class $\mathcal{K}_{a/b}$ of K_4 -minor-free graphs with circular chromatic number a/b is universal if and only if $a/b \neq 2, 5/2$ or 3 . This shows yet another surprising richness of the K_4 -minor-free class that it contains universal classes as dense as the rational numbers.

Keywords: circular chromatic number, homomorphism, series-parallel graphs, universality

1 Introduction

We assume graphs are finite and simple (with no loops and parallel edges). Let G, G' be graphs. A *homomorphism* from G to G' is a mapping $f: V(G) \rightarrow V(G')$ which preserves adjacency. That is, $\{u, v\} \in E(G)$ implies $\{f(u), f(v)\} \in E(G')$. We write $G \leq G'$ if there is a homomorphism from G to G' . The notation $G < G'$ means $G \leq G' \not\leq G$, whereas $G \sim G'$ means $G \leq G' \leq G$. If $G \sim G'$, we say G and G' are *hom-equivalent*. The smallest graph H for which $G \sim H$ is called the *core* of G . For finite graphs, the core is uniquely determined up to an isomorphism. It can also be seen that H is an induced subgraph of G . This will be denoted by $H \subseteq G$. Let \mathcal{C} and \mathcal{C}' be two classes of graphs. We also write $\mathcal{C} \sim \mathcal{C}'$ if for each graph $G \in \mathcal{C}$ there exists a $G' \in \mathcal{C}'$ such that $G \sim G'$ and vice versa. See [2] for introduction to graphs and their homomorphisms.

Let $k \geq d \geq 1$ be integers. The *circular chromatic number* of G , written $\chi_c(G)$, is the smallest rational k/d such that $G \leq K_{k/d}$, where $K_{k/d}$ is the *circular graph* with $V(K_{k/d}) = \{0, 1, 2, \dots, k-1\}$ and

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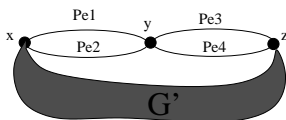


Fig. 1: Unavoidable configuration of G (a minimal counterexample to Lemma 6) with odd girth $2k+1$ and $l_{e_1} + l_{e_2} = l_{e_3} + l_{e_4} = 2k+1$.

$E(K_{k/d}) = \{\{i, j\} : d \leq |i - j| \leq k - d\}$. Note that when $d = 1$ we have the usual vertex coloring of G . Let $\mathcal{K}_{a/b} = \{G \in \mathcal{G}/K_4 : \chi_c(G) = a/b\}$. See [10] for some other equivalent definitions. It is trivial to see the following:

Theorem 1 $\mathcal{K}_2 \sim \{K_2\}$.

It is well known that graphs in \mathcal{G}/K_4 are 3-colorable. Hell and Zhu [3] have shown that triangle-free graphs in \mathcal{G}/K_4 have circular chromatic number at most $8/3$. Hence no graph in \mathcal{G}/K_4 has circular chromatic number in the interval $(8/3, 3)$. Hence, we have:

Theorem 2 $\mathcal{K}_3 \sim \{C_3\}$.

The main results of this note are the following two theorems establishing nice dichotomy between universality and homomorphism finiteness of the class $\mathcal{K}_{a/b}$:

Theorem 3 $\mathcal{K}_{5/2} \sim \{C_5\}$.

Somewhat surprisingly we show that Theorem 1, 2, and 3 cover all cases when $\mathcal{K}_{a/b}$ is a finite set.

Theorem 4 $\mathcal{K}_{a/b}$ is universal if $a/b \in (2, 5/2) \cup (5/2, 8/3]$.

In section 2 we prove Theorem 3 using a folding lemma. In section 3 we prove Theorem 4.

2 $\mathcal{K}_{5/2}$ is equivalent to $\{C_5\}$

Let G be a graph. A *thread* in G is a path $P \subseteq G$ such that the two endpoints of P have degree at least 3 and all internal vertices of P are degree 2 in G . We shall often use the fact that if P and P' are two edge-disjoint paths and if the lengths of P and P' have same parity such that P is a thread and has at least equal length as P' , then there is a homomorphism that maps P to P' sending the two ends of P to the two ends of P' . Such a homomorphism is said to *fold* P to P' . Let G be a graph and let G^s denote the multi-graph we obtain from G by “smoothing” all degree 2 vertices of G . For each edge e of G^s , let P_e denote the thread of G represented by e in G^s , and let l_e denote the length of P_e .

The following Folding Lemma for K_4 -minor-free graphs is an analogy of the Folding Lemma of Klostermeyer and Zhang [6] for planar graphs. Its proof is easy (see [7]).

Lemma 5 (Edge folding lemma) *Let $G \in \mathcal{G}/\{K_4\}$ be of odd girth $2k+1$ and let e, e' be parallel edges in G^s , with common end vertices x, y . If G is not homomorphic to a strictly smaller graph of the same odd girth, then $l_e + l_{e'} = 2k+1$. Moreover, $P_e \cup P_{e'}$ is the unique cycle of length $2k+1$ containing both x and y .*

For short let K^m denote $K_{(7+5m)/(3+2m)}$. Recall that $V(K^m) = \{0, 1, \dots, 6+5m\}$.

Lemma 6 Let $G \in \mathcal{G}/\{K_4\}$ be of odd girth at least 7. Then $\chi_c(G) \leq (7 + 5m)/(3 + 2m) < 5/2$, for some $m < |V(G)|/2$.

Proof: Let $G \in \mathcal{G}/\{K_4\}$ be a core of odd girth $g \geq 7$. It suffices to show $G \leq K^m$ for some $m \geq 0$. Let \bar{G}^s be the graph we get by identifying parallel edges of G^s . Then, $\bar{G}^s \in \mathcal{G}/K_4$ and so there exists a $y \in V(\bar{G}^s)$ such that the degree $\deg_{\bar{G}^s}(y) = 2$. Then $3 \leq \deg_{G^s}(y) \leq 4$ (here we use a parity argument that, the multiplicity of edges of G^s is at most two, as G is a core). By Lemma 5, and assuming G is a minimal counterexample we can get $\deg_{G^s}(y) = 4$. Hence, a configuration depicted in Figure 1 is unavoidable. Let $G' = G - (\bigcup_{i=1}^4 P_{e_i} - \{x, z\})$. By induction $G' \leq K^m$, for some $m \geq 0$. We can assume $f(x) = 0$. By investigating a few cases for values of $f(y)$, it is not hard to see $G \leq K^{m+1}$, contrary to assumption (see [7] for detailed proof). \square **Proof of Theorem 3:** Let $G \in \mathcal{K}_{5/2}$ be

of odd girth g . Then $G \leq C_5$. By Lemma 6, we have $g \leq 5$. By Theorem 2, $g > 3$. Hence $g = 5$ and so $C_5 \leq G \leq C_5$. Hence $G \sim C_5$. The converse is obvious since $\chi_c(C_5) = 5/2$.

3 Universal sets of \mathcal{G}/K_4 are dense in $(2, 5/2) \cup (5/2, 8/3]$

In this section we shall show that we obtain a universal class $\mathcal{K}_{p/q} \subset \mathcal{G}/K_4$ for arbitrary $p/q \in (2, 5/2) \cup (5/2, 8/3]$. We use a graph $G_{p/q}$ with $\chi_c(G) = p/q$ as a generator of $\mathcal{K}_{p/q}$. We assume $G_{p/q}$ has the following two properties:

- (P1) $G_{p/q}$ is hom-equivalent neither to a cycle nor to a vertex.
- (P2) if $G' \in \mathcal{G}/K_4$ satisfies (P1) and $\chi_c(G') = p/q$, then $|V(G')| \geq |V(G)|$.

Lemma 7 Let $G \in \mathcal{G}/K_4$ have properties (P1) and (P2). Then, G is 2-connected. Moreover, G is a core and it is not vertex-transitive.

Proof: Since the circular graph $K_{k/d}$ is a vertex-transitive graph, for all k, d , we have $\chi_c(G) = \max_i(\chi_c(H_i)), 1 \leq i \leq p$, where each H_i is a 2-connected component of G . Here, (P2) implies that $p = 1$ and so G is 2-connected. Next, note that any graph $G \in \mathcal{G}/K_4$ is vertex-transitive if and only if G is an odd cycle or K_1 or K_2 . This is because all other 2-connected graphs in \mathcal{G}/K_4 have at least one degree-2 vertex and one non-degree-2 vertex. Hence by (P1), G is not vertex-transitive. Moreover, by (P1) the core of G also is not vertex-transitive. By (P2), we deduce G is a core. \square For any rational

number $p/q \in (2, 8/3]$, Pan and Zhu have shown in [9] a recursive method of constructing a 2-connected graph $G_{p/q}$ with $\chi_c(G) = p/q$. If $p/q \neq (2k + 1)/k$ then $G_{p/q}$ satisfies (P1). If $p/q = (2k + 1)/k$, the graph given in [9] is the cycle C_{2k+1} which is the natural candidate. Cycles do not satisfy (P1), hence we introduce a graph denoted by G^k of odd girth $2k + 3$ as follows: Take a triangle and double each edge to obtain a multi-graph H . For $i = 0, 1, 2$, let $\{e_1^i, e_2^i\}$, be the three parallel pairs of edges of H . To obtain a thread of length $k + 2$, subdivide e_1^0 and e_2^0 each $k + 1$ times. Next subdivide e_1^1 and e_2^1 each k times. Finally, subdivide e_1^2 three times and e_2^2 , $2k - 2$ times to obtain the graph G^k . We have:

Lemma 8 $\chi_c(G^k) = (2k + 1)/k$ for all $k \geq 3$.

Proof: It is easy to see that $G^k \leq C_{2k+1}$, and $G^k \not\leq C_{2k+3}$. Hence, we have $(2k + 3)/(k + 1) < \chi_c(G^k) \leq (2k + 1)/k$. Note that $\gcd(4k + 4, 2k + 1) = 1$. From basic number theory [8], using what is known as the *Farey sequence*, we can see that any rational strictly between $(2k + 3)/(k + 1)$ and $(2k + 1)/k$

has numerator $a \geq 4k + 4$. But then, if $k \geq 3$ the circumference of G^k is $4k + 3$. It is well known [10] that the numerator a of a circular chromatic number a/b of a graph G is at most its circumference. We deduce $\chi_c(G^k) = (2k + 1)/k$. \square

Corollary 9 *For every rational number $p/q \in (2, 5/2) \cup (5/2, 8/3]$ there is a graph $G_{p/q}$ satisfying (P1) and (P2).*

Next we prove that $\mathcal{K}_{p/q}$ inherits universality from the class \mathcal{P} of directed finite paths [5]. We take several isomorphic copies H_1, \dots, H_n of a fixed graph H , such that $\chi_c(H) = p/q$ and construct a ‘path-like’ structured graph H' by identifying a vertex of H_i with a vertex of H_{i+1} . Then $\chi_c(H') = \chi_c(H)$ because the circular graphs are vertex-transitive. We call such a construction K_1 -concatenation. A more precise definition of ‘ K_1 -concatenation’ of a graph H :

Let $P \in \mathcal{P}$ be an oriented path of length $n \geq 1$, $V(P) = \{v_1, v_2, \dots, v_{n+1}\}$. Then either $v_i v_{i+1}$ or $v_{i+1} v_i \in E(P)$ (but not both). Let H be a graph and $a, b \in V(H)$. Let H_1, H_2, \dots, H_n be isomorphic copies of H and let a_i, b_i be the vertices of H_i corresponding to a and b . The K_1 -concatenation of H by P is a graph $P * (H, a, b)$ constructed as follows: For $i = 1, \dots, n$, if $v_i v_{i+1} \in E(P)$, choose b_i (otherwise choose a_i). Then identify every chosen vertex of H_i with the unchosen vertex of H_{i+1} . To make the construction non-trivial, we choose a and b so that there is no automorphism sending a to b or b to a . If H satisfies (P1), then we know there exists such a pair.

Lemma 10 *Let $G_{p/q} \in \mathcal{G}/K_4$ satisfy (P1), (P2). Then, $\mathcal{K}_{p/q}$ is universal.*

Proof: Since the class of oriented paths \mathcal{P} is universal we show for every $P, P' \in \mathcal{P}$, we have $P \leq P'$ if and only if $P * (G_{p/q}, a, b) \leq P' * (G_{p/q}, a, b)$. This proves the lemma.

The forward implication is straightforward. To prove the reverse implication, let $P, P' \in \mathcal{P}$ of length n and n' such that P is a core and suppose there exists a homomorphism $f : P * (G_{p/q}, a, b) \rightarrow P' * (G_{p/q}, a, b)$. Assume further that P is not an edge, since this case is trivial, and without loss of generality, assume that the first edge of P is directed forward. Let H_1, \dots, H_n and $H'_1, \dots, H'_{n'}$ be isomorphic copies of $G_{p/q}$. First we show that $f|_{H_i}$ is induced by an automorphism of $G_{p/q}$ for each i . Suppose not. Then the image $f(H_i)$ is connected. If $f(H_i)$ is 2-connected then, it is isomorphic to $G_{p/q}$, since $G_{p/q}$ is a core. Suppose $f(H_i)$ is not 2-connected. Then each 2-connected component F of $f(H_i)$ is a proper subgraph of $G_{p/q}$. By (P2), we have $\chi_c(F) < p/q$. Then, $H \not\leq f(H_i)$, a contradiction. Hence $f|_{H_i}$ is induced by an automorphism of $G_{p/q}$.

Now we claim a stronger assertion that for any i , $f(a_i) = a'_j$ and $f(b_i) = b'_j$, for some $j, 1 \leq j \leq m$. let $H'_j = f(H_1)$. Since $f|_{H_1}$ is an automorphism, $f(b_1)$ must be in the same automorphism class of b'_j . Suppose that $f(b_1) \neq b'_j$, then $f(b_1)$ is not a cut-vertex of $P' * (G_{p/q}, a, b)$. As b_1 is a cut-vertex of $P * (G_{p/q}, a, b)$, it is identified with either a_2 or b_2 . If it were identified with a_2 , then $f|_{H_2}$ would be an automorphism of $G_{p/q}$ such that $f(a_2) = b'_j$, contrary to the choice of a, b in $V(G_{p/q})$. Hence b_1 is identified with b_2 . Similarly, we get a_2 is identified with a_3 , and b_3 with b_4 and so on. This implies P is a ‘zig-zag’ which is hom-equivalent to an edge, contrary to P being a core. So $f(b_1) = b'_j$. The claim follows by induction on the length of P .

We define a homomorphism $g : V(P) \rightarrow V(P')$ so that if $f(a_i) = a'_j$ then $g(v_i) = v'_j$ and similarly for b_i . By our construction g preserves the adjacency condition and so $P \leq P'$. \square **Proof of Theorem 4:** Let $a/b \in (2, 5/2) \cup (5/2, 8/3]$. By Lemma 9, there is a graph $G_{a/b}$ with properties (P1),(P2). By Lemma 10, $\mathcal{K}_{a/b}$ is universal. This concludes our result.

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