



**HAL**  
open science

## Density of universal classes of series-parallel graphs

Jaroslav Nešetřil, Yared Nigussie

► **To cite this version:**

Jaroslav Nešetřil, Yared Nigussie. Density of universal classes of series-parallel graphs. 2005 European Conference on Combinatorics, Graph Theory and Applications (EuroComb '05), 2005, Berlin, Germany. pp.245-250, 10.46298/dmtcs.3407 . hal-01184364

**HAL Id: hal-01184364**

**<https://inria.hal.science/hal-01184364v1>**

Submitted on 14 Aug 2015

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Density of universal classes of series-parallel graphs

Jaroslav Nešetřil<sup>1†</sup> and Yared Nigussie<sup>1‡</sup>

<sup>1</sup> Department of Applied Mathematics  
Institute for Theoretical Computer Science(ITI)  
Charles University  
Malostranské nám.25  
11800 Praha 1 Czech Republic

---

A class of graphs  $\mathcal{C}$  ordered by the homomorphism relation is *universal* if every countable partial order can be embedded in  $\mathcal{C}$ . It was shown in [1] that the class  $\mathcal{C}_k$  of  $k$ -colorable graphs, for any fixed  $k \geq 3$ , induces a universal partial order. In [4], a surprisingly small subclass of  $\mathcal{C}_3$  which is a proper subclass of  $K_4$ -minor-free graphs ( $\mathcal{G}/K_4$ ) is shown to be universal. In another direction, a density result was given in [9], that for each rational number  $a/b \in [2, 8/3] \cup \{3\}$ , there is a  $K_4$ -minor-free graph with circular chromatic number equal to  $a/b$ . In this note we show for each rational number  $a/b$  within this interval the class  $\mathcal{K}_{a/b}$  of  $K_4$ -minor-free graphs with circular chromatic number  $a/b$  is universal if and only if  $a/b \neq 2, 5/2$  or  $3$ . This shows yet another surprising richness of the  $K_4$ -minor-free class that it contains universal classes as dense as the rational numbers.

**Keywords:** circular chromatic number, homomorphism, series-parallel graphs, universality

---

## 1 Introduction

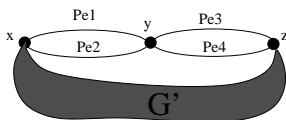
We assume graphs are finite and simple (with no loops and parallel edges). Let  $G, G'$  be graphs. A *homomorphism* from  $G$  to  $G'$  is a mapping  $f: V(G) \rightarrow V(G')$  which preserves adjacency. That is,  $\{u, v\} \in E(G)$  implies  $\{f(u), f(v)\} \in E(G')$ . We write  $G \leq G'$  if there is a homomorphism from  $G$  to  $G'$ . The notation  $G < G'$  means  $G \leq G' \not\leq G$ , whereas  $G \sim G'$  means  $G \leq G' \leq G$ . If  $G \sim G'$ , we say  $G$  and  $G'$  are *hom-equivalent*. The smallest graph  $H$  for which  $G \sim H$  is called the *core* of  $G$ . For finite graphs, the core is uniquely determined up to an isomorphism. It can also be seen that  $H$  is an induced subgraph of  $G$ . This will be denoted by  $H \subseteq G$ . Let  $\mathcal{C}$  and  $\mathcal{C}'$  be two classes of graphs. We also write  $\mathcal{C} \sim \mathcal{C}'$  if for each graph  $G \in \mathcal{C}$  there exists a  $G' \in \mathcal{C}'$  such that  $G \sim G'$  and vice versa. See [2] for introduction to graphs and their homomorphisms.

Let  $k \geq d \geq 1$  be integers. The *circular chromatic number* of  $G$ , written  $\chi_c(G)$ , is the smallest rational  $k/d$  such that  $G \leq K_{k/d}$ , where  $K_{k/d}$  is the *circular graph* with  $V(K_{k/d}) = \{0, 1, 2, \dots, k-1\}$  and

---

<sup>†</sup>Supported by a Grant 1M0021620808 of Czech Ministry of Education

<sup>‡</sup>Partially supported by DIMATIA and 1M0021620808



**Fig. 1:** Unavoidable configuration of  $G$  (a minimal counterexample to Lemma 6) with odd girth  $2k+1$  and  $l_{e_1} + l_{e_2} = l_{e_3} + l_{e_4} = 2k + 1$ .

$E(K_{k/d}) = \{\{i, j\} : d \leq |i - j| \leq k - d\}$ . Note that when  $d = 1$  we have the usual vertex coloring of  $G$ . Let  $\mathcal{K}_{a/b} = \{G \in \mathcal{G}/K_4 : \chi_c(G) = a/b\}$ . See [10] for some other equivalent definitions. It is trivial to see the following:

**Theorem 1**  $\mathcal{K}_2 \sim \{K_2\}$ .

It is well known that graphs in  $\mathcal{G}/K_4$  are 3-colorable. Hell and Zhu [3] have shown that triangle-free graphs in  $\mathcal{G}/K_4$  have circular chromatic number at most  $8/3$ . Hence no graph in  $\mathcal{G}/K_4$  has circular chromatic number in the interval  $(8/3, 3)$ . Hence, we have:

**Theorem 2**  $\mathcal{K}_3 \sim \{C_3\}$ .

The main results of this note are the following two theorems establishing nice dichotomy between universality and homomorphism finiteness of the class  $\mathcal{K}_{a/b}$ :

**Theorem 3**  $\mathcal{K}_{5/2} \sim \{C_5\}$ .

Somewhat surprisingly we show that Theorem 1, 2, and 3 cover all cases when  $\mathcal{K}_{a/b}$  is a finite set.

**Theorem 4**  $\mathcal{K}_{a/b}$  is universal if  $a/b \in (2, 5/2) \cup (5/2, 8/3]$ .

In section 2 we prove Theorem 3 using a folding lemma. In section 3 we prove Theorem 4.

## 2 $\mathcal{K}_{5/2}$ is equivalent to $\{C_5\}$

Let  $G$  be a graph. A *thread* in  $G$  is a path  $P \subseteq G$  such that the two endpoints of  $P$  have degree at least 3 and all internal vertices of  $P$  are degree 2 in  $G$ . We shall often use the fact that if  $P$  and  $P'$  are two edge-disjoint paths and if the lengths of  $P$  and  $P'$  have same parity such that  $P$  is a thread and has at least equal length as  $P'$ , then there is a homomorphism that maps  $P$  to  $P'$  sending the two ends of  $P$  to the two ends of  $P'$ . Such a homomorphism is said to *fold*  $P$  to  $P'$ . Let  $G$  be a graph and let  $G^s$  denote the multi-graph we obtain from  $G$  by “smoothing” all degree 2 vertices of  $G$ . For each edge  $e$  of  $G^s$ , let  $P_e$  denote the thread of  $G$  represented by  $e$  in  $G^s$ , and let  $l_e$  denote the length of  $P_e$ .

The following Folding Lemma for  $K_4$ -minor-free graphs is an analogy of the Folding Lemma of Klostermeyer and Zhang [6] for planar graphs. Its proof is easy (see [7]).

**Lemma 5 (Edge folding lemma)** *Let  $G \in \mathcal{G}/\{K_4\}$  be of odd girth  $2k + 1$  and let  $e, e'$  be parallel edges in  $G^s$ , with common end vertices  $x, y$ . If  $G$  is not homomorphic to a strictly smaller graph of the same odd girth, then  $l_e + l_{e'} = 2k + 1$ . Moreover,  $P_e \cup P_{e'}$  is the unique cycle of length  $2k + 1$  containing both  $x$  and  $y$ .*

For short let  $K^m$  denote  $K_{(7+5m)/(3+2m)}$ . Recall that  $V(K^m) = \{0, 1, \dots, 6 + 5m\}$ .

**Lemma 6** Let  $G \in \mathcal{G}/\{K_4\}$  be of odd girth at least 7. Then  $\chi_c(G) \leq (7 + 5m)/(3 + 2m) < 5/2$ , for some  $m < |V(G)|/2$ .

**Proof:** Let  $G \in \mathcal{G}/\{K_4\}$  be a core of odd girth  $g \geq 7$ . It suffices to show  $G \leq K^m$  for some  $m \geq 0$ . Let  $\bar{G}^s$  be the graph we get by identifying parallel edges of  $G^s$ . Then,  $\bar{G}^s \in \mathcal{G}/K_4$  and so there exists a  $y \in V(\bar{G}^s)$  such that the degree  $\deg_{\bar{G}^s}(y) = 2$ . Then  $3 \leq \deg_{G^s}(y) \leq 4$  (here we use a parity argument that, the multiplicity of edges of  $G^s$  is at most two, as  $G$  is a core). By Lemma 5, and assuming  $G$  is a minimal counterexample we can get  $\deg_{G^s}(y) = 4$ . Hence, a configuration depicted in Figure 1 is unavoidable. Let  $G' = G - (\bigcup_{i=1}^4 P_{e_i} - \{x, z\})$ . By induction  $G' \leq K^m$ , for some  $m \geq 0$ . We can assume  $f(x) = 0$ . By investigating a few cases for values of  $f(y)$ , it is not hard to see  $G \leq K^{m+1}$ , contrary to assumption (see [7] for detailed proof).  $\square$  **Proof of Theorem 3:** Let  $G \in \mathcal{K}_{5/2}$  be

of odd girth  $g$ . Then  $G \leq C_5$ . By Lemma 6, we have  $g \leq 5$ . By Theorem 2,  $g > 3$ . Hence  $g = 5$  and so  $C_5 \leq G \leq C_5$ . Hence  $G \sim C_5$ . The converse is obvious since  $\chi_c(C_5) = 5/2$ .

### 3 Universal sets of $\mathcal{G}/K_4$ are dense in $(2, 5/2) \cup (5/2, 8/3]$

In this section we shall show that we obtain a universal class  $\mathcal{K}_{p/q} \subset \mathcal{G}/K_4$  for arbitrary  $p/q \in (2, 5/2) \cup (5/2, 8/3]$ . We use a graph  $G_{p/q}$  with  $\chi_c(G) = p/q$  as a generator of  $\mathcal{K}_{p/q}$ . We assume  $G_{p/q}$  has the following two properties:

- (P1)  $G_{p/q}$  is hom-equivalent neither to a cycle nor to a vertex.
- (P2) if  $G' \in \mathcal{G}/K_4$  satisfies (P1) and  $\chi_c(G') = p/q$ , then  $|V(G')| \geq |V(G)|$ .

**Lemma 7** Let  $G \in \mathcal{G}/K_4$  have properties (P1) and (P2). Then,  $G$  is 2-connected. Moreover,  $G$  is a core and it is not vertex-transitive.

**Proof:** Since the circular graph  $K_{k/d}$  is a vertex-transitive graph, for all  $k, d$ , we have  $\chi_c(G) = \max_i(\chi_c(H_i)), 1 \leq i \leq p$ , where each  $H_i$  is a 2-connected component of  $G$ . Here, (P2) implies that  $p = 1$  and so  $G$  is 2-connected. Next, note that any graph  $G \in \mathcal{G}/K_4$  is vertex-transitive if and only if  $G$  is an odd cycle or  $K_1$  or  $K_2$ . This is because all other 2-connected graphs in  $\mathcal{G}/K_4$  have at least one degree-2 vertex and one non-degree-2 vertex. Hence by (P1),  $G$  is not vertex-transitive. Moreover, by (P1) the core of  $G$  also is not vertex-transitive. By (P2), we deduce  $G$  is a core.  $\square$  For any rational

number  $p/q \in (2, 8/3]$ , Pan and Zhu have shown in [9] a recursive method of constructing a 2-connected graph  $G_{p/q}$  with  $\chi_c(G) = p/q$ . If  $p/q \neq (2k + 1)/k$  then  $G_{p/q}$  satisfies (P1). If  $p/q = (2k + 1)/k$ , the graph given in [9] is the cycle  $C_{2k+1}$  which is the natural candidate. Cycles do not satisfy (P1), hence we introduce a graph denoted by  $G^k$  of odd girth  $2k + 3$  as follows: Take a triangle and double each edge to obtain a multi-graph  $H$ . For  $i = 0, 1, 2$ , let  $\{e_1^i, e_2^i\}$ , be the three parallel pairs of edges of  $H$ . To obtain a thread of length  $k + 2$ , subdivide  $e_1^0$  and  $e_2^0$  each  $k + 1$  times. Next subdivide  $e_1^1$  and  $e_2^1$  each  $k$  times. Finally, subdivide  $e_1^2$  three times and  $e_2^2$ ,  $2k - 2$  times to obtain the graph  $G^k$ . We have:

**Lemma 8**  $\chi_c(G^k) = (2k + 1)/k$  for all  $k \geq 3$ .

**Proof:** It is easy to see that  $G^k \leq C_{2k+1}$ , and  $G^k \not\leq C_{2k+3}$ . Hence, we have  $(2k + 3)/(k + 1) < \chi_c(G^k) \leq (2k + 1)/k$ . Note that  $\gcd(4k + 4, 2k + 1) = 1$ . From basic number theory [8], using what is known as the *Farey sequence*, we can see that any rational strictly between  $(2k + 3)/(k + 1)$  and  $(2k + 1)/k$

has numerator  $a \geq 4k + 4$ . But then, if  $k \geq 3$  the circumference of  $G^k$  is  $4k + 3$ . It is well known [10] that the numerator  $a$  of a circular chromatic number  $a/b$  of a graph  $G$  is at most its circumference. We deduce  $\chi_c(G^k) = (2k + 1)/k$ .  $\square$

**Corollary 9** *For every rational number  $p/q \in (2, 5/2) \cup (5/2, 8/3]$  there is a graph  $G_{p/q}$  satisfying (P1) and (P2).*

Next we prove that  $\mathcal{K}_{p/q}$  inherits universality from the class  $\mathcal{P}$  of directed finite paths [5]. We take several isomorphic copies  $H_1, \dots, H_n$  of a fixed graph  $H$ , such that  $\chi_c(H) = p/q$  and construct a ‘path-like’ structured graph  $H'$  by identifying a vertex of  $H_i$  with a vertex of  $H_{i+1}$ . Then  $\chi_c(H') = \chi_c(H)$  because the circular graphs are vertex-transitive. We call such a construction  $K_1$ -concatenation. A more precise definition of ‘ $K_1$ -concatenation’ of a graph  $H$ :

Let  $P \in \mathcal{P}$  be an oriented path of length  $n \geq 1$ ,  $V(P) = \{v_1, v_2, \dots, v_{n+1}\}$ . Then either  $v_i v_{i+1}$  or  $v_{i+1} v_i \in E(P)$  (but not both). Let  $H$  be a graph and  $a, b \in V(H)$ . Let  $H_1, H_2, \dots, H_n$  be isomorphic copies of  $H$  and let  $a_i, b_i$  be the vertices of  $H_i$  corresponding to  $a$  and  $b$ . The  $K_1$ -concatenation of  $H$  by  $P$  is a graph  $P * (H, a, b)$  constructed as follows: For  $i = 1, \dots, n$ , if  $v_i v_{i+1} \in E(P)$ , choose  $b_i$  (otherwise choose  $a_i$ ). Then identify every chosen vertex of  $H_i$  with the unchosen vertex of  $H_{i+1}$ . To make the construction non-trivial, we choose  $a$  and  $b$  so that there is no automorphism sending  $a$  to  $b$  or  $b$  to  $a$ . If  $H$  satisfies (P1), then we know there exists such a pair.

**Lemma 10** *Let  $G_{p/q} \in \mathcal{G}/K_4$  satisfy (P1), (P2). Then,  $\mathcal{K}_{p/q}$  is universal.*

**Proof:** Since the class of oriented paths  $\mathcal{P}$  is universal we show for every  $P, P' \in \mathcal{P}$ , we have  $P \leq P'$  if and only if  $P * (G_{p/q}, a, b) \leq P' * (G_{p/q}, a, b)$ . This proves the lemma.

The forward implication is straightforward. To prove the reverse implication, let  $P, P' \in \mathcal{P}$  of length  $n$  and  $n'$  such that  $P$  is a core and suppose there exists a homomorphism  $f : P * (G_{p/q}, a, b) \rightarrow P' * (G_{p/q}, a, b)$ . Assume further that  $P$  is not an edge, since this case is trivial, and without loss of generality, assume that the first edge of  $P$  is directed forward. Let  $H_1, \dots, H_n$  and  $H'_1, \dots, H'_{n'}$  be isomorphic copies of  $G_{p/q}$ . First we show that  $f|_{H_i}$  is induced by an automorphism of  $G_{p/q}$  for each  $i$ . Suppose not. Then the image  $f(H_i)$  is connected. If  $f(H_i)$  is 2-connected then, it is isomorphic to  $G_{p/q}$ , since  $G_{p/q}$  is a core. Suppose  $f(H_i)$  is not 2-connected. Then each 2-connected component  $F$  of  $f(H_i)$  is a proper subgraph of  $G_{p/q}$ . By (P2), we have  $\chi_c(F) < p/q$ . Then,  $H \not\leq f(H_i)$ , a contradiction. Hence  $f|_{H_i}$  is induced by an automorphism of  $G_{p/q}$ .

Now we claim a stronger assertion that for any  $i$ ,  $f(a_i) = a'_j$  and  $f(b_i) = b'_j$ , for some  $j, 1 \leq j \leq m$ . let  $H'_j = f(H_1)$ . Since  $f|_{H_1}$  is an automorphism,  $f(b_1)$  must be in the same automorphism class of  $b'_j$ . Suppose that  $f(b_1) \neq b'_j$ , then  $f(b_1)$  is not a cut-vertex of  $P' * (G_{p/q}, a, b)$ . As  $b_1$  is a cut-vertex of  $P * (G_{p/q}, a, b)$ , it is identified with either  $a_2$  or  $b_2$ . If it were identified with  $a_2$ , then  $f|_{H_2}$  would be an automorphism of  $G_{p/q}$  such that  $f(a_2) = b'_j$ , contrary to the choice of  $a, b$  in  $V(G_{p/q})$ . Hence  $b_1$  is identified with  $b_2$ . Similarly, we get  $a_2$  is identified with  $a_3$ , and  $b_3$  with  $b_4$  and so on. This implies  $P$  is a ‘zig-zag’ which is hom-equivalent to an edge, contrary to  $P$  being a core. So  $f(b_1) = b'_j$ . The claim follows by induction on the length of  $P$ .

We define a homomorphism  $g : V(P) \rightarrow V(P')$  so that if  $f(a_i) = a'_j$  then  $g(v_i) = v'_j$  and similarly for  $b_i$ . By our construction  $g$  preserves the adjacency condition and so  $P \leq P'$ .  $\square$  **Proof of Theorem 4:** Let  $a/b \in (2, 5/2) \cup (5/2, 8/3]$ . By Lemma 9, there is a graph  $G_{a/b}$  with properties (P1),(P2). By Lemma 10,  $\mathcal{K}_{a/b}$  is universal. This concludes our result.

## References

- [1] Z. Hedrlín, On universal partly ordered sets and classes, *J. Algebra* 11(1969),503-509.
- [2] P.Hell, J. Nešetřil, *Graphs and Homomorphisms*, Oxford University Press, 2004.
- [3] P.Hell, X. Zhu, The circular chromatic number of series-parallel graphs, *J. Graph Theory*, 33(2000),14-24.
- [4] J. Hubička, J. Nešetřil, Universal Partial Order Represented by Means of Trees and Other Simple Graphs, (to appear in *European J. Comb.*)
- [5] J. Hubička, J. Nešetřil, Finite Paths are Universal, ITI Series 2003-129, Charles University, 2003 (to appear in *Order*)
- [6] W. Klostermeyer, C.Q. Zhang,  $(2 + \epsilon)$ -coloring of planar graphs with prescribed girth, *J. Graph Theory*, 33(2000) (2):109-119.
- [7] J. Nešetřil, Y. Nigussie, Density of universal classes of series-parallel graphs (KAM Series 2004-717)
- [8] I. Niven, H. Zuckerman and H. Montgomery, *An introduction to the Theory of Numbers*, Hojn Wiley & Sons, Inc., Fifth Edition, 1991.
- [9] Z. Pan, X. Zhu, Density of the circular chromatic number series-parallel graphs, *J. Graph Theory*, 46(2004), 57-68.
- [10] X. Zhu, Circular Chromatic Number, a survey. *Discrete Mathematics*, Vol.229 (1-3)(2001), 371-410.

