



**HAL**  
open science

## Connected $\tau$ -critical hypergraphs of minimal size

Matěj Stehlík

► **To cite this version:**

Matěj Stehlík. Connected  $\tau$  -critical hypergraphs of minimal size. 2005 European Conference on Combinatorics, Graph Theory and Applications (EuroComb '05), 2005, Berlin, Germany. pp.157-160, 10.46298/dmtcs.3397 . hal-01184353

**HAL Id: hal-01184353**

**<https://inria.hal.science/hal-01184353>**

Submitted on 14 Aug 2015

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Connected $\tau$ -critical hypergraphs of minimal size

Matěj Stehlík

Instituto de Matemáticas, UNAM, Circuito Exterior, 04510 México, D.F., Mexico

---

A hypergraph  $\mathcal{H}$  is  $\tau$ -critical if  $\tau(\mathcal{H} - E) < \tau(\mathcal{H})$  for every edge  $E \in \mathcal{H}$ , where  $\tau(\mathcal{H})$  denotes the transversal number of  $\mathcal{H}$ . It can be shown that a connected  $\tau$ -critical hypergraph  $\mathcal{H}$  has at least  $2\tau(\mathcal{H}) - 1$  edges; this generalises a classical theorem of Gallai on  $\chi$ -vertex-critical graphs with connected complements. In this paper we study connected  $\tau$ -critical hypergraphs  $\mathcal{H}$  with exactly  $2\tau(\mathcal{H}) - 1$  edges. We prove that such hypergraphs have at least  $2\tau(\mathcal{H}) - 1$  vertices, and characterise those with  $2\tau(\mathcal{H}) - 1$  vertices using a directed odd ear decomposition of an associated digraph. Using Seymour's characterisation of  $\chi$ -critical 3-chromatic square hypergraphs, we also show that a connected square hypergraph  $\mathcal{H}$  with fewer than  $2\tau(\mathcal{H})$  edges is  $\tau$ -critical if and only if it is  $\chi$ -critical 3-chromatic. Finally, we deduce some new results on  $\chi$ -vertex-critical graphs with connected complements.

**Keywords:**  $\tau$ -critical hypergraph,  $\chi$ -critical 3-chromatic hypergraph

---

## 1 Introduction

A hypergraph  $\mathcal{H}$  is a finite set of finite non-empty sets called the *edges* of  $\mathcal{H}$ . The *vertices* of  $\mathcal{H}$  are the elements of the set  $V(\mathcal{H}) = \bigcup_{E \in \mathcal{H}} E$ . A set  $T \subseteq V(\mathcal{H})$  is a *transversal* (also *vertex cover* or *blocking set*) of  $\mathcal{H}$  if  $T \cap E \neq \emptyset$  for every  $E \in \mathcal{H}$ . The smallest cardinality of a transversal of  $\mathcal{H}$  is the *transversal number*  $\tau(\mathcal{H})$ . A *k-colouring* of a hypergraph  $\mathcal{H}$  is an assignment of at most  $k$  colours to  $V(\mathcal{H})$  such that no edge is monochromatic. The chromatic number  $\chi(\mathcal{H})$  is the smallest  $k$  such that  $\mathcal{H}$  admits a  $k$ -colouring. A hypergraph  $\mathcal{H}$  is  *$\tau$ -critical* (resp.  *$\chi$ -critical*) if  $\tau(\mathcal{H} - E) < \tau(\mathcal{H})$  (resp.  $\chi(\mathcal{H} - E) < \chi(\mathcal{H})$ ) for every  $E \in \mathcal{H}$ .

A number of authors have studied  $\tau$ -critical hypergraphs; see for example [1, 2, 3]. It is trivial to verify that a hypergraph is  $\tau$ -critical if and only if all its components are  $\tau$ -critical. So what can be said about *connected*  $\tau$ -critical hypergraphs? In particular, it seems natural to ask what is the minimal possible number of edges in a connected  $\tau$ -critical hypergraph.

We first present a sharp lower bound on the number of edges in a connected  $\tau$ -critical hypergraph, and then investigate the cases where equality is attained. We exhibit a surprising connection with  $\chi$ -critical 3-chromatic square hypergraphs studied by Seymour [7], and show how our results relate to the work of Gallai [4] on  $\chi$ -vertex-critical graphs with connected complements.

## 2 Main results

The following two results were proved in [9]. (A hypergraph is a *star* if all its edges have a common vertex.)

**Theorem 1** *If  $\mathcal{H}$  is a connected  $\tau$ -critical hypergraph, then for every  $E \in \mathcal{H}$  the edges of  $\mathcal{H} - E$  can be partitioned into  $\tau(\mathcal{H}) - 1$  stars of size at least two.*

**Corollary 2** *If  $\mathcal{H}$  is a connected  $\tau$ -critical hypergraph, then  $|\mathcal{H}| \geq 2\tau(\mathcal{H}) - 1$ .*

The bound in Corollary 2 is sharp, as can be seen by considering odd cycles. Hypergraphs attaining equality in Corollary 2 are called *minimal connected  $\tau$ -critical hypergraphs*. We might hope that such hypergraphs would be of an analysable form. Indeed, since a partition into 2-stars of a hypergraph corresponds to a matching of its line graph, Theorem 1 implies the following useful result. (A graph  $G$  *factor-critical* if  $G - x$  has a perfect matching, for every  $x \in V(G)$ .)

**Corollary 3** *If  $\mathcal{H}$  is a minimal connected  $\tau$ -critical hypergraph, then  $L(\mathcal{H})$  is factor-critical.*

Lovász [5] proved that every factor-critical graph has an *odd ear decomposition*: it can be built up from a single vertex by successively attaching the end vertices of odd paths. So by Corollary 3 the line graph of a minimal connected  $\tau$ -critical hypergraph has an odd ear decomposition. This fact can be used to prove the following two results.

**Theorem 4** *If  $\mathcal{H}$  is a minimal connected  $\tau$ -critical hypergraph, then  $\mathcal{H}$  has a system of distinct representatives.*

**Corollary 5** *If  $\mathcal{H}$  is a minimal connected  $\tau$ -critical hypergraph, then  $|V(\mathcal{H})| \geq |\mathcal{H}|$ .*

Again, considering odd cycles shows that the bound in Corollary 5 is sharp. A hypergraph with an equal number of edges and vertices is said to be *square*. As might be expected, the minimal connected  $\tau$ -critical hypergraphs which are square have particularly nice properties. Indeed, they can be characterised in terms of an odd ear decomposition of an associated digraph.

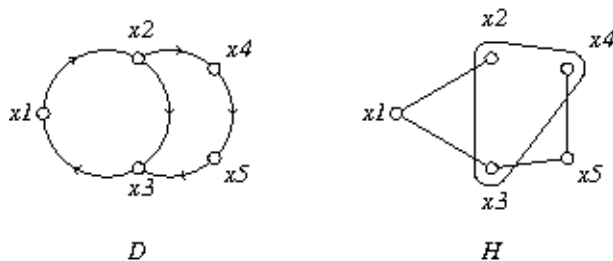
With any digraph  $D$  we can associate the hypergraph  $\mathcal{H}_D = \{\{x\} \cup N^+(x) \mid x \in V(D)\}$  where  $N^+(x)$  denotes the set of outneighbours of  $x$  in  $D$ . Note that  $\mathcal{H}_D$  is square and has a system of distinct representatives. Conversely, if  $\mathcal{H}$  is a square hypergraph with a system of distinct representatives  $f : \mathcal{H} \rightarrow V(\mathcal{H})$ , then  $\mathcal{H} = \mathcal{H}_D$ , where  $D$  is the digraph with vertex set  $V(\mathcal{H})$  and arc set  $\{(x, y) \mid x \in V(\mathcal{H}), y \in f^{-1}(x) \setminus \{x\}\}$ .

A *directed odd ear* with respect to a digraph  $D$  consists of a directed odd path such that the two end vertices are in  $V(D)$  but no internal vertices belong to  $V(D)$ . A *directed odd ear decomposition* of a digraph  $D$  is a sequence  $D_0, \dots, D_p$  of digraphs such that  $D_0$  is a single vertex,  $D_p = D$ , and for  $i = 1, \dots, p$ ,  $D_i$  is obtained from  $D_{i-1}$  by adding a directed odd ear joining two not necessarily distinct vertices of  $D_{i-1}$ .

Seymour [7] proved that a square hypergraph  $\mathcal{H}$  is  $\chi$ -critical 3-chromatic if and only if  $\mathcal{H} = \mathcal{H}_D$ , where  $D$  is a strongly connected digraph with no directed even circuits. The following result can be proved using Seymour's theorem, Corollary 3 and Theorem 4. The *absorption number*  $\beta(D)$  of a digraph  $D$  is the minimal size of a set  $S \subseteq V(D)$  such that every  $x \in V(D) \setminus S$  has an outneighbour in  $S$ .

**Theorem 6** *For any square hypergraph  $\mathcal{H}$ , the following conditions are equivalent:*

1.  $\mathcal{H}$  is minimal connected  $\tau$ -critical;
2.  $\mathcal{H}$  is  $\chi$ -critical 3-chromatic and  $|\mathcal{H}| < 2\tau(\mathcal{H})$ ;
3.  $\mathcal{H} = \mathcal{H}_D$ , where  $D$  has a directed odd ear decomposition, contains no directed even circuits and  $|V(D)| < 2\beta(D)$ .



**Fig. 1:**  $\mathcal{H} = \mathcal{H}_D$ , where  $D$  has a directed odd ear decomposition and contains no directed even circuits; the associated square hypergraph  $\mathcal{H}_D$  is minimal connected  $\tau$ -critical by Theorem 6.

Let  $\mathcal{H}^*$  denote the vertex-edge dual of  $\mathcal{H}$ . The following result can be proved using Corollary 5 and Theorem 6.

**Corollary 7** *If  $\mathcal{H}$  is a minimal connected  $\tau$ -critical hypergraph, then so is  $\mathcal{H}^*$  if and only if  $\mathcal{H}$  is square.*

### 3 Application to $\chi$ -vertex-critical graphs

A hypergraph has the *Helly property* if all its intersecting partial hypergraphs are stars. There is a useful link between the chromatic number of graphs and the transversal number of Helly hypergraphs. Namely, given a graph  $G$ , let  $\mathcal{A}(G)$  be the hypergraph formed with the maximal independent sets of  $G$ , and denote its dual by  $\mathcal{A}^*(G)$ . It is not difficult to check that  $\mathcal{A}^*(G)$  has the Helly property and  $\chi(G) = \tau(\mathcal{A}^*(G))$ . A graph  $G$  is  $\chi$ -vertex-critical if  $\chi(G - x) < \chi(G)$ , for every vertex  $x \in V(G)$ ; note that a graph  $G$  is  $\chi$ -vertex-critical if and only if  $\mathcal{A}^*(G)$  is  $\tau$ -critical. Hence the restriction to Helly hypergraphs of Corollary 2 is equivalent to the following classical result of Gallai [4], also proved in [6, 8].

**Theorem 8 (Gallai 1963)** *A  $\chi$ -vertex-critical graph  $G$  with a connected complement has at least  $2\chi(G) - 1$  vertices.*

The restriction to Helly hypergraphs of Corollary 5 is equivalent to the following result.

**Theorem 9** *A  $\chi$ -vertex-critical graph  $G$  with a connected complement and  $2\chi(G) - 1$  vertices has at least  $2\chi(G) - 1$  maximal independent sets.*

Finally, Theorem 6 implies the following.

**Theorem 10** *If  $G$  is a graph with a connected complement,  $2\chi(G) - 1$  vertices and  $2\chi(G) - 1$  maximal independent sets, then  $G$  is  $\chi$ -vertex-critical if and only if  $\mathcal{A}^*(G)$  is a  $\chi$ -critical 3-chromatic Helly hypergraph.*

## References

[1] C. Berge. *Hypergraphs: Combinatorics of Finite Sets*. Elsevier, Amsterdam, 1989.

- [2] P. Duchet. Hypergraphs. In R. L. Graham, M. Grötschel, and L. Lovász, editors, *Handbook of Combinatorics*, pages 381–432. North-Holland, New York, 1995.
- [3] P. Frankl. Extremal set systems. In R. L. Graham, M. Grötschel, and L. Lovász, editors, *Handbook of Combinatorics*, pages 1293–1329. North-Holland, New York, 1995.
- [4] T. Gallai. Kritische Graphen II. *Magyar Tud. Akad. Kutató Int. Közl.*, **8**:373–395, 1963.
- [5] L. Lovász. A note on factor-critical graphs. *Studia Sci. Math. Hungar.*, **7**:279–280, 1972.
- [6] M. Molloy. Chromatic neighborhood sets. *J. Graph Theory*, **31**:303–311, 1999.
- [7] P. D. Seymour. On the two-colouring of hypergraphs. *Quart. J. Math. Oxford*, **25**(3):303–312, 1974.
- [8] M. Stehlík. Critical graphs with connected complements. *J. Combin. Theory Ser. B*, **89**:189–194, 2003.
- [9] M. Stehlík. Minimal connected  $\tau$ -critical hypergraphs. Submitted, .