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► **To cite this version:**

Andrew D. King, Bruce A. Reed, Adrian R. Vetta. An upper bound for the chromatic number of line graphs. 2005 European Conference on Combinatorics, Graph Theory and Applications (EuroComb '05), 2005, Berlin, Germany. pp.151-156, 10.46298/dmtcs.3401 . hal-01184357

HAL Id: hal-01184357

<https://inria.hal.science/hal-01184357>

Submitted on 14 Aug 2015

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An upper bound for the chromatic number of line graphs

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It was conjectured by Reed [12] that for any graph G , the graph's chromatic number $\chi(G)$ is bounded above by $\left\lceil \frac{\Delta(G)+1+\omega(G)}{2} \right\rceil$, where $\Delta(G)$ and $\omega(G)$ are the maximum degree and clique number of G , respectively. In this paper we prove that this bound holds if G is the line graph of a multigraph. The proof yields a polynomial time algorithm that takes a line graph G and produces a colouring that achieves our bound.

1 Introduction

The *chromatic number* of a graph G , denoted by $\chi(G)$, is the minimum number of colours required to colour the vertex set of G so that no two adjacent vertices are assigned the same colour. That is, the vertices of a given colour form a *stable set*. Determining the exact chromatic number of a graph efficiently is very difficult, and for this reason it has proven fruitful to explore the relationships between $\chi(G)$ and other invariants of G . The *clique number* of G , denoted by $\omega(G)$, is the largest set of mutually adjacent vertices in G and the *degree* of a vertex v , written $\deg(v)$, is the number of vertices to which v is adjacent; the maximum degree over all vertices in G is denoted by $\Delta(G)$. It is easy to see that $\omega(G) \leq \chi(G) \leq \Delta(G) + 1$. Brooks' Theorem (see [1]) tightens this:

Brooks' Theorem $\chi(G) \leq \Delta(G)$ unless G contains a clique of size $\Delta(G) + 1$ or $\Delta(G) = 2$ and G contains an odd cycle.

So for $\chi(G)$ we have a trivial upper bound in terms of $\Delta(G)$ and a trivial lower bound in terms of $\omega(G)$. We are interested in exploring upper bounds on $\chi(G)$ in terms of a convex combination of $\Delta(G) + 1$ and $\omega(G)$. In [12], Reed conjectured a bound on the chromatic number of any graph G :

Conjecture 1 For any graph G , $\chi(G) \leq \left\lceil \frac{\Delta(G)+1+\omega(G)}{2} \right\rceil$.

Several related results exist. In the same paper, Reed proved that the conjecture holds if $\Delta(G)$ is sufficiently large and $\omega(G)$ is sufficiently close to $\Delta(G)$. Using this, he proved that there exists a positive constant α such that $\chi(G) \leq \alpha(\omega(G)) + (1 - \alpha)(\Delta(G) + 1)$ for all graphs. Some results are also known for generalizations of the chromatic number.

[†]Corresponding author: king@cs.mcgill.ca. Research supported by NSERC and Tomlinson doctoral fellowships.

[‡]Research supported in part by a Canada Research Chair.

[§]Research supported in part by NSERC grant 28833-04 and FQRNT grant NC-98649.

A *fractional vertex c -colouring* of a graph G can be described as a set of stable sets $\{S_1, S_2, \dots, S_l\}$ with weights $\{w_1, w_2, \dots, w_l\}$ such that for every vertex v , $\sum_{S_i: v \in S_i} w_i = 1$ and $\sum_{i=1}^l w_i = c$. The *fractional chromatic number* of G , written $\chi^*(G)$, is the smallest c for which G has a *fractional vertex c -colouring*. Note that it is always bounded above by the chromatic number. The *list chromatic number* of a graph G , written $\chi_l(G)$, is the smallest r such that if each vertex is assigned any list of r colours, the graph has a colouring in which every vertex is coloured with a colour on its list. For any graph we clearly have $\chi^*(G) \leq \chi(G) \leq \chi_l(G)$.

In [10], Molloy and Reed proved the fractional analogue of Conjecture 1 for all graphs, i.e. that

$$\chi^*(G) \leq \left\lceil \frac{\Delta(G) + 1 + \omega(G)}{2} \right\rceil \text{ for any graph } G. \quad (1)$$

In fact, the round-up is not needed in the fractional case. In this paper we prove that Conjecture 1 holds for line graphs, which are defined in the next section.

2 Fractional and Integer Colourings in Line Graphs of Multigraphs

A *multigraph* is a graph in which multiple edges are permitted between any pair of vertices – all multigraphs in this paper are loopless. Given a multigraph $H = (V, E)$, the *line graph* of H , denoted by $L(H)$, is a graph with vertex set E ; two vertices of $L(H)$ are adjacent if and only if their corresponding edges in H share at least one endpoint. We say that G is a line graph if there is a multigraph H for which $G = L(H)$.

The *chromatic index* of H , written $\chi'(H)$, is the chromatic number of $L(H)$. Similarly, the *fractional chromatic index* $\chi'^*(H)$ is equal to the fractional chromatic number of $L(H)$. In [6], Holyer proved that determining the chromatic index of an arbitrary multigraph is NP-complete, so practically speaking we are bound to the task of approximating the chromatic index of multigraphs and hence the chromatic number of line graphs.

Vizing's Theorem (see [14]) bounds the chromatic index of a multigraph in terms of its maximum degree, stating that $\Delta(H) \leq \chi'(H) \leq \Delta(H) + d$, where d is the maximum number of edges between any two vertices in H . Both bounds are achievable, but a more meaningful bound should consider other invariants of H . Of course, $\chi'(H)$ is always bounded below by $\chi'^*(H)$, and Edmond's theorem for matching polytopes (presented in [3], also mentioned in [8]) tells us that given

$$\Gamma(H) = \max \left\{ \frac{2|E(W)|}{|V(W)| - 1} : W \subseteq H, |V(W)| \text{ is odd} \right\},$$

$$\chi'^*(H) = \max\{\Delta(H), \Gamma(H)\}. \quad (2)$$

Does this necessarily translate into a good upper bound on the chromatic index of a multigraph? The following long-standing conjecture, proposed by Goldberg [4] and Seymour [13], implies that $\chi'^*(H) \leq \chi'(H) \leq \chi'^*(H) + 1$:

Goldberg-Seymour Conjecture *For a multigraph H for which $\chi'(H) > \Delta(H) + 1$, $\chi'(H) = \lceil \Gamma(H) \rceil$.*

Asymptotic results are known: Kahn [7] proved that the fractional chromatic index asymptotically agrees with the integral chromatic index, i.e. that $\chi'(H) \leq (1 + o(1))\chi'^*(H)$. This implies the Goldberg-Seymour Conjecture asymptotically. He later proved that in fact, the fractional chromatic index asymptotically agrees with the list chromatic index [8].

Another result that supports the Goldberg-Seymour Conjecture is the following theorem:

Theorem 2 (Caprara and Rizzi [2]) For any multigraph H , $\chi'(H) \leq \max\{\lfloor 1.1\Delta(H)+0.7 \rfloor, \lceil \Gamma(H) \rceil\}$.

This theorem is a slight improvement of an earlier result of Nishizeki and Kashiwagi [11], lowering the additive factor from 0.8 to 0.7. Note that this implies the Goldberg-Seymour Conjecture for any multigraph H with $\Delta(H) \leq 12$, since in this case we have $\lfloor 1.1\Delta(H) + 0.7 \rfloor \leq \Delta(H) + 1$.

3 The Main Result

We will now prove our main result:

Theorem 3 For any line graph G , $\chi(G) \leq \left\lceil \frac{\Delta(G)+1+\omega(G)}{2} \right\rceil$.

Consider a multigraph H for which $G = L(H)$. The proof consists of two cases: the case where $\Delta(G)$ is large compared to $\Delta(H)$, and the case where $\Delta(G)$ is close to $\Delta(H)$. In both cases we use the fact that $\omega(G) \geq \Delta(H)$. The first case is given by the following lemma, which follows easily from Theorem 2.

Lemma 4 If G is the line graph of a multigraph H , and $\Delta(G) \geq \frac{3}{2}\Delta(H) - 1$, then $\chi(G) \leq \left\lceil \frac{\Delta(G)+1+\omega(G)}{2} \right\rceil$.

Proof of Theorem 3:

Consider a counterexample $G = L(H)$ such that the theorem holds for every line graph on fewer vertices. We know that $\Delta(G) < \frac{3}{2}\Delta(H) - 1$. Our approach is as follows: We find a maximal stable set $S \subset V(G)$ that has a vertex in every maximum clique in G , and let G' be the subgraph of G induced on $V(G) \setminus S$. We can see that $\Delta(G') \leq \Delta(G) - 1$ (since S is maximal) and $\omega(G') = \omega(G) - 1$, and that the theorem holds for G' , as any induced subgraph of a line graph is clearly a line graph. So we know that $\chi(G') \leq \left\lceil \frac{\Delta(G')+1+\omega(G')}{2} \right\rceil - 1$. We can now construct a proper $\chi(G') + 1$ -colouring of $V(G)$ by taking a proper $\chi(G')$ -colouring of G' and letting S be the final colour class, hence $\chi(G) \leq \left\lceil \frac{\Delta(G)+1+\omega(G)}{2} \right\rceil$, a contradiction.

It suffices, then, to show the existence of such a stable set S in G . We actually need only find a stable set that hits all the maximum cliques of G , as we can extend any such stable set until it is maximal. We will do this in terms of a *matching* in H , i.e. a set of edges in $E(H)$, no two of which share an endpoint – a matching in H exactly represents a stable set in G . We need some notation first. For a pair of vertices $u, v \in V(H)$, the *multiplicity* of uv is the number of edges in $E(H)$ between u and v ; we denote it by $\mu(u, v)$. A *triangle* in H is a set of three mutually adjacent vertices, and we denote the maximum number of edges of any triangle in H by $\text{tri}(H)$; the edges of a triangle are those edges in $E(H)$ joining the triangle's vertices. Note the following facts that relate invariants of H and G :

Fact 1 $\Delta(G) = \max_{uv \in E(H)} \{\deg(u) + \deg(v) - \mu(u, v) - 1\}$.

Fact 2 $\omega(G) = \max\{\Delta(H), \text{tri}(H)\}$.

We say that a matching *hits* a vertex v if v is an endpoint of an edge in the matching. We will find a maximal matching M in H which corresponds to a desired stable set because it hits every vertex of maximum degree in H and contains an edge of every triangle with $\max\{\Delta(H), \text{tri}(H)\}$ edges in H .

To this end, let S_Δ be the set of vertices of degree $\Delta(H)$ in H and let T be the set of triangles in H that contain $\max\{\Delta(H), \text{tri}(H)\}$ edges. It is instructive to consider how the elements of T interact; we omit the straightforward proofs of these observations from this extended abstract.

Observation 1 *If two triangles of T intersect in exactly the vertices a and b then ab has multiplicity greater than $\Delta(H)/2$.*

Observation 2 *If abc is a triangle of T intersecting another triangle ade of T in exactly the vertex a then $\mu(b, c)$ is greater than $\Delta(H)/2$.*

Observation 3 *If there is an edge of H joining two vertices a and b of S_Δ then $\mu(a, b) > \Delta(H)/2$.*

Guided by these observations, we let T' be those triangles in T that contain no pair of vertices of multiplicity $> \Delta(H)/2$ and S'_Δ be those elements of S_Δ which are in no pair of vertices of multiplicity greater than $\Delta(H)/2$. We treat $T' \cup S'_\Delta$ and $(T \setminus T') \cup (S_\Delta \setminus S'_\Delta)$ separately. A few more observations regarding S'_Δ and T' will serve us well. Again, we omit the proofs.

Observation 4 *For any $S \subseteq S'_\Delta$, $|N(S)| \geq |S|$.*

Observation 5 *If an edge ab in H has exactly one endpoint in a triangle bcd of T' , then the degree of a is less than $\Delta(H)$.*

Observation 6 *If an edge ab in H has exactly one endpoint in a triangle bcd of T' , then $\mu(a, b) \leq \Delta(H)/2$.*

Observation 7 *For any vertex v with two neighbours u and w , $\deg(u) + \mu(vw) \leq \frac{3}{2}\Delta(H)$.*

It is now straightforward to show that the desired matching exists. We begin with a matching M consisting of one edge between each vertex pair with multiplicity greater than $\Delta(H)/2$ – this hits $S_\Delta \setminus S'_\Delta$ and contains an edge of each triangle in $T \setminus T'$. Observation 4 tells us that we can apply Hall's Theorem (see [5]) to get a matching in H that hits S'_Δ ; Observation 7 dictates that this matching cannot hit M , so the union M' of these two matchings is a matching in H that hits S_Δ and contains an edge of each triangle in $T \setminus T'$. Every edge in this matching either hits a maximum-degree vertex in H or has endpoints with multiplicity greater than $\Delta(H)/2$.

What, then, can prevent us from extending this M' to contain an edge of every triangle in T' ? Observations 1 and 2 tell us that any two triangles in T' are vertex-disjoint, so our only worry is that M' hits two vertices of some triangle in T' . Observations 3, 5 and 6 guarantee that at most one such vertex in a given triangle is hit, and if there is such a vertex, it has degree $\Delta(H)$. We can therefore extend M' to contain an edge of every triangle in T' . The result is a matching that satisfies all of our requirements, so the proof of the theorem is complete. \square

4 Algorithmic Considerations

We have presented a new upper bound for the chromatic number of line graphs, i.e. $\chi(G) \leq \left\lceil \frac{\Delta(G)+1+\omega(G)}{2} \right\rceil$. Our proof of the bound yields an algorithm for constructing a colour class in G but we have an initial condition in the proof (i.e. $\Delta(G) < \frac{3}{2}\Delta(H) - 1$) that does not necessarily remain if we remove these vertices. However, the bound given by Caprara and Rizzi in Theorem 2 can be achieved in $O(|E(H)|(|V(H)| + \Delta(H)))$ time [2]. It is easy to see that in the proof of Theorem 3 we can find our matching in polynomial time, so we can formulate a polytime algorithm for $\left\lceil \frac{\Delta(G)+1+\omega(G)}{2} \right\rceil$ -colouring a line graph G with root graph H as follows.

1. While $\Delta(L(H)) < \frac{3}{2}\Delta(H) - 1$, remove a matching M from H as in the proof of Theorem 3 (and let it be a colour class).
2. Employ Caprara and Rizzi's algorithm to complete the edge colouring of H .

This, of course, assumes that we have the root graph H such that $G = L(H)$. Lehot provides an $O(|E(G)|)$ algorithm that detects whether or not G is the line graph of a simple graph H and outputs H if possible [9]. Two vertices u and v in G are *twins* if they are adjacent and their neighbourhoods are otherwise identical. We can extend Lehot's algorithm to line graphs of multigraphs by contracting each set of k mutually twin vertices in G into a single vertex, which we say has multiplicity k . This can be done trivially in $O(|E(G)|\Delta(G))$ time. The resulting graph G' is the line graph of a simple graph H' if and only if G is the line graph of a multigraph H ; we can generate H from H' by considering the multiplicities of the vertices in G' and duplicating edges in H' accordingly.

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