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# An upper bound for the chromatic number of line graphs

A. D. King<sup>†</sup>, B. A. Reed<sup>‡</sup> and A. Vetta<sup>§</sup>

*School of Computer Science, McGill University, 3480 University Ave., Montréal, Québec, H3A 2A7, Canada*

It was conjectured by Reed [12] that for any graph  $G$ , the graph's chromatic number  $\chi(G)$  is bounded above by  $\left\lceil \frac{\Delta(G)+1+\omega(G)}{2} \right\rceil$ , where  $\Delta(G)$  and  $\omega(G)$  are the maximum degree and clique number of  $G$ , respectively. In this paper we prove that this bound holds if  $G$  is the line graph of a multigraph. The proof yields a polynomial time algorithm that takes a line graph  $G$  and produces a colouring that achieves our bound.

## 1 Introduction

The *chromatic number* of a graph  $G$ , denoted by  $\chi(G)$ , is the minimum number of colours required to colour the vertex set of  $G$  so that no two adjacent vertices are assigned the same colour. That is, the vertices of a given colour form a *stable set*. Determining the exact chromatic number of a graph efficiently is very difficult, and for this reason it has proven fruitful to explore the relationships between  $\chi(G)$  and other invariants of  $G$ . The *clique number* of  $G$ , denoted by  $\omega(G)$ , is the largest set of mutually adjacent vertices in  $G$  and the *degree* of a vertex  $v$ , written  $\deg(v)$ , is the number of vertices to which  $v$  is adjacent; the maximum degree over all vertices in  $G$  is denoted by  $\Delta(G)$ . It is easy to see that  $\omega(G) \leq \chi(G) \leq \Delta(G) + 1$ . Brooks' Theorem (see [1]) tightens this:

**Brooks' Theorem**  $\chi(G) \leq \Delta(G)$  unless  $G$  contains a clique of size  $\Delta(G) + 1$  or  $\Delta(G) = 2$  and  $G$  contains an odd cycle.

So for  $\chi(G)$  we have a trivial upper bound in terms of  $\Delta(G)$  and a trivial lower bound in terms of  $\omega(G)$ . We are interested in exploring upper bounds on  $\chi(G)$  in terms of a convex combination of  $\Delta(G) + 1$  and  $\omega(G)$ . In [12], Reed conjectured a bound on the chromatic number of any graph  $G$ :

**Conjecture 1** For any graph  $G$ ,  $\chi(G) \leq \left\lceil \frac{\Delta(G)+1+\omega(G)}{2} \right\rceil$ .

Several related results exist. In the same paper, Reed proved that the conjecture holds if  $\Delta(G)$  is sufficiently large and  $\omega(G)$  is sufficiently close to  $\Delta(G)$ . Using this, he proved that there exists a positive constant  $\alpha$  such that  $\chi(G) \leq \alpha(\omega(G)) + (1 - \alpha)(\Delta(G) + 1)$  for all graphs. Some results are also known for generalizations of the chromatic number.

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A *fractional vertex  $c$ -colouring* of a graph  $G$  can be described as a set of stable sets  $\{S_1, S_2, \dots, S_l\}$  with weights  $\{w_1, w_2, \dots, w_l\}$  such that for every vertex  $v$ ,  $\sum_{S_i: v \in S_i} w_i = 1$  and  $\sum_{i=1}^l w_i = c$ . The *fractional chromatic number* of  $G$ , written  $\chi^*(G)$ , is the smallest  $c$  for which  $G$  has a *fractional vertex  $c$ -colouring*. Note that it is always bounded above by the chromatic number. The *list chromatic number* of a graph  $G$ , written  $\chi_l(G)$ , is the smallest  $r$  such that if each vertex is assigned any list of  $r$  colours, the graph has a colouring in which every vertex is coloured with a colour on its list. For any graph we clearly have  $\chi^*(G) \leq \chi(G) \leq \chi_l(G)$ .

In [10], Molloy and Reed proved the fractional analogue of Conjecture 1 for all graphs, i.e. that

$$\chi^*(G) \leq \left\lceil \frac{\Delta(G) + 1 + \omega(G)}{2} \right\rceil \text{ for any graph } G. \quad (1)$$

In fact, the round-up is not needed in the fractional case. In this paper we prove that Conjecture 1 holds for line graphs, which are defined in the next section.

## 2 Fractional and Integer Colourings in Line Graphs of Multigraphs

A *multigraph* is a graph in which multiple edges are permitted between any pair of vertices – all multigraphs in this paper are loopless. Given a multigraph  $H = (V, E)$ , the *line graph* of  $H$ , denoted by  $L(H)$ , is a graph with vertex set  $E$ ; two vertices of  $L(H)$  are adjacent if and only if their corresponding edges in  $H$  share at least one endpoint. We say that  $G$  is a line graph if there is a multigraph  $H$  for which  $G = L(H)$ .

The *chromatic index* of  $H$ , written  $\chi'(H)$ , is the chromatic number of  $L(H)$ . Similarly, the *fractional chromatic index*  $\chi'^*(H)$  is equal to the fractional chromatic number of  $L(H)$ . In [6], Holyer proved that determining the chromatic index of an arbitrary multigraph is NP-complete, so practically speaking we are bound to the task of approximating the chromatic index of multigraphs and hence the chromatic number of line graphs.

Vizing's Theorem (see [14]) bounds the chromatic index of a multigraph in terms of its maximum degree, stating that  $\Delta(H) \leq \chi'(H) \leq \Delta(H) + d$ , where  $d$  is the maximum number of edges between any two vertices in  $H$ . Both bounds are achievable, but a more meaningful bound should consider other invariants of  $H$ . Of course,  $\chi'(H)$  is always bounded below by  $\chi'^*(H)$ , and Edmond's theorem for matching polytopes (presented in [3], also mentioned in [8]) tells us that given

$$\Gamma(H) = \max \left\{ \frac{2|E(W)|}{|V(W)| - 1} : W \subseteq H, |V(W)| \text{ is odd} \right\},$$

$$\chi'^*(H) = \max\{\Delta(H), \Gamma(H)\}. \quad (2)$$

Does this necessarily translate into a good upper bound on the chromatic index of a multigraph? The following long-standing conjecture, proposed by Goldberg [4] and Seymour [13], implies that  $\chi'^*(H) \leq \chi'(H) \leq \chi'^*(H) + 1$ :

**Goldberg-Seymour Conjecture** *For a multigraph  $H$  for which  $\chi'(H) > \Delta(H) + 1$ ,  $\chi'(H) = \lceil \Gamma(H) \rceil$ .*

Asymptotic results are known: Kahn [7] proved that the fractional chromatic index asymptotically agrees with the integral chromatic index, i.e. that  $\chi'(H) \leq (1 + o(1))\chi'^*(H)$ . This implies the Goldberg-Seymour Conjecture asymptotically. He later proved that in fact, the fractional chromatic index asymptotically agrees with the list chromatic index [8].

Another result that supports the Goldberg-Seymour Conjecture is the following theorem:

**Theorem 2 (Caprara and Rizzi [2])** For any multigraph  $H$ ,  $\chi'(H) \leq \max\{\lfloor 1.1\Delta(H)+0.7 \rfloor, \lceil \Gamma(H) \rceil\}$ .

This theorem is a slight improvement of an earlier result of Nishizeki and Kashiwagi [11], lowering the additive factor from 0.8 to 0.7. Note that this implies the Goldberg-Seymour Conjecture for any multigraph  $H$  with  $\Delta(H) \leq 12$ , since in this case we have  $\lfloor 1.1\Delta(H) + 0.7 \rfloor \leq \Delta(H) + 1$ .

### 3 The Main Result

We will now prove our main result:

**Theorem 3** For any line graph  $G$ ,  $\chi(G) \leq \left\lceil \frac{\Delta(G)+1+\omega(G)}{2} \right\rceil$ .

Consider a multigraph  $H$  for which  $G = L(H)$ . The proof consists of two cases: the case where  $\Delta(G)$  is large compared to  $\Delta(H)$ , and the case where  $\Delta(G)$  is close to  $\Delta(H)$ . In both cases we use the fact that  $\omega(G) \geq \Delta(H)$ . The first case is given by the following lemma, which follows easily from Theorem 2.

**Lemma 4** If  $G$  is the line graph of a multigraph  $H$ , and  $\Delta(G) \geq \frac{3}{2}\Delta(H) - 1$ , then  $\chi(G) \leq \left\lceil \frac{\Delta(G)+1+\omega(G)}{2} \right\rceil$ .

#### Proof of Theorem 3:

Consider a counterexample  $G = L(H)$  such that the theorem holds for every line graph on fewer vertices. We know that  $\Delta(G) < \frac{3}{2}\Delta(H) - 1$ . Our approach is as follows: We find a maximal stable set  $S \subset V(G)$  that has a vertex in every maximum clique in  $G$ , and let  $G'$  be the subgraph of  $G$  induced on  $V(G) \setminus S$ . We can see that  $\Delta(G') \leq \Delta(G) - 1$  (since  $S$  is maximal) and  $\omega(G') = \omega(G) - 1$ , and that the theorem holds for  $G'$ , as any induced subgraph of a line graph is clearly a line graph. So we know that  $\chi(G') \leq \left\lceil \frac{\Delta(G')+1+\omega(G')}{2} \right\rceil - 1$ . We can now construct a proper  $\chi(G') + 1$ -colouring of  $V(G)$  by taking a proper  $\chi(G')$ -colouring of  $G'$  and letting  $S$  be the final colour class, hence  $\chi(G) \leq \left\lceil \frac{\Delta(G)+1+\omega(G)}{2} \right\rceil$ , a contradiction.

It suffices, then, to show the existence of such a stable set  $S$  in  $G$ . We actually need only find a stable set that hits all the maximum cliques of  $G$ , as we can extend any such stable set until it is maximal. We will do this in terms of a *matching* in  $H$ , i.e. a set of edges in  $E(H)$ , no two of which share an endpoint – a matching in  $H$  exactly represents a stable set in  $G$ . We need some notation first. For a pair of vertices  $u, v \in V(H)$ , the *multiplicity* of  $uv$  is the number of edges in  $E(H)$  between  $u$  and  $v$ ; we denote it by  $\mu(u, v)$ . A *triangle* in  $H$  is a set of three mutually adjacent vertices, and we denote the maximum number of edges of any triangle in  $H$  by  $\text{tri}(H)$ ; the edges of a triangle are those edges in  $E(H)$  joining the triangle's vertices. Note the following facts that relate invariants of  $H$  and  $G$ :

**Fact 1**  $\Delta(G) = \max_{uv \in E(H)} \{\deg(u) + \deg(v) - \mu(u, v) - 1\}$ .

**Fact 2**  $\omega(G) = \max\{\Delta(H), \text{tri}(H)\}$ .

We say that a matching *hits* a vertex  $v$  if  $v$  is an endpoint of an edge in the matching. We will find a maximal matching  $M$  in  $H$  which corresponds to a desired stable set because it hits every vertex of maximum degree in  $H$  and contains an edge of every triangle with  $\max\{\Delta(H), \text{tri}(H)\}$  edges in  $H$ .

To this end, let  $S_\Delta$  be the set of vertices of degree  $\Delta(H)$  in  $H$  and let  $T$  be the set of triangles in  $H$  that contain  $\max\{\Delta(H), \text{tri}(H)\}$  edges. It is instructive to consider how the elements of  $T$  interact; we omit the straightforward proofs of these observations from this extended abstract.

**Observation 1** *If two triangles of  $T$  intersect in exactly the vertices  $a$  and  $b$  then  $ab$  has multiplicity greater than  $\Delta(H)/2$ .*

**Observation 2** *If  $abc$  is a triangle of  $T$  intersecting another triangle  $ade$  of  $T$  in exactly the vertex  $a$  then  $\mu(b, c)$  is greater than  $\Delta(H)/2$ .*

**Observation 3** *If there is an edge of  $H$  joining two vertices  $a$  and  $b$  of  $S_\Delta$  then  $\mu(a, b) > \Delta(H)/2$ .*

Guided by these observations, we let  $T'$  be those triangles in  $T$  that contain no pair of vertices of multiplicity  $> \Delta(H)/2$  and  $S'_\Delta$  be those elements of  $S_\Delta$  which are in no pair of vertices of multiplicity greater than  $\Delta(H)/2$ . We treat  $T' \cup S'_\Delta$  and  $(T \setminus T') \cup (S_\Delta \setminus S'_\Delta)$  separately. A few more observations regarding  $S'_\Delta$  and  $T'$  will serve us well. Again, we omit the proofs.

**Observation 4** *For any  $S \subseteq S'_\Delta$ ,  $|N(S)| \geq |S|$ .*

**Observation 5** *If an edge  $ab$  in  $H$  has exactly one endpoint in a triangle  $bcd$  of  $T'$ , then the degree of  $a$  is less than  $\Delta(H)$ .*

**Observation 6** *If an edge  $ab$  in  $H$  has exactly one endpoint in a triangle  $bcd$  of  $T'$ , then  $\mu(a, b) \leq \Delta(H)/2$ .*

**Observation 7** *For any vertex  $v$  with two neighbours  $u$  and  $w$ ,  $\deg(u) + \mu(vw) \leq \frac{3}{2}\Delta(H)$ .*

It is now straightforward to show that the desired matching exists. We begin with a matching  $M$  consisting of one edge between each vertex pair with multiplicity greater than  $\Delta(H)/2$  – this hits  $S_\Delta \setminus S'_\Delta$  and contains an edge of each triangle in  $T \setminus T'$ . Observation 4 tells us that we can apply Hall's Theorem (see [5]) to get a matching in  $H$  that hits  $S'_\Delta$ ; Observation 7 dictates that this matching cannot hit  $M$ , so the union  $M'$  of these two matchings is a matching in  $H$  that hits  $S_\Delta$  and contains an edge of each triangle in  $T \setminus T'$ . Every edge in this matching either hits a maximum-degree vertex in  $H$  or has endpoints with multiplicity greater than  $\Delta(H)/2$ .

What, then, can prevent us from extending this  $M'$  to contain an edge of every triangle in  $T'$ ? Observations 1 and 2 tell us that any two triangles in  $T'$  are vertex-disjoint, so our only worry is that  $M'$  hits two vertices of some triangle in  $T'$ . Observations 3, 5 and 6 guarantee that at most one such vertex in a given triangle is hit, and if there is such a vertex, it has degree  $\Delta(H)$ . We can therefore extend  $M'$  to contain an edge of every triangle in  $T'$ . The result is a matching that satisfies all of our requirements, so the proof of the theorem is complete.  $\square$

## 4 Algorithmic Considerations

We have presented a new upper bound for the chromatic number of line graphs, i.e.  $\chi(G) \leq \left\lceil \frac{\Delta(G)+1+\omega(G)}{2} \right\rceil$ . Our proof of the bound yields an algorithm for constructing a colour class in  $G$  but we have an initial condition in the proof (i.e.  $\Delta(G) < \frac{3}{2}\Delta(H) - 1$ ) that does not necessarily remain if we remove these vertices. However, the bound given by Caprara and Rizzi in Theorem 2 can be achieved in  $O(|E(H)|(|V(H)| + \Delta(H)))$  time [2]. It is easy to see that in the proof of Theorem 3 we can find our matching in polynomial time, so we can formulate a polytime algorithm for  $\left\lceil \frac{\Delta(G)+1+\omega(G)}{2} \right\rceil$ -colouring a line graph  $G$  with root graph  $H$  as follows.

1. While  $\Delta(L(H)) < \frac{3}{2}\Delta(H) - 1$ , remove a matching  $M$  from  $H$  as in the proof of Theorem 3 (and let it be a colour class).
2. Employ Caprara and Rizzi's algorithm to complete the edge colouring of  $H$ .

This, of course, assumes that we have the root graph  $H$  such that  $G = L(H)$ . Lehot provides an  $O(|E(G)|)$  algorithm that detects whether or not  $G$  is the line graph of a simple graph  $H$  and outputs  $H$  if possible [9]. Two vertices  $u$  and  $v$  in  $G$  are *twins* if they are adjacent and their neighbourhoods are otherwise identical. We can extend Lehot's algorithm to line graphs of multigraphs by contracting each set of  $k$  mutually twin vertices in  $G$  into a single vertex, which we say has multiplicity  $k$ . This can be done trivially in  $O(|E(G)|\Delta(G))$  time. The resulting graph  $G'$  is the line graph of a simple graph  $H'$  if and only if  $G$  is the line graph of a multigraph  $H$ ; we can generate  $H$  from  $H'$  by considering the multiplicities of the vertices in  $G'$  and duplicating edges in  $H'$  accordingly.

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