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# Position of the maximum in a sequence with geometric distribution

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As a sequel to [1], the position of the maximum in a geometrically distributed sample is investigated. Samples of length  $n$  are considered, where the maximum is required to be in the first  $d$  positions. The probability that the maximum occurs in the first  $d$  positions is sought for  $d$  dependent on  $n$  (as opposed to  $d$  fixed in [1]). Two scenarios are discussed. The first is when  $d = \alpha n$  for  $0 < \alpha \leq 1$ , where Mellin transforms are used to obtain the asymptotic results. The second is when  $1 \leq d = o(n)$ .

**Keywords:** Mellin transforms, generating functions, geometric distribution.

## 1 Introduction

Consider a word whose letters are natural numbers. Assume that these letters occur independently and with geometric probability. So for  $p + q = 1$ , each letter  $j$  appears in the word with probability  $pq^{j-1}$ . We write  $Q := q^{-1}$  and  $L := \log Q$ .

We address the question: “What is the probability that the maximum in a word occurs in the first  $d$  positions?” We take words of length  $n$  and require  $d \leq n$ . Previously (in [1]),  $d$  was considered fixed. Now,  $d$  is allowed to grow with  $n$ . First, we assume that  $d$  is proportional to  $n$  and then we consider the case when  $d$  is  $o(n)$  (but at least 1). The latter produces the same solutions as when  $d$  is fixed (see [1]).

We distinguish between two cases: ‘strict’ and ‘weak’. A ‘strict’ maximum never recurs, whereas a ‘weak’ maximum can recur any number of times. These apply separately to the two parts of the word (the first  $d$  letters, and the remaining  $n - d$  letters). This means that in total, there are four different cases to be dealt with: (strict, strict), (weak, strict), (strict, weak) and (weak, weak), where the first entry refers to the first  $d$  letters in the word, and the second entry to the rest of the word. The results in the (strict, strict) case for  $d$  fixed will hold for the other scenarios too (i.e., it is not required that  $d$  is independent of  $n$ ). This is because the relevant calculations in [1] still go through when we take the limits as  $n \rightarrow \infty$ . This is not true for the remaining three cases which are dealt with in Section 3.

## 2 Results

**Theorem 1** *The probability that the maximum value in a word of length  $n$  is in the first  $d$  positions (for  $d = \alpha n$ ) is*

$$\text{Max}_{(w,s)}(n) \sim \frac{1}{L} \log \left( \frac{1}{1 - \alpha(Q-1)} \right) + \frac{1}{L} (\psi_0(n(1 - \alpha p)) - \psi_0(n)),$$

$$\text{Max}_{(s,w)}(n) \sim \frac{\alpha(Q-1)}{L(1 + \alpha(Q-1))} (1 + \psi(n(q + p\alpha))),$$

$$\text{Max}_{(w,w)}(n) \sim \frac{\log(1 + \alpha(Q-1))}{L} + \frac{1}{L} \left( \psi_0(n) - \psi_0 \left( n \left( \frac{q + \alpha p}{q} \right) \right) \right),$$

as  $n \rightarrow \infty$  where  $Q := q^{-1}$ ,  $L := \log Q$ , and the fluctuations are defined by

$$\psi(x) := \sum_{k \neq 0} \Gamma(1 - \chi_k) e^{2k\pi i x}, \quad \text{for } k \in \mathbb{Z},$$

and

$$\psi_0(x) := \sum_{k \neq 0} \Gamma(-\chi_k) e^{2k\pi i \log_Q x}, \quad \text{for } k \in \mathbb{Z}.$$

Note that we write  $\mathbf{i}$  rather than the  $i$  we use as an index.

**Theorem 2** *The probability that the maximum value in a word of length  $n$  is in the first  $d$  positions, for  $1 \leq d = o(n)$ , is*

$$\begin{aligned} \text{Max}_{(w,s)}(n) &\sim \frac{(1-Q^{-1})d}{Ln} (1 + \psi(n)), \\ \text{Max}_{(s,w)}(n) &\sim \frac{(Q-1)d}{Ln} (1 + \psi(n)), \\ \text{Max}_{(w,w)}(n) &\sim \frac{(Q-1)d}{Ln} (1 + \psi(n)), \end{aligned}$$

as  $n \rightarrow \infty$ , where  $\psi(x) = \sum_{k \neq 0} \Gamma(1 - \chi_k) e^{2k\pi i \log_Q x}$ ,  $k \in \mathbb{Z}$ .

### 3 Suppose $d = \alpha n$

Suppose we now consider  $d = \alpha n$  where  $0 < \alpha \leq 1$ . The ‘ $d$  fixed’ results from [1] continue to hold only in the (strict, strict) case. For the other three cases we make use of a technique from complex analysis called the ‘Mellin’ transform. The rules used below can be found in [2] and [5], among others. The first case is done in greater detail than the other two, as a similar process is used in each case.

#### 3.1 Case (weak, strict), for $d = \alpha n$

For this scenario (finding the probability with which the maximum,  $k$ , occurs in the first  $d$  letters of a word) we require that there is at least one  $k$  in the first  $d$  places, and possibly more. However, the rest of the word may only have letters from the set  $\{1, \dots, k-1\}$ . Now suppose that  $d = \alpha n$ , for  $0 < \alpha \leq 1$ . The generating function is explained in [1], yielding the following coefficients (with  $d$  replaced by  $\alpha n$ ).

$$f_{(w,s)}(n) := \sum_{k \geq 1} \sum_{i=0}^{\alpha n - 1} (1 - q^{k-1})^{i+n-\alpha n} p q^{k-1} (1 - q^k)^{\alpha n - 1 - i}. \quad (1)$$

Now note that

$$\sum_{i=0}^{\alpha n - 1} \left( \frac{1 - q^{k-1}}{1 - q^k} \right)^i = \frac{(1 - q^k)^{\alpha n} - (1 - q^{k-1})^{\alpha n}}{(1 - q^k)^{\alpha n - 1} q^{k-1} (1 - q)}, \quad (2)$$

and thus, since  $p = 1 - q$  and  $(1 - a)^n \sim e^{-an}$  for small  $a$ ,

$$\begin{aligned} f_{(w,s)}(n) &= \sum_{k \geq 1} (1 - q^{k-1})^{n(1-\alpha)} [(1 - q^k)^{\alpha n} - (1 - q^{k-1})^{\alpha n}] \\ &\sim \sum_{k \geq 1} [e^{-nq^{k-1}(1-\alpha p)} - e^{-nq^{k-1}}]. \end{aligned}$$

We are now in a position to use Mellin transforms to find an approximation. We shift the fundamental strip from  $\langle 0, \infty \rangle$  to  $\langle -1, 0 \rangle$  and define a new function:

$$f_{ws}(x) := \sum_{k \geq 1} [(e^{-nq^{k-1}(1-\alpha p)} - 1) - (e^{-nq^{k-1}} - 1)]. \quad (3)$$

Then the Mellin transform of this function is:

$$\begin{aligned} f_{ws}^*(s) &= \sum_{k \geq 1} [(q^{k-1}(1 - \alpha p))^{-s} \Gamma(s) - (q^{k-1})^{-s} \Gamma(s)] \\ &= \sum_{k \geq 1} q^{s(1-k)} \Gamma(s) [(1 - \alpha p)^{-s} - 1] \\ &= q^s \Gamma(s) [(1 - \alpha p)^{-s} - 1] \sum_{k \geq 1} (q^{-s})^k \\ &= \Gamma(s) [(1 - \alpha p)^{-s} - 1] \frac{1}{1 - q^{-s}}, \quad \text{for } \Re(s) < 0, \end{aligned} \quad (4)$$

where  $\Re(s)$  represents the real part of the complex number  $s$ . The reason for shifting the fundamental strip is that the transform exists in the intersection of the domain of convergence of the generalised Dirichlet series and the fundamental strip of  $f^*(s)$ . The intersection  $\langle -\infty, 0 \rangle \cap \langle 0, \infty \rangle$  is empty, but with the shift we have a final fundamental strip of  $\langle -1, 0 \rangle$ . We choose a value inside this, say  $-\frac{1}{2}$ , with which to perform our inverse Mellin transform:

$$f_{ws}(x) = \frac{1}{2\pi i} \int_{(-\frac{1}{2})} \Gamma(s) [(1 - \alpha p)^{-s} - 1] \frac{1}{1 - q^{-s}} x^{-s} ds. \quad (5)$$

The notation  $(-\frac{1}{2})$  under the integral sign means an integral from  $-\frac{1}{2} - i\infty$  to  $-\frac{1}{2} + i\infty$ . This can be approximated by moving the contour to the right (and thus collecting negative residues) since we are interested in  $x$  large. The first poles we encounter are the simple pole at  $s = 0$  (which would be a double pole except that one cancels with the factor  $(1 - \alpha p)^{-s} - 1$ ) and the simple poles at  $s = \chi_k := \frac{2k\pi i}{L}$ ,  $k \in \mathbb{Z} \setminus \{0\}$  where  $L := \log Q$ . The former contributes the main term and the rest contribute the fluctuations which are comparatively extremely small. As  $s \rightarrow 0$ ,

$$\Gamma(s) \sim \frac{1}{s},$$

$$(1 - \alpha p)^{-s} - 1 \sim 1 - s \log(1 - \alpha p) - 1 = -s \log(1 - \alpha p),$$

$$\frac{1}{1 - q^{-s}} = \frac{1}{1 - e^{-s \log q}} \sim \frac{1}{1 - (1 - s \log q)} = \frac{1}{s \log q},$$

and

$$x^{-s} = e^{-s \log x} \sim 1.$$

Thus the negative residue is

$$-[s^{-1}] \frac{1}{s} (-s \log(1 - \alpha p)) \frac{1}{s \log q} = \frac{\log(1 - \alpha p)}{\log q}.$$

We also have simple poles at  $s = \chi_k$ , for  $k \neq 0$ . Let  $\varepsilon := s - \chi_k$  then expanding around  $\varepsilon = 0$  gives

$$\frac{1}{1 - q^{-s}} = \frac{1}{1 - q^{-\chi_k - \varepsilon}} = \frac{1}{1 - q^{-\varepsilon}} = \frac{1}{1 - e^{-\varepsilon \log q}} \sim \frac{1}{1 - (1 - \varepsilon \log q)} = \frac{1}{\varepsilon \log q}.$$

So the negative residues for all non-zero  $k$  are

$$\begin{aligned} \sum_{k \neq 0} (-1) [\varepsilon^{-1}] \Gamma(\chi_k) [(1 - \alpha p)^{-\chi_k} - 1] \frac{1}{\varepsilon \log q} x^{-\chi_k} &= \frac{1}{L} \sum_{k \neq 0} \Gamma(\chi_k) [(1 - \alpha p)^{-\chi_k} - 1] x^{-\chi_k} \\ &= \frac{1}{L} \sum_{k \neq 0} \Gamma(\chi_k) [e^{-\chi_k \log(1 - \alpha p)} - 1] e^{-\chi_k \log x} \\ &= \frac{1}{L} \sum_{k \neq 0} \Gamma(\chi_k) [e^{-\chi_k \log(x(1 - \alpha p))} - e^{-\chi_k \log x}] \\ &= \frac{1}{L} \sum_{k \neq 0} \Gamma(-\chi_k) [e^{2k\pi i \log_Q(x(1 - \alpha p))} - e^{2k\pi i \log_Q x}]. \end{aligned}$$

Now put these together to get the probability of having a weak maximum in the first  $d$  positions which does not repeat in the rest of the word and where  $d = \alpha n$  grows with  $n$  as  $n \rightarrow \infty$ :

$$\begin{aligned} &\frac{\log(1 - \alpha p)}{\log q} + \frac{1}{L} \sum_{k \neq 0} \Gamma(-\chi_k) [e^{2k\pi i \log_Q(n(1 - \alpha p))} - e^{2k\pi i \log_Q n}] \\ &= \frac{\log(1 - \alpha(1 - Q^{-1}))}{-L} + \frac{1}{L} (\psi_0(n(1 - \alpha p)) - \psi_0(n)) \end{aligned} \quad (6)$$

for  $Q = q^{-1}$ ,  $L = \log Q$  and  $\psi_0(x)$  as in Theorem 1. Note that if  $\alpha = 1$  then  $d = n$  and the main term yields a probability of 1, confirming our result on a word with no restrictions.

### 3.2 Case (strict, weak), for $d = \alpha n$

In this case, we allow the maximum,  $k$ , to occur only once in the first  $d$  letters, but any number of times in the rest of the word. Again, [1] provides us with the generating function whose coefficients are

$$\begin{aligned} f_{(s,w)}(n) &= \sum_{k \geq 1} \alpha n p q^{k-1} (1 - q^{k-1})^{\alpha n - 1} (1 - q^k)^{n(1-\alpha)} \\ &\sim \sum_{k \geq 1} \alpha n p q^{k-1} e^{-nq^{k-1}(q+p\alpha)}. \end{aligned} \quad (7)$$

If we define the function

$$f_{sw}(x) := \sum_{k \geq 1} \alpha x p q^{k-1} e^{-xq^{k-1}(q+p\alpha)},$$

then the Mellin transform will be

$$\begin{aligned} f_{sw}^*(s) &= \sum_{k \geq 1} \alpha p q^{k-1} (q^{k-1})^{-(s+1)} (q + p\alpha)^{-(s+1)} \Gamma(s+1) \\ &= \alpha p (q + p\alpha)^{-(s+1)} \Gamma(s+1) \frac{1}{1 - q^{-s}}, \quad \text{for } \Re(s) < 0, \end{aligned} \quad (8)$$

and the fundamental strip is the overlap of the interval  $(-\infty, 0)$  and the fundamental strip of  $xe^{-x}$  which is  $\langle -1, \infty \rangle$ , i.e.,  $\langle -1, 0 \rangle$ . Hence we pick our contour integral from  $-\frac{1}{2} - i\infty$  to  $-\frac{1}{2} + i\infty$ , and perform the inverse Mellin transform to get:

$$f_{sw}(x) = \frac{1}{2\pi i} \int_{(-\frac{1}{2})} \alpha p (q + p\alpha)^{-(s+1)} \Gamma(s+1) \frac{1}{1 - q^{-s}} x^{-s} ds. \quad (9)$$

By moving the contour to the right, the first poles we reach are at  $s = 0$  and  $s = \chi_k$ ,  $k \neq 0$ . For the main term, the negative residue is

$$-[s^{-1}] \alpha p (q + p\alpha)^{-1} \Gamma(1) \frac{1}{s \log q} = \frac{\alpha p}{L(q + p\alpha)}.$$

The fluctuations come from the negative residues of the poles at  $s = \chi_k$ ,  $k \neq 0$ . Let  $\varepsilon := s - \chi_k$ , then around  $\varepsilon = 0$  we get  $\frac{1}{1 - q^{-s}} \sim \frac{1}{\varepsilon \log q}$ , and so these poles contribute:

$$-\sum_{k \neq 0} [\varepsilon^{-1}] \alpha p (q + p\alpha)^{-(\chi_k+1)} \Gamma(\chi_k + 1) \frac{1}{\varepsilon \log q} x^{-\chi_k} = \frac{\alpha p}{L(q + p\alpha)} \sum_{k \neq 0} \Gamma(1 - \chi_k) e^{2k\pi i \log_Q(x(q+p\alpha))}.$$

This gives a total probability in the (strict, weak) case asymptotic to

$$\frac{\alpha(Q-1)}{L(1 + \alpha(Q-1))} (1 + \psi(n(q+p\alpha))), \quad (10)$$

as  $n \rightarrow \infty$  where  $\psi(x) = \sum_{k \neq 0} \Gamma(1 - \chi_k) e^{2k\pi i \log_Q x}$ .

(For  $\alpha = 1$ , the dominant term gives  $\frac{p}{L}$ , which is the same as the probability of having one winner among  $n$  players in a game where each player tosses a coin until a head appears, and the winner is the player who takes the longest to toss a head, see [3].)

### 3.3 Case (weak, weak), for $d = \alpha n$

Here, the maximum can recur anywhere, having first appeared at least once in the first  $d$  letters. From [1], and by (2), we can approximate as in the (w,s) case to get:

$$f_{(w,w)}(n) \sim \sum_{k \geq 1} [(e^{-nq^k} - 1) - (e^{-nq^{k-1}(q+p\alpha)} - 1)]. \quad (11)$$

We define an exact function in terms of  $x$  to be:

$$f_{ww}(x) := \sum_{k \geq 1} [(e^{-xq^k} - 1) - (e^{-xq^{k-1}(q+p\alpha)} - 1)].$$

The transform of this function is

$$\begin{aligned} f_{ww}^*(s) &= \sum_{k \geq 1} [q^{-sk} \Gamma(s) - (q^{k-1})^{-s} (q + \alpha p)^{-s} \Gamma(s)] \\ &= \Gamma(s) [1 - q^s (q + \alpha p)^{-s}] \frac{1}{q^s - 1}, \end{aligned} \quad (12)$$

which exists in the strip  $\langle -1, 0 \rangle$ . We can thus rewrite  $f_{ww}(x)$  as a contour integral

$$f_{ww}(x) = \frac{1}{2\pi i} \int_{(-\frac{1}{2})} \Gamma(s) [1 - q^s (q + \alpha p)^{-s}] \frac{1}{q^s - 1} x^{-s} ds. \quad (13)$$

The relevant simple poles occur at  $s = 0$  and  $s = \chi_k, k \neq 0$ . The negative residue at  $s = 0$  is

$$-[s^{-1}] \frac{1}{s} \log \left( \frac{q + \alpha p}{q} \right) \frac{1}{s \log q} = \frac{1}{L} \log \left( \frac{q + \alpha p}{q} \right), \quad (14)$$

which, as in the (weak, strict) case, is one for  $\alpha = 1$ . For the poles at  $s = \chi_k$ , let  $\varepsilon := s - \chi_k$ . Then expanding around  $\varepsilon = 0$  gives  $\frac{1}{q^s - 1} \sim \frac{1}{\varepsilon \log q}$  and so the negative residues are

$$-\sum_{k \neq 0} [\varepsilon^{-1}] \Gamma(\chi_k) [1 - q^{\chi_k} (q + \alpha p)^{-\chi_k}] \frac{1}{\varepsilon \log q} x^{-\chi_k} = \frac{1}{L} \sum_{k \neq 0} \Gamma(-\chi_k) \left[ e^{2k\pi i \log_Q x} - e^{2k\pi i \log_Q(x(\frac{q+\alpha p}{q}))} \right]. \quad (15)$$

By summing (14) and (15) and replacing  $x$  by  $n$  the asymptotic result for the (weak, weak) case in Theorem 1 is found.

## 4 Suppose $1 \leq d = o(n)$

Initially, in this section  $d$  was considered to be dependent on  $n$  according to the relationship  $d = \alpha n^\gamma$ , where  $0 < \gamma < 1$ , and  $0 < \alpha \leq 1$ . Mellin transforms were used to obtain the same results as found in the  $d$  fixed case (see [1]). However, it was then found (as suggested by a referee) that in fact any  $d$  such that  $1 \leq d = o(n)$  will produce the same results. The explanation is given below.

We show that the results are the same as when  $d$  is fixed by referring back to the step in the calculations for  $d$  fixed (see [1]) where the  $d = \alpha n$  calculations failed. The important stage is when the main term of the probability is given by the expression

$$\sum_{i=0}^{d-1} \sum_{l=0}^{d-1-i} \binom{d-1-i}{l} (-1)^l \frac{Q^{-l}(1-Q^{-1})}{L} \frac{1}{(l+1+N) \binom{N+l}{l}}. \quad (16)$$

This is the (weak, strict) case, but the others are similar. Since  $N$  grows like  $n$ , we use  $n$  instead of  $N$  for simplicity. For  $d$  fixed, it can be seen that the  $l = 0$  term dominates, since each term in the sum on  $l$  is of order  $O(\frac{1}{n^{l+1}})$ . For  $d$  proportional to  $n$ , each term in the inner sum is of order  $O(\frac{1}{n})$ , so none clearly dominates, and Mellin transforms are required to find the result (see Section 3). But what if  $d = \alpha n^\gamma$  for  $0 < \gamma < 1$ , or even  $d = \frac{n}{\log n}$ ?

Suppose we let  $f(n) = o(n)$  for some  $f(n)$  such that  $f(n) \rightarrow \infty$  as  $n \rightarrow \infty$ . Then we can write  $d = \frac{n}{f(n)} (= o(n))$ . In general, a typical term in the sum on  $l$  is of order  $O(\frac{1}{n f^l(n)})$ . For  $l = 0$ , we again have an order of  $O(\frac{1}{n})$ . In fact, this term can be expressed as  $\frac{c}{n}$  where  $c$  is a constant. This will dominate all other terms, since even the infinite sum on  $l$  (a geometric series) is:

$$\frac{1}{n} \sum_{l=1}^{\infty} (f(n))^{-l} = \frac{1}{n(f(n) - 1)} = o\left(\frac{1}{n}\right). \quad (17)$$

The results in Theorem 2 follow from the above.

## 5 Conclusion

Table 1 is a summary of results from this paper. (The  $d$  fixed results are from [1]). This table shows only the dominant term for the results, expressed in terms of  $Q$ .

| Case              | (s,s)                        | (w,s)                                 | (s,w)                                  | (w,w)                           |
|-------------------|------------------------------|---------------------------------------|--|---------------------------------|
| $1 \leq d = o(n)$ | $\frac{(1-Q^{-1})d}{Ln}$     | $\frac{(1-Q^{-1})d}{Ln}$              | $\frac{(Q-1)d}{Ln}$                    | $\frac{(Q-1)d}{Ln}$             |
| $d = \alpha n$    | $\frac{(1-Q^{-1})\alpha}{L}$ | $\frac{\log(1-\alpha(1-Q^{-1}))}{-L}$ | $\frac{\alpha(Q-1)}{L(1+\alpha(Q-1))}$ | $\frac{\log(1+\alpha(Q-1))}{L}$ |

**Tab. 1:** Table of results for the two categories

If we consider  $\alpha$  small (i.e., close to zero) in the second category, we should get similar solutions to category one (in which  $d$  is always small relative to  $n$  for  $n$  large). We thus determine what these dominant terms look like asymptotically as  $\alpha \rightarrow 0$ . We use the approximations  $\log(1+x) \sim x$  and  $\frac{1}{1-x} \sim 1$  as  $x \rightarrow 0$  ([4]). Suppose  $d = \alpha n$ , then for the (weak, strict) case, we have

$$\frac{\log(1 - \alpha(1 - Q^{-1}))}{\log Q^{-1}} \sim \frac{-\alpha(1 - Q^{-1})}{\log Q^{-1}} = \frac{\alpha(1 - Q^{-1})}{L}.$$

For the (strict, weak) case, we find that

$$\frac{\alpha(Q-1)}{L(1+\alpha(Q-1))} \sim \frac{\alpha(Q-1)}{L},$$

and for the (weak, weak) case,

$$\frac{\log(1 + \alpha(Q-1))}{L} \sim \frac{\alpha(Q-1)}{L}.$$

By replacing each  $\alpha$  by  $\frac{d}{n}$ , it can be seen that each of these corresponds to the  $d$  fixed case in Table 1 above.

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