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# Mader Tools

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The deep theorem of Mader concerning the number of internally disjoint  $H$ -paths is a very powerful tool. Nevertheless its use is very difficult, because one has to deal with a very reach family of separators. This paper shows several ways to strengthen Mader's theorem by certain additional restrictions of the appearing separators.

**Keywords:** graph,  $H$ -path, separator

## 1 Preliminaries and Results

For notations not defined here we refer to (1). Unless otherwise stated,  $k$  is an arbitrary integer,  $G$  is an arbitrary finite simple graph (loops and multiple edges are forbidden),  $U$  is an arbitrary subgraph of  $G$ ,  $X$  and  $H$  are arbitrary disjoint subsets of  $V(G)$  and  $Y$  is an arbitrary subset of  $E(G - X - H)$ . A path having exactly its endvertices in  $H$  is called an  $H$ -path. The maximum number of independent  $H$ -paths we denote by  $p_G(H)$ .  $[Y]$  denotes the graph with edge set  $Y$  whose vertex set is the set of all vertices incident with at least one edge of  $Y$ . Let  $\mathcal{C}(G)$  denote the set of components of  $G$  and  $\partial_G(U)$  denote the set of vertices of  $U$  incident with at least one edge of  $G - E(U)$ . A pair  $(X, Y)$  is called  $H$ -separator of  $G$ , if each  $H$ -path of  $G$  contains a vertex of  $X$  or an edge of  $Y$ . Let  $\mathcal{S}$  be the set of all  $H$ -separators of  $G - E(G[H])$ . A vertex  $x'$  of  $G$  is called *big brother* of a vertex  $x$  of  $G$ , if the neighborhood of  $x'$  in  $G$  contains the neighborhood of  $x$  in  $G - x'$ .

According to (1) we define the permeability of a pair  $(X, Y)$  by:

$$M_G(X, Y) = |X| + \sum_{C \in \mathcal{C}([Y])} \left\lfloor \frac{|\partial_{G-X}(C)|}{2} \right\rfloor$$

Mader's Theorem (cf. (2)) can be rewritten as follows (cf. (1).)

**Theorem 1 (Mader, 1978)**

$$p_G(H) = |E(G[H])| + \min\{M_G(X, Y) \mid (X, Y) \in \mathcal{S}\}$$

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Note, that here  $H$  is a set of vertices. To get this from the version of Mader's theorem in (1), you have to apply the version of (1) with the graph  $G[H]$  instead of  $H$ .

Let a subset  $S'$  of  $S$  be a *Mader-Set*, whenever Theorem 1 remains valid if  $S$  is replaced by  $S'$ . In other words, a subset  $S'$  of  $S$  is a Mader-Set, iff for each element  $(X, Y)$  of  $S$  there is an element  $(X', Y')$  of  $S'$  with  $M_G(X', Y') \leq M_G(X, Y)$ . Note that a subset of  $S$  containing a Mader-Set is a Mader-Set, too.

The following conditions for elements  $(X, Y)$  of  $S$  will be discussed:

- *Odd Border Condition (OB)*

For each component  $C$  of  $[Y]$  the number  $|\partial_{G-X}C|$  is odd.

- *Big Brother Vertex Condition (BV)*: If  $x \in X$  and  $x'$  is a big brother of  $x$ , then  $x' \in X$ .
- *Symmetric Edge Condition (SE)*: If  $v$  and  $v'$  are two vertices of  $G - H - X$  such that the neighborhood of  $v'$  in  $G - v$  equals the neighborhood of  $v$  in  $G - v'$ , then the neighborhood of  $v'$  in  $[Y] - v$  equals the neighborhood of  $v$  in  $[Y] - v'$ .
- *Edge Component Condition (EC)*: For each edge  $e$  of  $G - H - X - Y$  and each component  $C$  of  $[Y] \cup (V(G - H - X), \emptyset)$  there is a path  $P$  in  $G - X - Y - C$  containing an element of  $H$  and an endvertex of  $e$ .
- *Half Border Condition (HB)*: For each  $C \in [Y]$  and each  $B \subseteq \partial_{G-X}C$  with  $2|B| \geq |\partial_{G-X}C|$  there are two vertexdisjoint *HB*-paths in  $G - X$ .

For a subset  $Q$  of the set of conditions  $\{OB, BV, SE, EC\}$  let  $\mathcal{S}(Q)$  be the subset of  $S$  satisfying all conditions in  $Q$ . Our main results are as follows:

**Theorem 2**  $\mathcal{S}(\{OB, SE, HB, EC\})$  is a Mader-Set.

**Theorem 3**  $\mathcal{S}(\{BV, SE, HB, EC\})$  is a Mader-Set.

**Theorem 4** There is a graph  $G$  and a subset  $H$  of  $V(G)$  such that  $\mathcal{S}(\{OB, BV\})$  is not a Mader-Set.

In other words, Theorem 2 and Theorem 3 state, that for each graph  $G$  and each subset  $H$  of  $V(G)$  the set  $\mathcal{S}^*(G, H)$  of  $H$ -separators of  $G$  with minimal permeability has (possibly equal) elements  $(X_1, Y_1)$  and  $(X_2, Y_2)$  such that  $(X_1, Y_1)$  satisfies the Odd Border Condition, the Symmetric Edge Condition, the Half Border Condition and the Edge Component Condition, and  $(X_2, Y_2)$  satisfies the Big Brother Vertex Condition, the Symmetric Edge Condition, the Half Border Condition and the Edge Component Condition.

Theorem 4 states, that there is a graph  $G$  and a subset  $H$  of  $V(G)$ , such that none of the elements of  $\mathcal{S}^*(G, H)$  satisfies the Odd Border Condition and the Big Brother Vertex Condition.

## 2 Motivation

Why dealing with such mysterious conditions? The Odd Border Condition helps to simplify the formula for the permeability of an  $H$  separator:

**Theorem 5** Let  $G$  be a graph,  $H$  be a subset of  $G$ , and  $(X, Y)$  be an  $H$ -separator of  $G$  satisfying the Odd Border Condition. Then for the permeability of  $(X, Y)$  the following equation holds:

$$M_G(X, Y) = |X| + \frac{|\partial_{G-X}[Y]| - |\mathcal{C}([Y])|}{2}$$

In order to motivate the remaining three conditions, we regard an application of Mader's Theorem: Suppose, a function  $f$  mapping  $H$  into the set of nonnegative integers is given. We are interested in a 'separator-like' condition for the existence of a set of  $p$  independent  $H$ -paths such that in the graph  $U$  being the union of all this paths  $f(h) \geq d_U(h)$  holds for all  $h \in H$ . Such a problem appears for instance, if one wants to prove the  $f$ -factor theorem with help of Mader's Theorem.

Let the graph  $R(G, f)$  be obtained from  $G$  by the following procedure: Let  $G'$  be the graph obtained from  $G$  by intersecting each edge  $e$  of  $G[H]$  by a vertex  $h_e$ . In  $G'$  sequentially replace each vertex  $v$  of  $H$  by a complete bipartite graph  $R_v$  whose partition classes  $A_v$  and  $B_v$  satisfy  $|A_v| = d_G(v) + 1$  and  $|B_v| = f(v)$ . In each step each edge incident with  $v$  (say  $(u, v)$ ) of  $G'$  has to be replaced by an edge  $(u, a)$  with  $a \in A_v$  such that in the resulting graph  $R(G, f)$  only one vertex  $a_v$  of  $A_v$  has all its neighbors in  $B_v$ . We call  $R(G, f)$  the  $f$ -replacement of  $G$ .

The set  $H_R(G, f) = \{a_v | v \in H\}$  we call  $f$ -replacement of  $H$  in  $G$ . With this definitions we find the following lemma:

**Lemma 6**  $G$  has a set of  $p$  independent  $H$ -paths such that each vertex  $v$  of  $H$  is contained in at most  $f(v)$  of this paths if and only if  $R(G, f)$  has a set of  $p$  independent  $H_R(G, f)$ -paths.

Using Mader's Theorem for  $R(G, f)$  instead of  $G$  and  $H_R(G, f)$  instead of  $H$  we get

**Lemma 7**  $G$  has a set of  $p$  independent  $H$ -paths such that each vertex  $v$  of  $H$  is contained in at most  $f(v)$  of this paths if and only if each  $H_R(G, f)$ -separator  $(X, Y)$  of  $R(G, f)$  satisfies  $p < p(X, Y)$ .

Now, we are nearly done. We have to retranslate this condition to a condition for the graph  $G$ , the set  $H$ , and the function  $f$  only. To reconstruct  $G$  from  $R(G, f)$ , for each  $v \in H$  we have to contract the graph  $R_v$  to the vertex  $v$ , and after that for each  $e \in E(G[H])$  we have to delete  $h_e$  and to add  $e$ . But, without any knowledge about a special structure of  $H_R(G, f)$ -separators in  $R(G, f)$ , we loose too much information by doing the contractions.

The situation changes rapidly, if we first apply Theorem 3 with  $R(G, f)$  instead of  $G$  and  $H_R(G, f)$  instead of  $H$ . Using this we prove the following Lemma:

**Lemma 8**  $G$  has a set of  $p$  independent  $H$ -paths such that each vertex  $v$  of  $H$  is contained in at most  $f(v)$  of this paths if and only if each  $H_R(G, f)$ -separator  $(X, Y)$  of  $R(G, f)$  that satisfies the following conditions also satisfies  $p < p(X, Y)$ .

Here are the conditions:

For each element  $v$  of  $H$  one of the following statements holds:

1.  $V(R_v) \cap X = B_v$  and no edge of  $Y$  is incident with  $A_v$ ,
2.  $V(R_v) \cap X = \emptyset$  and  $Y$  contains each edge of  $R(G, f)$  incident with  $A_v \setminus \{a_v\}$ .
3.  $V(R_v) \cap X = \emptyset$  and no edge of  $Y$  is incident with  $A_v$ .

For each edge  $e$  of  $G[H]$  we have  $h_e \in X$  if and only if for each edge  $v$  incident with  $e$  the third statement ( $V(R_v) \cap X = \emptyset$  and no edge of  $Y$  is incident with  $A_v$ ) holds.

By Lemma 8, it is possible to interpret the resulting structure in  $G$ , directly. For this, let a pair  $(X, Y)$  be  $(G, H)$ -valid, if  $G - X - Y$  has no  $H$ -path and  $\partial_{G-X}[Y]$  is disjoint to  $H$ .

We derive the following Theorem:

**Theorem 9** Given a graph  $G$ , a subset  $H$  of its vertex set, and a function  $f$  that maps  $H$  to the set of non-negative integers.

The maximum number of independent  $H$ -paths, for which each vertex  $v$  of  $H$  is contained in at most  $f(v)$  of these paths, equals the minimum of

$$|E(G[H \setminus (X \cup V([Y]))])| + |X \setminus H| + \sum_{x \in H \cap X} f(x) + \sum_{C \in \mathcal{C}([Y])} \left[ \frac{1}{2} \left( |\partial_{G-X} C| + \sum_{v \in H \cap V(C)} f(v) \right) \right]$$

taken over all  $(G, H)$ -valid pairs  $(X, Y)$ .

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