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Algorithmic and combinatoric aspects of multiple harmonic sums

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Ordinary generating series of *multiple* harmonic sums admit a *full* singular expansion in the basis of functions $\{(1-z)^\alpha \log^\beta(1-z)\}_{\alpha \in \mathbb{Z}, \beta \in \mathbb{N}}$, near the singularity $z = 1$. A *constructive* proof of this result is given, and, by *combinatoric* aspects, an explicit evaluation of Taylor coefficients of functions in some *polylogarithmic* algebra is obtained. In particular, the *asymptotic expansion* of multiple harmonic sums is easily deduced.

Keywords: polylogarithms, polyzêtas, multiple harmonic sums, singular expansion, shuffle algebra, Lyndon words

1 Introduction

Hierarchical data structure occur in numerous domains, like computer graphics, image processing or biology (pattern matching). Among them, quadrees, whose construction is based on a recursive definition of space, constitute a classical data structure for storing and accessing collection of points in multidimensional space. Their characteristics (depth of a node, number of nodes in a given subtree, number of leaves) are studied by Laforest [12], with probabilistic tools. In particular, she shows, for a quadree of size N in a d -dimension space, that the probability $\pi_{N,k}$ for the first subtree to have size k can be expressed as an algebraic combination of j -th order harmonic numbers $H_j(N)$ and $H_j(k)$, $j \geq 1$, defined by

$$H_j(n) = \sum_{m=1}^n \frac{1}{m^j}. \quad (1)$$

For instance, for $d = 3$, one has

$$\pi_{N,k} = \frac{[H_1(N) - H_1(k)]^2 + H_2(N) - H_2(k)}{2N}. \quad (2)$$

Flajolet et al. [2] give this general expression for the splitting probability

$$\pi_{N,k} = \sum_{N \geq i_1 \dots \geq i_{d-1} > k} \frac{1}{i_1 \dots i_{d-1}}. \quad (3)$$

The probability $\pi_{N,k}$ appears as a particular case of the following sum $A_{\mathbf{s}}(N)$ associated to the *multi-index* $\mathbf{s} = (s_1, \dots, s_r)$, which is strongly related to multiple harmonic sums $H_{\mathbf{s}}(N)$:

$$A_{\mathbf{s}}(N) = \sum_{N \geq n_1 \geq \dots \geq n_r > 0} \frac{1}{n_1^{s_1} \dots n_r^{s_r}} \quad \text{and} \quad H_{\mathbf{s}}(N) = \sum_{N \geq n_1 > \dots > n_r > 0} \frac{1}{n_1^{s_1} \dots n_r^{s_r}}. \quad (4)$$

Let us note that there exist explicit relations, given by Hoffman [10] between the $A_{\mathbf{s}}(N)$ and $H_{\mathbf{s}}(N)$. Indeed, let $\text{Comp}(n)$ be the *set of compositions* of n , i.e. sequences (i_1, \dots, i_r) of positive integers summing to n . If $I = (i_1, \dots, i_r)$ (resp. $J = (j_1, \dots, j_p)$) is a composition of n (resp. of r) then $J \circ I = (i_1 + \dots + i_{j_1}, i_{j_1+1} + \dots + i_{j_1+j_2}, \dots, i_{k-j_p+1} + \dots + i_k)$ is a composition of n . By Möbius inversion, one has

$$A_{\mathbf{s}}(N) = \sum_{J \in \text{Comp}(r)} H_{J \circ \mathbf{s}}(N) \quad \text{and} \quad H_{\mathbf{s}}(N) = \sum_{J \in \text{Comp}(r)} (-1)^{l(J)-r} A_{J \circ \mathbf{s}}(N), \quad (5)$$

where $l(J)$ is the number of parts of J .

Example 1. For $\mathbf{s} = (1, 1, 1)$, since the set of compositions of 3 is $\{(1, 1, 1), (1, 2), (2, 1), (3)\}$, we get

$$A_{1,1,1}(N) = H_{1,1,1}(N) + H_{1,2}(N) + H_{2,1}(N) + H_3(N),$$

$$H_{1,1,1}(N) = A_{1,1,1}(N) - A_{1,2}(N) - A_{2,1}(N) + A_3(N).$$

Therefore, the $A_{\mathbf{s}}(N)$ are \mathbb{Z} -linear combinations on $H_{\mathbf{s}}(N)$ (and *vice versa*). Thus, the remaining problem is to know the asymptotic behaviour of $\pi_{N,k}$, for $N \rightarrow \infty$ [11]. For that, in this work, we are interested in the *combinatorial* aspects of these sums by use of a symbolic encoding by words. This enables then to transfer *shuffle relations* on words into algebraic relations between multiple harmonic sums, or between *polylogarithm* functions, defined for a multi-index $\mathbf{s} = (s_1, \dots, s_r)$ by

$$\text{Li}_{\mathbf{s}}(z) = \sum_{n_1 > \dots > n_r > 0} \frac{z^{n_1}}{n_1^{s_1} \dots n_r^{s_r}}, \quad \text{for } |z| < 1. \quad (6)$$

These relations are recalled in Section 2. The reason to call upon polylogarithms is given in Section 3, since the generating function $P_{\mathbf{s}}(z) = \sum_{n \geq 0} H_{\mathbf{s}}(n)z^n$ of $\{H_{\mathbf{s}}(n)\}_{n \geq 0}$, verifies

$$P_{\mathbf{s}}(z) = \frac{1}{1-z} \text{Li}_{\mathbf{s}}(z). \quad (7)$$

So, we set the polylogarithmic algebra of $\{P_{\mathbf{s}}\}_{\mathbf{s}}$, with coefficients in $\mathcal{C} = \mathbb{C}[z, z^{-1}, (z-1)^{-1}]$, and we then establish *exact* transfer results between a function g in this algebra and its Taylor coefficients $[z^N]g(z)$, in the \mathbb{C} -algebra generated by $\{N^k H_{\mathbf{s}}(N)\}_{\mathbf{s}, k \in \mathbb{Z}}$ in both directions. The main result of this paper is finally stated in Section 4, which gives a computation of the *full* singular expansion of g , in the basis of functions $\{(1-z)^\alpha \log^\beta(1-z)\}_{\alpha \in \mathbb{Z}, \beta \in \mathbb{N}}$, near the singularity $z = 1$. We deduce from this a *full* asymptotic expansion of its Taylor coefficients. These results are based on the analysis of the *noncommutative* generating series of functions of the form (7), in particular on its infinite factorization indexed by *Lyndon words*.

2 Background

2.1 Combinatorics on words

To the multi-index \mathbf{s} we can canonically associate the word $u = x_0^{s_1-1} x_1 \dots x_0^{s_r-1} x_1$ over the finite alphabet $X = \{x_0, x_1\}$. In the same way, \mathbf{s} can be canonically associated to the word $v = y_{s_1} \dots y_{s_r}$ over the infinite alphabet $Y = \{y_i\}_{i \geq 1}$. Moreover, in both alphabets, the empty multi-index will correspond to the empty word ϵ . We shall henceforth identify the multi-index \mathbf{s} with its encoding by the word u (resp. v). We denote by X^* (resp. Y^*) the free monoid generated by X (resp. Y), which is the set of words over X (resp. Y). Noting $\mathbb{C}\langle X \rangle$ (resp. $\mathbb{C}\langle Y \rangle$) the algebra of noncommutative polynomials with coefficients in \mathbb{C} , we obtain so a concatenation isomorphism from the \mathbb{C} -algebra of multi-indexes into the algebra $\mathbb{C}\langle X \rangle$ (resp. $\mathbb{C}\langle Y \rangle$). The coefficient of $w \in X^*$ in a polynomial $S \in \mathbb{C}\langle X \rangle$ is denoted by $(S|w)$ or S_w . The duality between polynomials is defined as follows

$$(S|p) = \sum_{w \in X^*} S_w p_w, \quad p \in \mathbb{C}\langle X \rangle. \quad (8)$$

The set of Lie monomials is defined by induction: the letters in X are Lie monomials and the Lie bracket $[a, b] = ab - ba$ of two Lie monomials a and b is a Lie monomial. A Lie polynomial is a \mathbb{C} -linear combination of Lie monomials. The set of Lie polynomials is called the *free Lie algebra*.

2.2 Shuffle products

Let $a, b \in X$ (resp. $y_i, y_j \in Y$) and $u, v \in X^*$ (resp. Y^*). The *shuffle* (resp. *stuffle*) of $u = au'$ and $v = bv'$ (resp. $u = y_i u'$ and $v = y_j v'$) is the polynomial recursively defined by

$$\epsilon \sqcup u = u \sqcup \epsilon = u \quad \text{and} \quad u \sqcup v = a(u' \sqcup v) + b(u \sqcup v'), \quad (9)$$

$$\text{(resp. } \epsilon \sqcup u = u \sqcup \epsilon = u \quad \text{and} \quad u \sqcup v = y_i(u' \sqcup v) + y_j(u \sqcup v') + y_{i+j}(u' \sqcup v')). \quad (10)$$

Example 2.

$$\begin{aligned} x_0 x_1 \sqcup x_1 &= x_1 x_0 x_1 + 2x_0 x_1^2 \\ y_2 \sqcup y_1 &= y_1 y_2 + y_2 y_1 + y_3. \end{aligned} \quad (11)$$

This product is extended to $\mathbb{C}\langle X \rangle$ (resp. $\mathbb{C}\langle Y \rangle$) by linearity. With this product, $\mathbb{C}\langle X \rangle$ (resp. $\mathbb{C}\langle Y \rangle$) is a commutative and associative \mathbb{C} -algebra.

l	Q_l	S_l
x_0	x_0	x_0
x_1	x_1	x_1
x_0x_1	$[x_0, x_1]$	x_0x_1
$x_0^2x_1$	$[x_0, [x_0, x_1]]$	$x_0^2x_1$
$x_0x_1^2$	$[[x_0, x_1], x_1]$	$x_0x_1^2$
$x_0^3x_1$	$[x_0, [x_0, [x_0, x_1]]]$	$x_0^3x_1$
\vdots	\vdots	\vdots
$x_0^3x_1^3$	$[x_0, [x_0, [[[x_0, x_1], x_1], x_1]]]$	$x_0^3x_1^3$
$x_0^2x_1x_0x_1^2$	$[x_0, [[x_0, x_1], [[x_0, x_1], x_1]]]$	$3x_0^3x_1^3 + x_0^2x_1x_0x_1^2$
$x_0^2x_1^2x_0x_1$	$[[x_0, [[x_0, x_1], x_1]], [x_0, x_1]]$	$6x_0^3x_1^3 + 3x_0^2x_1x_0x_1^2 + x_0^2x_1^2x_0x_1$
$x_0^2x_1^4$	$[x_0, [[[[x_0, x_1], x_1], x_1], x_1]]]$	$x_0^2x_1^4$
$x_0x_1x_0x_1^3$	$[[x_0, x_1], [[[x_0, x_1], x_1], x_1]]]$	$4x_0^2x_1^4 + x_0x_1x_0x_1^3$
$x_0x_1^5$	$[[[[[x_0, x_1], x_1], x_1], x_1], x_1]]]$	$x_0x_1^5$

Tab. 1: Lyndon words, bracket forms and dual basis

2.3 Lyndon words and Radford's theorem

By definition, a *Lyndon word* is a non empty word $l \in X^*$ (resp. $\in Y^*$) which is lower than any of its proper right factors [14] (for the lexicographical ordering) i.e. for all $u, v \in X^* \setminus \{\epsilon\}$ (resp. $\in Y^* \setminus \{\epsilon\}$), $l = uv \Rightarrow l < v$. The set of Lyndon words of X (resp. Y) is denoted by $\mathcal{Lyn}(X)$ (resp. $\mathcal{Lyn}(Y)$).

Example 3. For $X = \{x_0, x_1\}$ with the order $x_0 < x_1$ the Lyndon words of length ≤ 5 on X^* are (in lexicographical increasing order)

$$\{x_0, x_0^4x_1, x_0^3x_1, x_0^3x_1^2, x_0^2x_1, x_0^2x_1x_0x_1, x_0^2x_1^2, x_0^2x_1^3, x_0x_1, x_0x_1x_0x_1^2, x_0x_1^2, x_0x_1^3, x_0x_1^4, x_1\}.$$

For $Y = \{y_i, i \geq 1\}$, with the order $y_i < y_j$ when $i > j$, here are the corresponding Lyndon words over Y

$$\{y_5, y_4, y_4y_1, y_3, y_3y_2, y_3y_1, y_3y_1^2, y_2, y_2^2y_1, y_2y_1, y_2y_1^2, y_2y_1^3, y_1\}.$$

Theorem 1 (Radford, [13, 14]). Let

$$C_1 = \mathbb{C} \oplus (\mathbb{C}\langle X \rangle \setminus x_0\mathbb{C}\langle X \rangle x_1) \quad \text{and} \quad C_2 = \mathbb{C} \oplus (\mathbb{C}\langle Y \rangle \setminus y_1\mathbb{C}\langle Y \rangle)$$

be the sets of convergent polynomials over X and Y respectively. Then,

$$\begin{aligned} (\mathbb{C}\langle X \rangle, \sqcup) &\simeq (\mathbb{C}[\mathcal{Lyn}(X)], \sqcup) = (C_1[x_0, x_1], \sqcup), \\ (\mathbb{C}\langle Y \rangle, \sqcup) &\simeq (\mathbb{C}[\mathcal{Lyn}(Y)], \sqcup) = (C_2[y_1], \sqcup). \end{aligned}$$

Example 4.

$$\begin{aligned} y_2y_4y_1 + y_2y_1y_4 + y_1y_2y_4 + y_2y_5 + y_3y_4 &= y_4 \sqcup y_2 \sqcup y_1 - y_4y_2 \sqcup y_1 - y_6 \sqcup y_1 \in \mathbb{C}[\mathcal{Lyn}(Y)] \\ &= y_2y_4 \sqcup y_1 \in C_2[y_1] \end{aligned}$$

2.4 Bracket forms and the dual basis

The *bracket form* Q_l of a Lyndon word $l = uv$, with $l, u, v \in \mathcal{Lyn}(X)$ and the word v being as long as possible, is defined recursively by

$$\begin{cases} Q_l &= [Q_u, Q_v] \\ Q_x &= x \quad \text{for each letter } x \in X, \end{cases}$$

It is classical that the set $\mathcal{B}_1 = \{Q_l ; l \in \mathcal{Lyn}(X)\}$, ordered lexicographically, is a basis for the free Lie algebra. Moreover, each word $w \in X^*$ can be expressed uniquely as a decreasing product of Lyndon words:

$$w = l_1^{\alpha_1} l_2^{\alpha_2} \dots l_k^{\alpha_k}, \quad l_1 > l_2 > \dots > l_k, \quad k \geq 0. \quad (12)$$

The Poincaré–Birkhoff–Witt basis $\mathcal{B} = \{Q_w; w \in X^*\}$ and its dual basis $\mathcal{B}^* = \{S_w; w \in X^*\}$ are obtained from (12) by setting [14]

$$\begin{cases} Q_w &= Q_{l_1}^{\alpha_1} Q_{l_2}^{\alpha_2} \cdots Q_{l_k}^{\alpha_k}, \\ S_w &= \frac{S_{l_1}^{\sqcup \alpha_1} \cdots S_{l_k}^{\sqcup \alpha_k}}{\alpha_1! \alpha_2! \cdots \alpha_k!}, \\ S_l &= x S_w, \quad \forall l \in \mathcal{L}yn(X), \text{ where } l = xw, x \in X, w \in X^*. \end{cases}$$

In [14], it is proved that \mathcal{B} and \mathcal{B}^* are dual bases of $\mathbb{C}\langle X \rangle$ i.e. $(Q_u | S_v) = \delta_u^v$, for all words $u, v \in X^*$ with $\delta_u^v = 1$ if $u = v$, otherwise 0.

Lemma 1. *For all $w \in x_0 X^* x_1$, one has $S_w \in x_0 \mathbb{Z}\langle X \rangle x_1$.*

Proof. The Lyndon words involved in the decomposition (12) of a word $w \in X^* x_1$ (resp. $w \in x_0 X^* x_1$) all belong to $X^* x_1$ (resp. $x_0 X^* x_1$). \square

2.5 Polylogarithms

Let $\mathcal{C} = \mathbb{C}[z, 1/z, 1/(z-1)]$ and let ω_0 and ω_1 be the two following differential forms

$$\omega_0(z) = \frac{dz}{z} \quad \text{and} \quad \omega_1(z) = \frac{dz}{1-z}. \quad (13)$$

One verifies the polylogarithm $\text{Li}_s(z)$, defined by Formula (6), is also the following *iterated integral* with respect to ω_0 and ω_1

$$\text{Li}_s(z) = \int_{0 \rightsquigarrow z} \omega_0^{s_1-1} \omega_1 \cdots \omega_0^{s_r-1} \omega_1. \quad (14)$$

Thanks to the bijection from Y^* to $X^* x_1$ previously explained, we can index the polylogarithms by the words of $X^* x_1$, or indistinctly by the words of Y^* . We can extend (14) over X^* by putting

$$\text{Li}_\epsilon(z) = 1, \quad \text{Li}_{x_0}(z) = \log z, \quad \text{Li}_{x_i w}(z) = \int_{0 \rightsquigarrow z} \omega_i(t) \text{Li}_w(t), \quad \text{for } x_i \in X, w \in X^*. \quad (15)$$

Therefore, Li_w verifies the following identity [4]

$$\forall u, v \in X^*, \quad \text{Li}_{u \sqcup v} = \text{Li}_u \text{Li}_v. \quad (16)$$

The extended definition enables to construct the noncommutative generating series [4]

$$L = \sum_{w \in X^*} \text{Li}_w w \quad (17)$$

as being the unique solution of the *Drinfel'd equation*, i.e. the differential equation [4]

$$dL = [x_0 \omega_0 + x_1 \omega_1] L, \quad (18)$$

satisfying the boundary condition

$$L(\varepsilon) = e^{x_0 \log \varepsilon} + o(\sqrt{\varepsilon}), \quad \text{when } \varepsilon \rightarrow 0^+. \quad (19)$$

Proposition 1 ([5]). *Let σ be the monoid morphism defined over X^* by $\sigma(x_0) = -x_1$ and $\sigma(x_1) = -x_0$. Then,*

$$L(1-z) = [\sigma L(z)] \prod_{l \in \mathcal{L}yn(X) \setminus \{x_0, x_1\}} e^{\zeta(S_l) Q_l}.$$

Example 5 ([5]).

$$\begin{aligned} \text{Li}_{x_0 x_1^2}(1-z) &= -\text{Li}_{x_0^2 x_1}(z) + \text{Li}_{x_0}(z) \text{Li}_{x_0 x_1}(z) - \frac{1}{2} \text{Li}_{x_0}^2(z) \text{Li}_{x_1}(z) + \zeta(3), \quad \text{i.e.} \\ \text{Li}_{2,1}(1-z) &= -\text{Li}_3(z) + \log(z) \text{Li}_2(z) + \frac{1}{2} \log^2(z) \log(1-z) + \zeta(3). \end{aligned}$$

2.6 Harmonic sums

Definition 1. Let $w = y_{s_1} \dots y_{s_r} \in Y^*$. For $N \geq r \geq 1$, the harmonic sum $H_w(N)$ is defined as

$$H_w(N) = \sum_{N \geq n_1 > \dots > n_r > 0} \frac{1}{n_1^{s_1} \dots n_r^{s_r}}.$$

For $0 \leq N < r$, $H_w(N) = 0$ and, for the empty word ϵ , we put $H_\epsilon(N) = 1$, for any $N \geq 0$.

Let $w = y_{s_1} \dots y_{s_r} \in Y^*$. If $s_1 > 1$ then, by an Abel's theorem,

$$\lim_{N \rightarrow \infty} H_w(N) = \lim_{z \rightarrow 1} \text{Li}_w(z) = \sum_{n_1 > \dots > n_r > 0} \frac{1}{n_1^{s_1} \dots n_r^{s_r}}.$$

That is nothing but the polyzêta (or MZV [16]) $\zeta(w)$ and the word $w \in Y^* \setminus y_1 Y^*$ is said to be *convergent*. A polynomial of $\mathbb{C}\langle Y \rangle$ is said to be convergent when it is a linear combination of convergent words. The double shuffle algebra of polyzêtas is already pointed out and extensively studied in [3].

For $w = y_s w'$, we have

$$\zeta(w) = \sum_{l \geq 1} \frac{H_{w'}(l-1)}{l^s}, \quad (20)$$

$$H_w(N+1) - H_w(N) = (N+1)^{-s} H_{w'}(N) \quad (21)$$

and, for any $u, v \in Y^*$ [9]

$$H_{u \sqcup v}(N) = H_u(N) H_v(N). \quad (22)$$

3 Generating series

3.1 Definition and first properties

Definition 2 ([8]). Let $w \in Y^*$ and let $P_w(z)$ be the ordinary generating series of $\{H_w(N)\}_{N \geq 0}$

$$P_w(z) = \sum_{N \geq 0} H_w(N) z^N.$$

Proposition 2 ([8]). Extended by linearity, the map $P : u \mapsto P_u$ is an isomorphism from $(\mathbb{C}\langle Y \rangle, \sqcup)$ to the Hadamard algebra of $(\{P_w\}_{w \in Y^*}, \odot)$. Therefore, the map $H : u \mapsto H_u = \{H_u(N)\}_{N \geq 0}$ is an isomorphism from $(\mathbb{C}\langle Y \rangle, \sqcup)$ to the algebra of $(\{H_w\}_{w \in Y^*}, \cdot)$.

Proof. The definition of the Hadamard product $\sum_{n=0}^{\infty} a_n z^n \odot \sum_{n=0}^{\infty} b_n z^n = \sum_{n=0}^{\infty} a_n b_n z^n$, and the formula (22) gives P as an algebra morphism. Since the functions $\{\text{Li}_w\}_{w \in X^*}$ are linearly independent over \mathbb{C} [4], P is the expected isomorphism. \square

Proposition 3 ([8]). For every word $w \in Y^*$ and for $z \in \mathbb{C}$ satisfying $|z| < 1$, one has $\text{Li}_w(z) = (1-z)P_w(z)$.

Proof. For $w = y_s w'$, since $P_w(z) = \sum_{N \geq 0} H_w(N) z^N$ and by using (21),

$$(1-z)P_w(z) = H_w(0) + \sum_{N \geq 1} \frac{H_{w'}(N-1)}{N^s} z^N = \text{Li}_w(z).$$

\square

A direct consequence of this proposition and Identity (16) is

Corollary 1. For all $u, v \in X^*$, for all $z \in \mathbb{C}$ satisfying $|z| < 1$, $P_u(z)P_v(z) = (1-z)^{-1}P_{u \sqcup v}(z)$.

Example 6. Since $x_1 \sqcup x_0 x_1 = x_1 x_0 x_1 + 2x_0 x_1^2$ then we get

$$P_{1,2}(z) = (1-z)P_1(z)P_2(z) - 2P_{2,1}(z).$$

Proposition 3 allows to extend the definition of P_w over X^* as we have already extended the definition of Li_w over X^* . Moreover,

Definition 3 ([8]). Let P be the noncommutative generating series of $\{P_w\}_{w \in X^*}$:

$$P = \sum_{w \in X^*} P_w w.$$

Proposition 4 ([8]). Let σ be the monoid morphism defined over X^* by $\sigma(x_0) = -x_1$ and $\sigma(x_1) = -x_0$. Then

$$P(1-z) = \frac{1-z}{z} [\sigma P(z)] \prod_{l \in \text{Lyn}(X) \setminus \{x_0, x_1\}} e^{\zeta(S_l) Q_l}.$$

Proof. It follows immediately from Proposition 1. \square

Example 7.

$$P_{2,1}(1-z) = \frac{1-z}{z} \left(-P_3(z) + \log(z)P_2(z) - \log^2(z)P_1(z) + \frac{\zeta(3)}{1-z} \right)$$

Thus,

$$P_{2,1}(z) = -\frac{z}{1-z} P_3(1-z) + \frac{z \log(1-z)}{1-z} P_2(1-z) - \frac{1}{2} \frac{z \log^2(1-z)}{1-z} P_1(1-z) + \frac{\zeta(3)}{1-z}.$$

By Formula (22) and Proposition 2, for $w \in Y^*$, there exist a finite set I and $(c_i)_{i \in I} \in C_2^I$ such that the three following identities are equivalent

$$w = \sum_{i \in I} c_i \sqcup y_1^{\sqcup i}, \quad (23)$$

$$P_w = \sum_{i \in I} P_{c_i} \odot P_{y_1^{\odot i}}, \quad (24)$$

$$H_w = \sum_{i \in I} H_{c_i} H_{y_1^i}. \quad (25)$$

In particular, for $w = y_1^k$, we have,

Lemma 2. Let $M = (m_{i,j})_{1 \leq i,j \leq k}$ be the matrix defined by $m_{i,j} = \delta_{i,j+1}$ (Kronecker symbol). Let $e_{i,j}$ the matrix of size $k \times k$, whose coefficients are all zero, except the one equal to 1 at line i and column j . Let

$$A = \begin{pmatrix} y_1 & 0 & \dots & 0 \\ -\frac{y_2}{2} & \frac{y_1}{2} & \dots & 0 \\ \vdots & \ddots & \ddots & 0 \\ \frac{(-1)^{k-1} y_k}{k} & \frac{(-1)^{k-2} y_{k-1}}{k} & \dots & \frac{y_1}{k} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} H_{y_1} & 0 & \dots & 0 \\ -\frac{H_{y_2}}{2} & \frac{H_{y_1}}{2} & \dots & 0 \\ \vdots & \ddots & \ddots & 0 \\ \frac{(-1)^{k-1} H_{y_k}}{k} & \frac{(-1)^{k-2} H_{y_{k-1}}}{k} & \dots & \frac{H_{y_1}}{k} \end{pmatrix}.$$

Then

$$\begin{pmatrix} y_1 \\ \vdots \\ y_1^k \end{pmatrix} = A \prod_{\ell=1}^{k-1} \left[M^\ell A({}^t M)^\ell + \sum_{\iota=1}^{\ell} e_{\iota, \ell} \right] \begin{pmatrix} \epsilon \\ \vdots \\ \epsilon \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} H_{y_1} \\ \vdots \\ H_{y_1^k} \end{pmatrix} = B \prod_{\ell=1}^{k-1} \left[M^\ell B({}^t M)^\ell + \sum_{\iota=1}^{\ell} e_{\iota, \ell} \right] \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}.$$

Proof. The formula $y_1^k = (-1)^{k-1} k^{-1} \sum_{l=0}^{k-1} (-1)^l y_1^l \sqcup y_{k-l}$ [6] can be written matrixially as follows

$$\begin{pmatrix} y_1 \\ y_1^2 \\ \vdots \\ y_1^k \end{pmatrix} = A \sqcup \begin{pmatrix} \epsilon \\ y_1 \\ \vdots \\ y_1^{k-1} \end{pmatrix} = A \sqcup \begin{pmatrix} \epsilon & 0 & \dots & 0 \\ 0 & y_1 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & \frac{(-1)^{k-2} y_{k-1}}{k-1} & \dots & \frac{y_1}{k-1} \end{pmatrix} \sqcup \begin{pmatrix} \epsilon \\ \epsilon \\ \vdots \\ y_1^{k-2} \end{pmatrix}.$$

Here all powers and products are carried out with the stuffle product. Successively, we get the expected result. \square

The word y_1^k appears then as a computable stuffle product of words of length 1. Hence,

Proposition 5. $H_{y_1^k}$ is a combination of $\{H_{y_r}\}_{1 \leq r < k}$ which are algebraically independent.

Proof. The $\{H_{y_r}\}_{1 \leq r < k}$ are algebraically independent according to Proposition 2, as image by the isomorphism H of the Lyndon words $\{y_r\}_{1 \leq r < k}$. By Lemma 2, we get the expected result. \square

Example 8. Since

$$y_1^2 = \frac{y_1 \sqcup y_1 - y_2}{2} \quad \text{and} \quad y_1^3 = \frac{2(y_3 - y_1 \sqcup y_2) + (y_1 \sqcup y_1 - y_2) \sqcup y_1}{6}$$

then we have

$$H_{y_1^2} = \frac{H_{y_1}^2 - H_{y_2}}{2} \quad \text{and} \quad H_{y_1^3} = \frac{2(H_{y_3} - H_{y_1}H_{y_2}) + (H_{y_1}^2 - H_{y_2})H_{y_1}}{6}.$$

Identities (23-25) give rise to two interpretations : (24) enables to decompose P_w in a basis of singular functions $(1-z)^\alpha \log^\beta(1-z)$ while (25) enables to compute an asymptotic expansion of its Taylor coefficients in terms of $N^a \log^b N$ (or equivalently in terms of $N^a H_{y_1}^b(N)$). Before stating a theorem linking these two interpretations, we are interested in the action of \mathcal{C} on Taylor coefficients; reciprocally, we are interested in the effects of changing Taylor coefficients on a function in $\mathcal{C}[\{P_w\}_{w \in Y^*}]$.

3.2 Operations on the generating functions P_w

For $f(z) = \sum_{n \geq 0} a_n z^n$, we will henceforth denote $[z^n]f(z) = a_n$ its n -th Taylor coefficient. Since multiplying or dividing by z acts very simply on $[z^n]f(z)$, we only have to study the effect of multiplying or dividing by $1-z$.

$$[z^n](1-z)P_w(z) = H_w(n) - H_w(n-1). \quad (26)$$

$$[z^n] \frac{P_w(z)}{1-z} = \sum_{k=0}^n H_w(k) \quad (27)$$

$$= \begin{cases} (n+1)H_w(n) - H_{y_{s-1}w'}(n) & \text{if } w = y_s w', \text{ with } s \neq 1 \\ (n+1)H_w(n) - \sum_{j=1}^n H_{w'}(j-1) & \text{if } w = y_1 w'. \end{cases} \quad (28)$$

and, more generally,

Proposition 6.

$$[z^n](1-z)^k P_w(z) = \sum_{j=0}^k \binom{k}{j} (-1)^j H_w(n-j) \quad \text{and} \quad [z^n] \frac{P_w(z)}{(1-z)^k} = \sum_{n \geq j_1 \geq \dots \geq j_k \geq 0} H_w(j_k).$$

3.3 Operations on Taylor coefficients of P_w

We are now to find how multiplying or dividing $H_w(N)$ by N acts on P_w .

3.3.1 A particular case : $w = \epsilon$

The simple case $w = \epsilon$, corresponding to $H_\epsilon(N) = 1$, can be studied and treated by the following

Proposition 7. For any $q \in \mathbb{Z}$, one has

$$n^q = \begin{cases} [z^n](1-z)P_{-q}(z) & \text{if } q < 0, \\ [z^n](1-z)^{-1} & \text{if } q = 0, \\ [z^n] \frac{z}{1-z} N_q \left(\frac{1}{1-z} \right) & \text{if } q > 0, \end{cases}$$

where N_q is defined by the following recurrence

$$N_0(X) = 1, \quad \text{and} \quad N_q(X) = X \left(\sum_{j=0}^{q-1} (-1)^{q-1-j} \binom{q}{j} N_j(X) \right).$$

Example 9.

$$\begin{aligned} n &= [z^n] \left(\frac{z}{(1-z)^2} \right) = [z^n] \left(\frac{1}{(1-z)^2} - \frac{1}{1-z} \right), \\ n^2 &= [z^n] \left(\frac{2z}{(1-z)^3} - \frac{z}{(1-z)^2} \right) = [z^n] \left(\frac{2}{(1-z)^3} - \frac{3}{(1-z)^2} + \frac{1}{1-z} \right). \end{aligned}$$

3.3.2 How to divide by n^k ?

Let $w = y_{s_1} \cdots y_{s_r}$ and $w' = y_{s_2} \cdots y_{s_r}$ be the suffix of w , of length $r-1$. The expression $n^{-k} \mathbf{H}_w(n)$, k positive integer, can be identified as follows

$$n^{-k} \mathbf{H}_w(n) = n^{-k} \mathbf{H}_w(n-1) + n^{-s_1-k} \mathbf{H}_{w'}(n-1) \quad (29)$$

$$= [z^n] \text{Li}_{y_k w + y_{s_1+k} w'}(z) \quad (30)$$

$$= [z^n] [(1-z) \mathbf{P}_{y_k w + y_{s_1+k} w'}(z)]. \quad (31)$$

3.3.3 How to multiply by n^k ?

In order to study the effect of multiplying by n^k , k positive integer, we denote by $\theta = z\partial/\partial z$ the Euler operator. Then for any integer k ,

$$n^k \mathbf{H}_w(n) = [z^n] \theta^k \mathbf{P}_w(z). \quad (32)$$

So, we just have to compute $\theta^k \mathbf{P}_w(z)$. As in [7], let us introduce

Definition 4. For any word $w = x_{i_1} \cdots x_{i_k}$ and for any composition $\mathbf{r} = (r_1, \dots, r_k)$, let $\tau_{\mathbf{r}}(w)$ be defined by $\tau_{\mathbf{r}}(w) = \tau_{r_1}(x_{i_1}) \cdots \tau_{r_k}(x_{i_k})$ with,

$$\tau_0(x_0) = x_0, \quad \tau_r(x_1) = x_1,$$

$$\text{and, for } r \in \mathbb{N}^*, \quad \tau_r(x_0) = \theta^r x_0 = 0 \quad \text{and} \quad \tau_r(x_1) = \theta^r \frac{zx_1}{1-z} = \frac{r!x_1}{(1-z)^{r+1}}.$$

We define the degree of \mathbf{r} by $\deg(\mathbf{r}) = k$ and its weight by $\text{wgt}(\mathbf{r}) = k + r_1 + \cdots + r_k$.

By applying successively the operator θ to L , we get

Lemma 3. $\theta^l L = A_l L$, where A_l is defined by

$$A_l(z) = \sum_{\text{wgt}(\mathbf{r})=l} \sum_{w \in X^{\deg(\mathbf{r})}} \prod_{i=1}^{\deg(\mathbf{r})} \binom{\sum_{j=1}^i r_j + j - 1}{r_i} \tau_{\mathbf{r}}(w).$$

Proof. This is a consequence of the recurrence relation verified by A_l , which is $A_0(z) = 1$, and, for all $l \in \mathbb{N}$, $A_{l+1}(z) = [\tau_0(x_0) + \tau_0(x_1)]A_l(z) + \theta A_l(z)$. \square

This lemma enables to extract the expression of $\theta^l \text{Li}_w$, for any word $w \in X^*$.

Example 10.

$$\begin{aligned} A_0(z) &= 1, \\ A_1(z) &= x_0 + \frac{z}{1-z} x_1, \\ A_2(z) &= x_0^2 + \frac{z}{1-z} x_0 \sqcup x_1 + \frac{z^2}{(1-z)^2} x_1^2 + \frac{1}{(1-z)^2} x_1. \end{aligned}$$

So, for $w = x_0^2 x_1$,

$$\begin{aligned} \theta \text{Li}_{x_0^2 x_1} &= \left((x_0 + \frac{z}{1-z} x_1) L(z) \mid x_0^2 x_1 \right) \\ &= \text{Li}_{x_0 x_1}, \\ \theta^2 \text{Li}_{x_0^2 x_1} &= \left((x_0^2 + \frac{z}{1-z} x_0 \sqcup x_1 + \frac{z^2}{(1-z)^2} x_1^2 + \frac{1}{(1-z)^2} x_1) L(z) \mid x_0^2 x_1 \right) \\ &= \text{Li}_{x_1}. \end{aligned}$$

Lemma 4. Let \perp be the linear operator of $\mathbb{Z}[X]$ defined by $\perp X^n = (n+1)X^{n+1} + nX^n$ and $\{B_l\}_{l \in \mathbb{N}} \in \mathbb{Z}[X]$ defined by $B_0(X) = 1$ and $B_{l+1}(X) = \perp B_l(X)$. Then

$$\theta^l(1-z)^{-1} = (1-z)^{-1}B_l(z(1-z)^{-1}).$$

Note that the head term of B_l , $l \geq 1$, is $l!X^l$ and its trail term is X .

Example 11. $B_0(X) = 1$, $B_1(X) = X$, $B_2(X) = 2X^2 + X$, $B_3(X) = 6X^3 + 6X^2 + X$.

Proposition 8. With the notations of Lemma 4,

$$\theta^k P(z) = \sum_{j=1}^k \sum_{\text{wgt}(\mathbf{r})} \sum_{w \in X^{\text{deg}(\mathbf{r})}} \prod_{i=1}^{\text{deg}(\mathbf{r})} \binom{\sum_{j=1}^i r_i + j - 1}{r_i} \binom{k}{j} \tau_{\mathbf{r}}(w) B_j\left(\frac{z}{1-z}\right) P(z).$$

Using Leibniz formula, one has

$$\theta^k P_w(z) = \sum_{j=0}^k \binom{k}{j} \theta^{k-j} \text{Li}_w(z) \theta^j \frac{1}{1-z} \quad (33)$$

$$= \sum_{j=0}^k \binom{k}{j} B_j\left(\frac{z}{1-z}\right) \frac{1}{1-z} \theta^{k-j} \text{Li}_w(z). \quad (34)$$

Thanks to Lemma 3, we can extract the coefficient $\theta^l \text{Li}_w$ of w in $\theta^l L$: this can be written as \mathcal{C} -linear combination of Li_v , with $|v| \leq |w| - l$ (where $|u|$ denotes the length of a word $u \in X^*$). We deduce so the expression of $\theta^k P_w$.

Example 12. For $w = x_0^2 x_1$ and $k = 2$,

$$\begin{aligned} \theta^2 P_{x_0^2 x_1}(z) &= \sum_{j=0}^2 \binom{2}{j} B_j\left(\frac{z}{1-z}\right) \frac{1}{1-z} \theta^{2-j} \text{Li}_w(z) \\ &= \frac{1}{1-z} \text{Li}_{x_1}(z) + 2 \frac{z}{1-z} \frac{1}{1-z} \text{Li}_{x_0 x_1}(z) + \left(2 \left(\frac{z}{1-z}\right)^2 + \frac{z}{1-z}\right) \text{Li}_{x_0^2 x_1}(z) \\ &= P_{x_1}(z) + \frac{2z}{1-z} P_{x_0 x_1}(z) + \frac{z^2 + z}{1-z} P_{x_0^2 x_1}(z). \end{aligned}$$

$$\text{So, } n^2 H_3(n) = [z^n] \left(P_1(z) + \frac{2z}{1-z} P_2(z) + \frac{z^2 + z}{1-z} P_3(z) \right).$$

4 The main theorem

Throughout the section, we will write

$$f_n \sim \sum_{i=0}^{\infty} g_i(n) \quad \text{for } n \rightarrow +\infty,$$

for a scale of functions $(g_i)_{i \in \mathbb{N}}$ - i.e. verifying $g_{i+1}(n) = O(g_i(n))$, for all i - to express that

$$f_n = \sum_{i=0}^I g_i(n) + O(g_{I+1}(n)), \quad \text{for any } I \geq 0.$$

In the same way, given a scale of functions $(h_i)_{i \in \mathbb{N}}$ around $z = 1$ (i.e. verifying $h_{i+1}(1-z) = O(h_i(1-z))$, when $z \rightarrow 1$) we will write

$$g(z) \sim \sum_{i=0}^{\infty} h_i(1-z) \quad \text{for } z \rightarrow 1,$$

to mean

$$g(z) = \sum_{i=0}^I h_i(1-z) + O(h_{I+1}(1-z)) \quad \text{for all } I \geq 0.$$

For $w = y_1^k$, we know the expression of $[z^N]P_{y_1^k}(z) = H_{y_1^k}(N)$ is given by Lemma 2. From the second form of Euler-MacLaurin formula, involving the Bernoulli numbers $\{B_k\}_{k \geq 0}$, we get the following asymptotic expansions

$$\begin{aligned} H_{y_1}(N) &\sim \log N + \gamma - \sum_{k=1}^{+\infty} \frac{B_k}{k} \frac{1}{N^k}, \\ H_{y_r}(N) &\sim \zeta(r) - \frac{1}{(r-1)N^{r-1}} - \sum_{k=r}^{+\infty} \frac{B_{k-r+1}}{k-r+1} \binom{k-1}{r-1} \frac{1}{N^k}, \quad \text{for } r > 1. \end{aligned}$$

Thus, we can deduce the asymptotic expansions of $H_{y_1^k}(N)$, for $N \rightarrow +\infty$, from the asymptotic expansions of $\{H_{y_r}(N)\}_{1 \leq r < k}$:

Example 13. From Example 8, we can deduce then

$$\begin{aligned} H_{y_1^2}(N) &= \frac{1}{2}(\log(N) + \gamma)^2 - \frac{1}{2}\zeta(2) + \frac{1}{2} \frac{\log(N) + \gamma + 1}{N} - \frac{1}{12N^2} + O\left(\frac{1}{N^2}\right), \\ H_{y_1^3}(N) &= \frac{1}{6} \log^3(N) + \frac{1}{2} \gamma \log^2(N) + \frac{1}{2}(\gamma^2 - \zeta(2)) \log(N) - \frac{1}{2}\zeta(2)\gamma + \frac{1}{3}\zeta(3) + \frac{1}{6}\gamma^3 + \frac{1}{4} \frac{\log^2(N)}{N} \\ &+ \frac{1}{2}(\gamma + 1) \frac{\log(N)}{N} + \frac{1}{4} (2\gamma + \gamma^2 - \zeta(2)) \frac{1}{N} - \frac{1}{24} \frac{\log^2(N)}{N^2} - \left(\frac{1}{8} + \frac{\gamma}{12}\right) \frac{\log(N)}{N^2} + O\left(\frac{1}{N^2}\right). \end{aligned}$$

Let us see in the general case how to reach the Taylor expansion of $g \in \mathcal{C}[(P_w)_{w \in Y^*}]$.

Theorem 2. Let $g \in \mathcal{C}[(P_w)_{w \in Y^*}]$. There exist $a_j \in \mathbb{C}$, $\alpha_j \in \mathbb{Z}$ and $\beta_j \in \mathbb{N}$ such that

$$g(z) \sim \sum_{j=0}^{+\infty} a_j (1-z)^{\alpha_j} \log^{\beta_j}(1-z), \quad \text{for } z \rightarrow 1.$$

Therefore, there exist $b_i \in \mathbb{C}$, $\eta_i \in \mathbb{Z}$ and $\kappa_i \in \mathbb{N}$ such that

$$[z^n]g(z) \sim \sum_{i=0}^{+\infty} b_i n^{\eta_i} \log^{\kappa_i}(n), \quad \text{for } n \rightarrow \infty.$$

Proof. Considering Corollary 1, we only have firstly to obtain the asymptotic expansion for the case $g(z) = P_w(z)$. Indeed, we get then the expansions of $f(z)g(z)$, for $f \in \mathcal{C}$ by remarking that $z = 1 - (1-z)$ and that $z^{-1} = \sum_{n \geq 0} (1-z)^n$.

The first expansion can be derived from Proposition 4 which links the behaviour of P_w around $z = 1$ to the behaviour of some algebraic combination of functions $\{P_u\}_{u \in X^*}$ around $z = 0$. Moreover, by Radford theorem 1, we can assume that each word u involved in this combination is a Lyndon word and so belongs to $x_0 X^* x_1 \cup \{x_0, x_1\}$. But, remind that, in this case, we have $P_u(z) = \sum_{n \geq 0} H_u(n) z^n$ and that $P_{x_0}(z) = (1-z)^{-1} \log(z)$. So, the expected first expansion follows.

From

$$(1-z)^\alpha \log(1-z)^\beta = (-1)^\beta \beta! (1-z)^{\alpha+1} P_{y_1^\beta}(z), \quad (35)$$

we derive the second expansion by computing the Taylor coefficient $[z^n](1-z)^\alpha \log^\beta(1-z)$. Since we have already explained how the multiplication by $(1-z)^\alpha$ acts on the Taylor coefficients, we just have then to compute $[z^n]P_{y_1^\beta} = H_{y_1^\beta}(n)$. For this, we use Lemma 2 which completes our proof. \square

Unfortunately, in the general case, knowing even the complete expansion of $[z^n]g(z)$ only enables to get an asymptotic expansion of $g(z)$, as $z \rightarrow 1$ up to order 0 (i.e. the *singular part* of the expansion). Indeed, Taylor coefficients of all functions $(1-z)^k$, $k \geq 0$ eventually vanish as in the following identity:

$$\frac{1}{n} = [z^n] \text{Li}_1(z) = [z^n][\text{Li}_1(z) + (1-z)^2], \quad \text{as soon as } n > 2. \quad (36)$$

In fact, to obtain this singular part, it is sufficient to know the asymptotic expansion of $[z^n]g(z)$ up to order $2 - \epsilon$, $\epsilon > 0$ [15].

Remark 1. In the case of a finite sum $\sum_{i \in I} b_i n^{\eta_i} H_1^{\kappa_i}(n)$, we are able to construct the unique function $f \in \mathcal{C}[(P_w)_{w \in Y^*}]$ such that,

$$\forall n \in \mathbb{N}, \quad [z^n]f(z) = \sum_{i \in I} b_i n^{\eta_i} H_1^{\kappa_i}(n), \quad (37)$$

as illustrated in Examples 9 and 12.

Remark 2. Note that the proof of Theorem 2 gives an effective construction of the asymptotic expansion of Taylor coefficients. In particular, applied to $g(z) = P_w(z)$ directly, it enables to find an asymptotic expansion of $H_w(N)$, as shown in the corollary below. Another algorithm, based on Euler Mac-Laurin formula, is available in [1].

Corollary 2. Let \mathcal{Z} be the \mathbb{Q} -algebra generated by convergent polyzêtas and let \mathcal{Z}' be the $\mathbb{Q}[\gamma]$ -algebra generated by \mathcal{Z} . Then there exist algorithmically computable coefficients $b_i \in \mathcal{Z}'$, $\kappa_i \in \mathbb{N}$ and $\eta_i \in \mathbb{Z}$ such that, for any $w \in Y^*$,

$$H_w(N) \sim \sum_{i=0}^{+\infty} b_i N^{\eta_i} \log^{\kappa_i}(N), \quad \text{for } N \rightarrow +\infty.$$

Example 14. From Example 7 we get, for $z \rightarrow 1$

$$P_{2,1}(z) = \frac{\zeta(3)}{1-z} + \log(1-z) - 1 - \frac{\log^2(1-z)}{2} + (1-z) \left(-\frac{\log^2(1-z)}{4} + \frac{\log(1-z)}{4} \right) + O(|1-z|).$$

But

$$\begin{aligned} [z^N]\zeta(3)(1-z)^{-1} &= \zeta(3), \\ [z^N]\log(1-z) &= -N^{-1}, \\ [z^N]\frac{\log^2(1-z)}{2} &= [z^N]\frac{2!(1-z)P_{y_1^2}(z)}{2} \\ &= [z^N](1-z)P_{y_1^2}(z) \\ &= H_{y_1^2}(N) - H_{y_1^2}(N-1), \\ &\vdots \end{aligned}$$

We find finally, using Example 13 :

$$[z^N]P_{2,1}(z) = H_{2,1}(N) = \zeta(3) - \frac{\log(N) + 1 + \gamma}{N} + \frac{1}{2} \frac{\log(N)}{N^2} + O\left(\frac{1}{N^2}\right).$$

Otherwise, by Example 6,

$$\begin{aligned} P_{1,2}(z) &= (1-z)P_1(z)P_2(z) - 2P_{2,1}(z) \\ &= (1-z) \frac{-\log(1-z)}{1-z} \frac{z}{1-z} \left(-P_2(1-z) + \log(1-z)P_1(1-z) + \frac{\zeta(2)}{z} \right) - 2P_{2,1}(z), \end{aligned}$$

calculated thanks to Proposition 4. So,

$$[z^N]P_{1,2}(z) = H_{1,2}(N) = \zeta(2)\gamma - 2\zeta(3) + \zeta(2)\log(N) + \frac{\zeta(2) + 2}{2N} + O\left(\frac{1}{N^2}\right).$$

Corollary 3 ([8]). For any $w \in Y^*$, the N -free term in the asymptotic expansion of $H_w(N)$, when $N \rightarrow +\infty$, is a polynomial q_w in $\mathcal{Z}[\gamma]$. This term is an element in \mathcal{Z} , if and only if w is a convergent word.

Example 15. $q_{y_1 y_2} = \zeta(2)\gamma - 2\zeta(3)$ and $q_{y_2 y_1} = \zeta(3) = \zeta(2, 1)$.

Question. For any convergent word w , are $\zeta(w)$ and γ algebraically independent ?

Now, let us go back to the A_s introduced in Section 1. We have seen that they are \mathbb{Z} -linear combinations on H_s , hence we get their asymptotic expansions with coefficients in \mathcal{Z}' .

Example 16. For $\mathbf{s} = (1, 1, 1)$,

$$\begin{aligned} A_{1,1,1}(N) &= H_{1,1,1}(N) + H_{1,2}(N) + H_{2,1}(N) + H_3(N), \\ &= \frac{1}{6} \log^3(N) + \frac{1}{2} \gamma \log^2(N) + \frac{1}{2} [\gamma^2 + \zeta(2)] \log(N) - \frac{1}{2} \zeta(2) \gamma + \frac{1}{3} \zeta(3) + \frac{1}{6} \gamma^3 + \frac{1}{4} \frac{\log^2(N)}{N} \\ &+ \frac{1}{2} (\gamma - 1) \frac{\log(N)}{N} + \frac{1}{4} [\gamma^2 - 2\gamma + \zeta(2)] \frac{1}{N} - \frac{1}{24} \frac{\log^2(N)}{N^2} + \frac{1}{24} (9 - 2\gamma) \frac{\log(N)}{N^2} + O\left(\frac{1}{N^2}\right). \end{aligned}$$

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