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# $K_\ell^-$ -factors in graphs

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Let  $K_\ell^-$  denote the graph obtained from  $K_\ell$  by deleting one edge. We show that for every  $\gamma > 0$  and every integer  $\ell \geq 4$  there exists an integer  $n_0 = n_0(\gamma, \ell)$  such that every graph  $G$  whose order  $n \geq n_0$  is divisible by  $\ell$  and whose minimum degree is at least  $\left(\frac{\ell^2 - 3\ell + 1}{\ell(\ell - 2)} + \gamma\right)n$  contains a  $K_\ell^-$ -factor, i.e. a collection of disjoint copies of  $K_\ell^-$  which covers all vertices of  $G$ . This is best possible up to the error term  $\gamma n$  and yields an approximate solution to a conjecture of Kawarabayashi.

**Keywords:** graph packing, factor, critical chromatic number

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## 1 Introduction

Given two graphs  $H$  and  $G$ , an  $H$ -packing in  $G$  is a collection of vertex-disjoint copies of  $H$  in  $G$ . An  $H$ -packing in  $G$  is called *perfect* if it covers all vertices of  $G$ . In this case, we also say that  $G$  contains an  $H$ -factor. The aim now is to find natural conditions on  $G$  which guarantee the existence of a perfect  $H$ -packing in  $G$ . For example, the famous theorem of Hajnal and Szemerédi [HS70] gives a best possible condition on the minimum degree of  $G$  which ensures that  $G$  has a perfect  $K_\ell$ -packing. More precisely, it states that every graph  $G$  whose order  $n$  is divisible by  $\ell$  and whose minimum degree is at least  $(1 - 1/\ell)n$  contains a perfect  $K_\ell$ -packing. (The case  $\ell = 3$  was proved earlier by Corrádi and Hajnal [CH63] and the case  $\ell = 2$  follows immediately from Dirac's theorem on Hamilton cycles.)

Alon and Yuster [AY96] proved an extension of this result to perfect packings of arbitrary graphs  $H$ .

**Theorem 1 [Alon and Yuster [AY96]]** *For every graph  $H$  and every  $\gamma > 0$  there exists an integer  $n_0 = n_0(\gamma, H)$  such that every graph  $G$  whose order  $n \geq n_0$  is divisible by  $|H|$  and whose minimum degree is at least  $(1 - 1/\chi(H) + \gamma)n$  contains a perfect  $H$ -packing.*

Alon and Yuster [AY96] observed that there are graphs  $H$  for which the error term  $\gamma n$  cannot be omitted completely, but conjectured that it could be replaced by a constant which depends only on  $H$ . This conjecture was proved by Komlós, Sárközy and Szemerédi [KSS01].

Thus one might think that just as in Turán theory – where instead of an  $H$ -packing one only asks for a single copy of  $H$  – the chromatic number of  $H$  is the crucial parameter when one considers  $H$ -packings. However, one indication that this is not the case is provided by the result of Komlós [Kom00], which states that if one only requires an *almost* perfect  $H$ -packing (i.e. one which covers almost all of the vertices of  $G$ ), then the relevant parameter is the critical chromatic number of  $H$ . Here the *critical chromatic number*  $\chi_{cr}(H)$  of a graph  $H$  is defined as  $(\chi(H) - 1)h / (h - \sigma(H))$ , where  $\sigma(H)$  denotes the minimum size of the smallest colour class in a colouring of  $H$  with  $\chi(H)$  colours and where  $h$  denotes the order of  $H$ . Note that  $\chi_{cr}(H)$  always satisfies  $\chi(H) - 1 < \chi_{cr}(H) \leq \chi(H)$  and is closer to  $\chi(H) - 1$  if  $\sigma(H)$  is comparatively small.

**Theorem 2 [Komlós [Kom00]]** *For every graph  $H$  and every  $\gamma_1 > 0$  there exists an integer  $n_1 = n_1(\gamma_1, H)$  such that every graph  $G$  of order  $n \geq n_1$  and minimum degree at least  $(1 - 1/\chi_{cr}(H))n$  contains an  $H$ -packing which covers all but at most  $\gamma_1 n$  vertices of  $G$ .*

Up to the error term  $\gamma_1 n$  this is best possible for all graphs  $H$ . Komlós conjectured that the error term  $\gamma_1 n$  could be replaced by a constant which depends only on  $H$ . This conjecture was proved by Shokoufandeh and Zhao [SZ03]. As in the above conjecture of Alon and Yuster, there are graphs  $H$  for which the error term cannot be omitted completely.

Komlós [Kom00] also observed that for every graph  $H$  the minimum degree required in Theorem 2 is necessary to guarantee a perfect  $H$ -packing:

**Proposition 3** *For every graph  $H$  and every integer  $n$  that is divisible by  $|H|$  there exists a graph  $G$  of order  $n$  and minimum degree  $\lceil (1 - 1/\chi_{cr}(H))n \rceil - 1$  which does not contain a perfect  $H$ -packing.*

Our main result shows that in the case when  $H = K_\ell^-$ , the critical chromatic number is indeed the parameter which governs the existence of perfect packings. (Recall that  $K_\ell^-$  denotes the graph obtained from  $K_\ell$  by deleting one edge.)

**Theorem 4** *For every  $\gamma > 0$  and every integer  $\ell \geq 4$  there exists an integer  $n_0 = n_0(\gamma, \ell)$  such that every graph  $G$  whose order  $n \geq n_0$  is divisible by  $\ell$  and whose minimum degree is at least*

$$\left(1 - \frac{1}{\chi_{cr}(K_\ell^-)} + \gamma\right)n$$

*contains a perfect  $K_\ell^-$ -packing.*

By Proposition 3, Theorem 4 is best possible up to the error term  $\gamma n$ . Our proof of Theorem 4 shows that the perfect  $K_\ell^-$ -packing can be found in polynomial time. Moreover, note that  $1 - 1/\chi_{cr}(K_\ell^-) = \frac{\ell^2 - 3\ell + 1}{\ell(\ell - 2)}$ . Thus Theorem 4 gives an approximate solution to the following conjecture of Kawarabayashi (it is approximate in the sense that we have the additional error term in the minimum degree condition and require  $n$  to be large).

**Conjecture 5 [Kawarabayashi [Kaw02]]** *Let  $\ell \geq 4$  be an integer. Suppose that  $G$  is a graph whose order  $n$  is divisible by  $\ell$  and whose minimum degree is at least*

$$\frac{\ell^2 - 3\ell + 1}{\ell(\ell - 2)}n.$$

*Then  $G$  contains a perfect  $K_\ell^-$ -packing.*

If true, the conjecture would be best possible. The case  $\ell = 4$  of the conjecture (and thus of Theorem 4) was proved by Kawarabayashi [Kaw02]. By a result of Enomoto, Kaneko and Tuza [EKT87], the conjecture also holds for the case  $\ell = 3$  under the additional assumption that  $G$  is connected. (Note that  $K_3^-$  is just a path on 3 vertices and that in this case the required minimum degree equals  $n/3$ .)

One question which is immediately raised by Theorem 4 is whether one can replace  $K_\ell^-$  with an arbitrary graph  $H$ . In [KO] we characterize the non-bipartite graphs  $H$  for which this is the case and show that for all other non-bipartite graphs as well as for all connected bipartite ones Theorem 1 is best possible up to the term  $\gamma n$ . This characterization depends on the sizes of the colour classes in the optimal colourings of  $H$ .

Unlike the proof in [Kaw02], our argument is based on the the Regularity lemma of Szemerédi and the Blow-up lemma of Komlós, Sárközy and Szemerédi [KSS97].

## 2 Sketch of the proof

In our sketch of the proof of Theorem 4 we assume that the reader is familiar with both the Regularity and the Blow-up lemma. The strategy of the proof of Theorem 4 is as follows. We first apply the Regularity lemma to our given graph  $G$  in order to obtain a reduced graph  $R$ . An application of Theorem 2 to  $R$  will give us a  $K_\ell^-$ -packing  $\mathcal{K}$  which covers almost all of the vertices of  $R$ . We then enlarge the exceptional set  $V_0$  by adding all the vertices of  $G$  that lie in clusters not covered by this  $K_\ell^-$ -packing. Next, for each exceptional vertex  $x \in V_0$  in turn, we choose a copy of  $K_\ell^-$  in  $G$  which consists of  $x$  together with  $\ell - 1$  vertices lying in some clusters. All these copies of  $K_\ell^-$  will be disjoint for distinct exceptional vertices  $x \in V_0$ . We delete all the vertices in these copies from the clusters they belong to. One can show that we can choose these  $K_\ell^-$  in such a way that from each cluster only a small fraction of vertices will be deleted.

Our aim now is to apply the Blow-up lemma to each of the copies  $K \in \mathcal{K}$  of  $K_\ell^-$  in order to find a  $K_\ell^-$ -packing in  $G$  which covers all the vertices belonging to (the modified) clusters in  $\mathcal{K}$ . (Then all these  $K_\ell^-$ -packings together with the copies of  $K_\ell^-$  chosen so far for the exceptional vertices will form a perfect  $K_\ell^-$ -packing in  $G$ .) However, a necessary condition for this is that the complete  $(\ell - 1)$ -partite graph  $K^*$  whose vertex classes are the clusters in  $K$  (where the two clusters which are not adjacent in  $K$  form one vertex class together) contains a perfect  $K_\ell^-$ -packing. It turns out that is the case if  $|K^*|$  is divisible by  $\ell$  and if the largest vertex class of  $K^*$  is a little less than twice as large as every other vertex class. We can satisfy the first condition by deleting a few carefully chosen further copies of  $K_\ell^-$  in  $G$ .

However, we cannot guarantee the second condition if we proceed as above. In fact, since we have changed the sizes of the clusters when choosing the copies of  $K_\ell^-$  for the exceptional vertices, the largest vertex class of  $K^*$  may now even be slightly more than twice as large as every other vertex class. In order to overcome this problem, we proceed a little differently. Instead of choosing an almost perfect  $K_\ell^-$ -packing in  $R$ , we will choose an almost perfect packing with copies of some complete  $(\ell - 1)$ -partite graph  $F$  which has  $\ell - 2$  vertex classes of equal size  $s$  and one vertex class of size  $(2 - \eta)s$  (where  $s$  is large and  $\eta$  is small). Moreover  $F$  will be chosen in such a way that it contains a perfect  $K_\ell^-$ -packing. Thus all these  $K_\ell^-$ -packings together form an almost perfect  $K_\ell^-$ -packing  $\mathcal{K}$  in  $R$ , as we had before. We now proceed similarly as described before, the only difference is that we aim to apply the Blow-up lemma to each copy of  $F$  in  $R$  (and not to the copies of  $K_\ell^-$ ). So consider one such copy  $F'$  and let  $F^*$  denote the ‘blown-up’ copy of  $F'$ . Thus  $F^*$  is a complete  $(\ell - 1)$ -partite graph whose  $i$ th vertex class is the union of all the clusters in the  $i$ th vertex class of  $F'$ . As before, we can achieve that  $|F^*|$  is divisible by  $\ell$ . However, this time we can also achieve that the largest vertex class of  $F^*$  is a little less than twice as large

as every other vertex class. Indeed, this holds for the vertex classes of  $F'$  with some room to spare and subsequently we only modified the cluster sizes by a comparatively small amount.

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