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# Norm Inequalities of Positive Semi-Definite Matrices

*Antoine Mhanna<sup>1</sup>*

*1 Dept of Mathematics, Lebanese University, Hadath, Beirut, Lebanon.*

*tmhanat@yahoo.com*

## Abstract

For positive block-matrix  $M$  we write  $M = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \in \mathbb{M}_{n+m}^+$ , with  $A \in \mathbb{M}_n^+$ ,  $B \in \mathbb{M}_m^+$ . The main result is first to study the consequences of a decomposition lemma due to C. Bourrin and second to extend the class of these P.S.D. matrices  $M$  written by blocks that satisfies the inequality:  $\|M\| \leq \|A + B\|$  for all symmetric norms and to give examples whenever it is necessary.

**Keywords :** Matrix Analysis, Hermitian matrices, symmetric norms.

## 1 Introduction

Let  $A$  be an  $n \times n$  matrix and  $F$  an  $m \times m$  matrix, ( $m > n$ ) written by blocks such that  $A$  is a diagonal block and all entries other than those of  $A$  are zeros, then the two matrices have the same singular values and for all unitarily invariant norms  $\|A\| = \|F\| = \|A \oplus 0\|$ , we say then that the symmetric norm on  $\mathbb{M}_m$  induces a symmetric norm on  $\mathbb{M}_n$ , so for square matrices we may assume that our norms are defined on all spaces  $\mathbb{M}_n$ ,  $n \geq 1$ . The spectral norm is denoted by  $\|\cdot\|_s$ , the Frobenius norm by  $\|\cdot\|_{(2)}$ , and the Ky Fan  $k$ -norms by  $\|\cdot\|_k$ . Let  $\mathbb{M}_n^+$  denote the set of positive and semi-definite part of the space of  $n \times n$  complex matrices and  $M$  be any positive block-matrices; that is,  $M = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \in \mathbb{M}_{n+m}^+$ , with  $A \in \mathbb{M}_n^+$ ,  $B \in \mathbb{M}_m^+$ .

## 2 Decomposition of block-matrices

**Lemma 2.1.** *For every matrix  $M$  in  $\mathbb{M}_{n+m}^+$  written in blocks, we have the decomposition:*

$$\begin{pmatrix} A & X \\ X^* & B \end{pmatrix} = U \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} U^* + V \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix} V^*$$

for some unitaries  $U, V \in \mathbb{M}_{n+m}$ .

*Proof.* Factorize the positive matrix as a square of positive matrices:

$$\begin{pmatrix} A & X \\ X^* & B \end{pmatrix} = \begin{pmatrix} C & Y \\ Y^* & D \end{pmatrix} \cdot \begin{pmatrix} C & Y \\ Y^* & D \end{pmatrix}$$

we verify that the right hand side can be written as  $T^*T + S^*S$  so :

$$\begin{pmatrix} C & Y \\ Y^* & D \end{pmatrix} \cdot \begin{pmatrix} C & Y \\ Y^* & D \end{pmatrix} = \underbrace{\begin{pmatrix} C & 0 \\ Y^* & 0 \end{pmatrix}}_{T^*} \cdot \underbrace{\begin{pmatrix} C & Y \\ 0 & 0 \end{pmatrix}}_T + \underbrace{\begin{pmatrix} 0 & Y \\ 0 & D \end{pmatrix}}_{S^*} \cdot \underbrace{\begin{pmatrix} 0 & 0 \\ Y^* & D \end{pmatrix}}_S.$$

Since  $TT^* = \begin{pmatrix} CC + YY^* & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$ ,  $SS^* = \begin{pmatrix} 0 & 0 \\ 0 & Y^*Y + DD \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}$  and  $AA^*$  is unitarily congruent to  $A^*A$  for any square matrix  $A$ , the lemma follows.  $\square$

**Remark 1.** *As a consequence of this lemma we have:*

$$\|M\| \leq \|A\| + \|B\|$$

for all symmetric norms.

Equations involving unitary matrices are called unitary orbits representations.

Recall that if  $A \in \mathbb{M}_n$ ,  $R(A) = \frac{A + A^*}{2}$  and  $I(A) = \frac{A - A^*}{2i}$ .

**Corollary 2.1.** *For every matrix in  $\mathbb{M}_{2n}^+$  written in blocks of the same size, we have the decomposition:*

$$\begin{pmatrix} A & X \\ X^* & B \end{pmatrix} = U \begin{pmatrix} \frac{A+B}{2} - R(X) & 0 \\ 0 & 0 \end{pmatrix} U^* + V \begin{pmatrix} 0 & 0 \\ 0 & \frac{A+B}{2} + R(X) \end{pmatrix} V^*$$

for some unitaries  $U, V \in \mathbb{M}_{2n}$ .

*Proof.* Let  $J = \frac{1}{\sqrt{2}} \begin{pmatrix} I & -I \\ I & I \end{pmatrix}$  where  $I$  is the identity of  $\mathbb{M}_n$ ,  $J$  is a unitary matrix, and we have:

$$J \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} J^* = \underbrace{\begin{pmatrix} \frac{A+B}{2} - R(X) & \frac{A-B}{2} + \frac{X^*-X}{2} \\ \frac{A-B}{2} - \frac{X-X^*}{2} & \frac{A+B}{2} + R(X) \end{pmatrix}}_N$$

Now we factorize  $N$  as a square of positive matrices:

$$\begin{pmatrix} A & X \\ X^* & B \end{pmatrix} = J^* \begin{pmatrix} L & M \\ M^* & F \end{pmatrix} \cdot \begin{pmatrix} L & M \\ M^* & F \end{pmatrix} J$$

and let:

$$\begin{aligned} \delta &= J^* \begin{pmatrix} L & M \\ M^* & F \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} L + M^* & M + F \\ M^* - L & F - M \end{pmatrix} \\ \psi &= \begin{pmatrix} L & M \\ M^* & F \end{pmatrix} J = \frac{1}{\sqrt{2}} \begin{pmatrix} L + M & M - L \\ F + M^* & F - M^* \end{pmatrix} \end{aligned}$$

A direct computation shows that:

$$\begin{aligned} \delta \cdot \psi &= \frac{1}{2} \begin{pmatrix} (L+M^*)(L+M)+(M+F)(F+M^*) & (L+M^*)(M-L)+(M+F)(F-M^*) \\ (M^*-L)(L+M)+(F-M)(F+M^*) & (M^*-L)(M-L)+(F-M)(F-M^*) \end{pmatrix} \\ &= \Gamma^* \Gamma + \Phi^* \Phi \end{aligned} \tag{1}$$

where:  $\Gamma = \frac{1}{\sqrt{2}} \begin{pmatrix} L+M & M-L \\ 0 & 0 \end{pmatrix}$ , and  $\Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 \\ F+M^* & F-M^* \end{pmatrix}$  to finish notice that for any square matrix  $A$ ,  $A^*A$  is unitarily congruent to  $AA^*$  and,  $\Gamma\Gamma^*$ ,  $\Phi\Phi^*$  have the required form.  $\square$

The previous corollary implies that  $\frac{A+B}{2} \geq R(X)$  and  $\frac{A+B}{2} \geq -R(X)$ .

**Corollary 2.2.** *For every matrix in  $\mathbb{M}_{2n}^+$  written in blocks of the same size, we have the decomposition:*

$$\begin{pmatrix} A & X \\ X^* & B \end{pmatrix} = U \begin{pmatrix} \frac{A+B}{2} + I(X) & 0 \\ 0 & 0 \end{pmatrix} U^* + V \begin{pmatrix} 0 & 0 \\ 0 & \frac{A+B}{2} - I(X) \end{pmatrix} V^*$$

for some unitaries  $U, V \in \mathbb{M}_{n+m}$ .

*Proof.* The proof is similar to Corollary 2.1, we have:  $J_1 \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} J_1^* = \begin{pmatrix} A & iX \\ -iX^* & B \end{pmatrix}$

where  $J_1 = \begin{pmatrix} I & 0 \\ 0 & -iI \end{pmatrix}$ , and

$$K = J J_1 \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} J_1^* J^* = \begin{pmatrix} \frac{A+B}{2} + I(X) & * \\ * & \frac{A+B}{2} - I(X) \end{pmatrix}$$

here  $(*)$  means an unspecified entry, the proof is similar to that in Corollary 2.1 but for reader's convenience we give the main headlines: first factorize  $K$  as a square of positive matrices; that is,  $M = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} = J_1^* J^* L^2 J J_1$  next decompose  $L^2$  as in Lemma 2.1 to obtain

$$M = J_1^* J^* (T^* T + S^* S) J J_1 = J_1^* J^* (T^* T) J J_1 + J_1^* J^* (S^* S) J J_1$$

where  $TT^* = \begin{pmatrix} \frac{A+B}{2} + I(X) & 0 \\ 0 & 0 \end{pmatrix}$  and  $SS^* = \begin{pmatrix} 0 & 0 \\ 0 & \frac{A+B}{2} - I(X) \end{pmatrix}$  finally the congruence property completes the proof.  $\square$

The existence of unitaries  $U$  and  $V$  in the decomposition process need not to be unique as one can take the special case; that is,  $M$  any diagonal matrix with diagonal entries equals a nonnegative number  $k$ , explicitly  $M = kI = U \left( \frac{k}{2} I \right) U^* + V \left( \frac{k}{2} I \right) V^*$  for any  $U$  and  $V$  unitaries.

**Remark 2.** Notice that from the Courant-Fischer theorem if  $A, B \in \mathbb{M}_n^+$ , then the eigenvalues of each matrix are the same as the singular values and  $A \leq B \implies \|A\|_k \leq \|B\|_k$ , for all  $k = 1, \dots, n$ , also  $A < B \implies \|A\|_k < \|B\|_k$ , for all  $k = 1, \dots, n$ .

**Corollary 2.3.** For every matrix in  $\mathbb{M}_{2n}^+$  written in blocks of the same size, we have:

$$\begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \leq \frac{1}{2} \left\{ U \begin{pmatrix} A + B + |X - X^*| & 0 \\ 0 & 0 \end{pmatrix} U^* + V \begin{pmatrix} 0 & 0 \\ 0 & A + B + |X - X^*| \end{pmatrix} V^* \right\}$$

for some unitaries  $U, V \in \mathbb{M}_{n+m}$ .

*Proof.* This is a consequence of the fact that  $I(X) \leq |I(X)|$ .  $\square$

### 3 Symmetric Norms and Inequalities

In [1] they found that if  $X$  is hermitian then

$$\|M\| \leq \|A + B\| \quad (2)$$

for all symmetric norms. It has been given counter-examples showing that this does not necessarily hold if  $X$  is a normal but not Hermitian matrix, the main idea of this section is to give examples and counter-examples in a general way and to extend the previous inequality to a larger class of P.S.D. matrices written by blocks satisfying (2).

**Theorem 3.1.** If  $A$  and  $B$  are positive definite matrices of same size. Then

$$\begin{pmatrix} A & X \\ X^* & B \end{pmatrix} > 0 \iff A \geq XB^{-1}X^*$$

*Proof.* Write  $\begin{pmatrix} A & X \\ X^* & B \end{pmatrix} = \begin{pmatrix} I & XB^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} A - XB^{-1}X^* & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} I & 0 \\ XB^{-1} & I \end{pmatrix}$  where  $I$  is the identity matrix, and that complete the proof since for any matrix  $A$ ,

$$A \geq 0 \iff X^*AX \geq 0, \forall X.$$

□

**Theorem 3.2.** Let  $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$  be any square matrix written by blocks of same size, if  $AC = CA$  then  $\det(M) = \det(AD - CB)$

*Proof.* Suppose first that  $A$  is invertible, let us write  $M$  as

$$M = \begin{pmatrix} Z & 0 \\ V & I \end{pmatrix} \begin{pmatrix} I & E \\ 0 & F \end{pmatrix} \quad (3)$$

upon calculation we find that:  $Z = A$ ,  $V = C$ ,  $E = A^{-1}B$ ,  $F = D - CA^{-1}B$  taking the determinant on each side of (3) we get:

$$\det(M) = \det(A(D - CA^{-1}B)) = \det(AD - CB)$$

the result follows by a continuity argument since the Determinant function is a continuous function. □

Given the matrix  $M = \begin{pmatrix} A & X \\ X^* & 0 \end{pmatrix}$  a matrix in  $\mathbb{M}_{2n}^+$  written by blocks of same size, we know that it  $M$  is not P.S.D., to see this notice that all the  $2 \times 2$  extracted principle submatrices of  $M$  are P.S.D if and only if  $X = 0$  and  $A$  is positive semi-definite. Even if a proof of this exists and would take two lines, it is quite nice to see a different constructive proof, a direct consequence of Lemma 2.1.

**Theorem 3.3.** Given  $\begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$  a matrix in  $\mathbb{M}_{2n}^+$  written in blocks of same size:

1. If  $\begin{pmatrix} A & X \\ X^* & 0 \end{pmatrix}$  is positive semi-definite,  $I(X) > 0$  or  $I(X) < 0$ , then there exist a matrix  $Y$  such that  $M = \begin{pmatrix} A & Y \\ Y^* & 0 \end{pmatrix}$  is positive semi-definite and:

$$\left\| \begin{pmatrix} A & Y \\ Y^* & 0 \end{pmatrix} \right\| > \|A\| \quad (4)$$

for all symmetric norms.

2. If  $\begin{pmatrix} 0 & X \\ X^* & B \end{pmatrix}$  is positive semi-definite,  $I(X) > 0$  or  $I(X) < 0$  then there exist a matrix  $Y$  such that  $M = \begin{pmatrix} 0 & Y \\ Y^* & B \end{pmatrix}$  is positive semi-definite and:

$$\left\| \begin{pmatrix} 0 & Y \\ Y^* & B \end{pmatrix} \right\| > \|B\| \quad (5)$$

The same result holds if we replaced  $I(X)$  by  $R(X)$  because  $\begin{pmatrix} A & iX \\ -iX^* & B \end{pmatrix}$  is unitarily congruent to  $\begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$ .

*Proof.* Without loss of generality we can consider  $I(X) > 0$  cause  $\begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$  and  $\begin{pmatrix} A & -X \\ -X^* & B \end{pmatrix}$  are unitarily congruent, we will show the first statement as the second one has a similar proof, from Corollary 2.2 we have:

$$\begin{pmatrix} A & X \\ X^* & 0 \end{pmatrix} \geq U \begin{pmatrix} \frac{A}{2} & 0 \\ 0 & 0 \end{pmatrix} U^* + U \begin{pmatrix} I(X) & 0 \\ 0 & 0 \end{pmatrix} U^* + V \begin{pmatrix} 0 & 0 \\ 0 & \frac{A}{2} \end{pmatrix} V^*$$

Since  $\begin{pmatrix} A & X \\ X^* & 0 \end{pmatrix}$  is congruent to  $L = \begin{pmatrix} A & lX \\ lX^* & 0 \end{pmatrix}$  for any  $l \in \mathbb{C}$ ,  $L$  is P.S.D.  $A$  is a fixed matrix, we have  $\left\| U \begin{pmatrix} \frac{A}{2} & 0 \\ 0 & 0 \end{pmatrix} U^* + V \begin{pmatrix} 0 & 0 \\ 0 & \frac{A}{2} \end{pmatrix} V^* \right\|_k = \beta \|A\|_k$  for some  $\beta \leq 1$  finally we set  $Y = lX$  where  $l \in \mathbb{R}$  is large enough to have  $\|M\|_k > \|A\|_k, \forall k$  thus  $\|M\| > \|A\|$  for all symmetric norms.

□

Notice that there exist a permutation matrix  $P$  such that  $P \begin{pmatrix} A & X \\ X^* & 0 \end{pmatrix} P^{-1} = \begin{pmatrix} 0 & X^* \\ X & A \end{pmatrix}$  and since  $I(X) > 0$  if and only if  $I(X^*) < 0$ , the two assertions of Theorem 3.3 are equivalent up to a permutation similarity.

**Corollary 3.1.** *If  $M = \begin{pmatrix} A & X \\ X^* & 0 \end{pmatrix}$ ,  $A$  a positive semi-definite matrix, and we have one of the following conditions:*

1.  $R(X) > 0$
2.  $R(X) < 0$
3.  $I(X) > 0$

4.  $I(X) < 0$

Then  $M$  can't be positive semi-definite.

*Proof.* By Remark 1 any positive matrix  $M$  written in blocks must satisfy  $\|M\| \leq \|A\| + \|B\|$  for all symmetric norms which is not the case of the matrix  $M$  constructed in Theorem 3.3.  $\square$

Finally we get:

**Theorem 3.4.** *If  $X \neq 0$  and  $B = 0$ ,  $A \geq 0$ , the matrix  $M = \begin{pmatrix} A & X \\ X^* & 0 \end{pmatrix}$  cannot be positive semi-definite.*

*Proof.* Suppose the converse, so  $M = \begin{pmatrix} A & X \\ X^* & 0 \end{pmatrix}$  is positive semi-definite, without loss of generality the only case we need to discuss is when  $R(X)$  has positive and negative eigenvalues, by Corollary 2.1 we can write:

$$M = U \begin{pmatrix} \frac{A}{2} - R(X) & 0 \\ 0 & 0 \end{pmatrix} U^* + V \begin{pmatrix} 0 & 0 \\ 0 & \frac{A}{2} + R(X) \end{pmatrix} V^*$$

for some unitaries  $U, V \in \mathbb{M}_{2n}$ .

Now if  $R(X)$  has  $-\alpha$  the smallest negative eigenvalue  $R(X) + (\alpha + \epsilon)I > 0$  consequently the matrix

$$H = U \begin{pmatrix} \frac{A}{2} - R(X) & 0 \\ 0 & 0 \end{pmatrix} U^* + V \begin{pmatrix} 0 & 0 \\ 0 & \frac{A}{2} + (\alpha + \epsilon)I + R(X) + (\alpha + \epsilon)I \end{pmatrix} V^* \quad (6)$$

$$= \begin{pmatrix} A + 2(\alpha + \epsilon)I & X + (\alpha + \epsilon)I \\ (X + (\alpha + \epsilon)I)^* & 0 \end{pmatrix} \quad (7)$$

is positive semi-definite with  $R(Y) > 0$ , where  $Y = X + (\alpha + \epsilon)I$ , by Corollary 3.1 this is a contradiction.  $\square$

A natural question would be how many are the nontrivial P.S.D.matrices written by blocks ? The following lemma will show us how to construct some of them.

**Lemma 3.1.** *Let  $A$  and  $B$  be any  $n \times n$  positive definite matrices, then there exist an integer  $t \geq 1$  such that the matrix  $F_t = \begin{pmatrix} tA & X \\ X^* & tB \end{pmatrix}$  is positive definite.*

*Proof.* Recall from Theorem 3.1 that  $F_1$  is positive definite if and only if  $A > XB^{-1}X^*$ , which is equivalent to  $x^*Ax > x^*XB^{-1}X^*x$  for all  $x \in \mathbb{C}^n$ . Set  $f(x) := x^*Ax$  and  $g(x) := x^*XB^{-1}X^*x$  and let us suppose, to the contrary, that there exist a vector  $z$  such that  $f(z) \leq g(z)$  since  $f(x)$  and  $g(x)$  are homogeneous functions of degree  $d = 2$  over  $\mathbb{R}$  if  $f(x) \geq g(x)$  for all  $x$  such that  $\|x\|_s = 1$  then  $f(x) \geq g(x)$  for any  $x \in \mathbb{C}^n$ . So let us set  $K = \max_{\|x\|_s=1} g(x)$ , and  $L = \min_{\|x\|_s=1} f(x)$  since  $g(x)$  and  $f(x)$  are continuous functions and  $\{x; \|x\|_s = 1\}$  is compact, there exist a vector  $w$  respectively  $v$  such that  $K = g(w)$ , respectively  $L = f(v)$ . Now choose  $t \geq 1$  such that  $tf(v) > \frac{g(w)}{t}$ , to obtain

$$x^*(tA)x \geq v^*(tA)v > w^*X(tB)^{-1}X^*w \geq x^*X(tB)^{-1}X^*x$$

for all  $x$  such that  $\|x\|_s = 1$ , thus  $x^*(tA)x > x^*X(tB)^{-1}X^*x$  for any  $x \in \mathbb{C}^n$  which completes the proof. □

**Theorem 3.5.** Let  $A = \text{diag}(\lambda_1, \dots, \lambda_n)$ ,  $B = \text{diag}(\nu_1, \dots, \nu_n)$  and  $M = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$  a given positive semi-definite matrix. If  $X^*$  commute with  $A$  and  $X^*X$  equals a diagonal matrix, then

$$\|M\| \leq \|A + B\|$$

for all symmetric norms. The same inequality holds if  $X$  commute with  $B$  and  $XX^*$  is diagonal.

*Proof.* It suffices to prove the inequality for the Ky Fan  $k$ -norms  $k = 1, \dots, n$ , let  $P = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$  where  $I_n$  is the identity matrix of order  $n$ , since  $\begin{pmatrix} B & X^* \\ X & A \end{pmatrix} = P \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} P^{-1}$  and  $\begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$  have same singular values, we will discuss only the first case; that is, when  $X^*$  commute with  $A$  and  $X^*X$  is diagonal, as the second case will follows. Let  $D := X^*X = \begin{pmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n \end{pmatrix}$ , as  $X^*$  commute with  $A$ , from Theorem 3.2 we conclude that the eigenvalues of  $\begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$  are the roots of

$$\det((A - \mu I_n)(B - \mu I_n) - D) = 0$$

Equivalently the eigenvalues are all the solutions of the  $n$  equations:

$$\begin{aligned}
1) \quad & (\lambda_1 - \mu)(\nu_1 - \mu) - d_1 = 0 \\
2) \quad & (\lambda_2 - \mu)(\nu_2 - \mu) - d_2 = 0 \\
3) \quad & (\lambda_3 - \mu)(\nu_3 - \mu) - d_3 = 0 \\
& \vdots \\
i) \quad & (\lambda_i - \mu)(\nu_i - \mu) - d_i = 0 \\
& \vdots \\
n) \quad & (\lambda_n - \mu)(\nu_n - \mu) - d_n = 0
\end{aligned}$$

Each equation is of  $2^{nd}$  degree, if we denote by  $a_i$  and  $b_i$  the two solutions of the  $i^{th}$  equation we deduce that:

$$\begin{aligned}
a_1 + b_1 &= \lambda_1 + \nu_1 \\
a_2 + b_2 &= \lambda_2 + \nu_2 \\
&\vdots \\
a_n + b_n &= \lambda_n + \nu_n
\end{aligned}$$

But

$$A + B = \begin{pmatrix} \lambda_1 + \nu_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 + \nu_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n + \nu_n \end{pmatrix}$$

and each diagonal entry of  $A + B$  is equal the sum of two nonnegative eigenvalues of  $M$ , thus we have necessarily:  $\|M\|_k \leq \|A + B\|_k$  for all  $k = 1, \dots, n$  which completes the proof.  $\square$

**Example 3.1.** *Let*

$$M_x = \begin{pmatrix} x & 0 & \frac{i}{2} & 0 \\ 0 & \frac{99}{100} & 0 & -\frac{i}{2} \\ -\frac{i}{2} & 0 & \frac{99}{100} & 0 \\ 0 & \frac{i}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

If  $\frac{3}{10} \leq x \leq \frac{1}{2}$ ,  $M_x$  is positive definite and we have:

$$\|M_x\| \leq \|A + B\| \tag{8}$$

for all symmetric norms, where  $A = \begin{pmatrix} x & 0 \\ 0 & \frac{99}{100} \end{pmatrix}$  and  $B = \begin{pmatrix} \frac{99}{100} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$ . If  $M_x$  is positive definite for  $x = \frac{3}{10}$  then  $M_x$  is P.D. for all  $x > \frac{3}{10}$ . The eigenvalues of  $M_{\frac{3}{10}}$  which are

the same as the singular values of  $M_{\frac{3}{10}}$  are:

$$\lambda_1 = \frac{149}{200} + \frac{\sqrt{12401}}{200} \approx 1.301 \quad (9)$$

$$\lambda_2 = \frac{129}{200} + \frac{\sqrt{14761}}{200} \approx 1.25 \quad (10)$$

$$\lambda_3 = \frac{149}{200} - \frac{\sqrt{12401}}{200} \approx 0.188 \quad (11)$$

$$\lambda_4 = \frac{129}{200} - \frac{\sqrt{14761}}{200} \approx 0.0375 \quad (12)$$

And the (8) inequality follows from Theorem 3.5.

Let us study the commutation condition in Theorem 3.5. First notice that any square matrix  $X = (x_{ij}) \in \mathbb{M}_n$  will commute with  $A = \text{diag}(a_1, \dots, a_n)$  if and only if :

$$X = \begin{pmatrix} x_{1,1}a_1 & x_{1,2}a_2 & \cdots & x_{1,n}a_n \\ x_{2,1}a_1 & x_{2,2}a_2 & \cdots & x_{2,n}a_n \\ \vdots & \vdots & \ddots & \vdots \\ x_{n,1}a_1 & x_{n,2}a_2 & \cdots & x_{n,n}a_n \end{pmatrix} = \begin{pmatrix} x_{1,1}a_1 & x_{1,2}a_1 & \cdots & x_{1,n}a_1 \\ x_{2,1}a_2 & x_{2,2}a_2 & \cdots & x_{2,n}a_2 \\ \vdots & \vdots & \ddots & \vdots \\ x_{n,1}a_n & x_{n,2}a_n & \cdots & x_{n,n}a_n \end{pmatrix} = Y$$

An  $(i, j)$  entry of  $X$  is equal to that of  $Y$  if and only if  $x_{i,j}a_j = x_{i,j}a_i$ , i.e. either  $a_i = a_j$  or  $x_{i,j} = 0$ .

**Corollary 3.2.** *Let  $A = \text{diag}(\lambda_1, \dots, \lambda_n)$ ,  $B = \text{diag}(\nu_1, \dots, \nu_n)$  and  $M = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$  a given positive semi-definite matrix. If  $X^*$  commute with  $A$ , or  $X$  commute with  $B$ , then  $\|M\| \leq \|A + B\|$  for all symmetric norms.*

*Proof.* As in Theorem 3.5, we will assume without loss of generality that  $X^*$  commute with  $A$ , as the other case is similar. If  $X^*$  is diagonal the result follows from Theorem 3.5, suppose there is an off diagonal entry  $x_{i,j}$  of  $X^*$  different from 0, from the commutation condition we have  $a_i = a_j$  and the same goes for all such entries, of course if  $AX = XA$  then

$$PAXP^{-1} = PXAP^{-1} = PAP^{-1}PXP^{-1} = PXP^{-1}PAP^{-1} = PXAP^{-1}$$

Take  $P$  to be the permutation matrix that will order the same diagonal entries of  $A$  in a one diagonal block and keeps the matrix  $B$  the same, since  $M$  is Hermitian so is  $PMP^{-1}$

because we can consider the permutation matrix as a product of transposition matrices  $P_1, \dots, P_n$  which are orthogonal; in other words

$$PMP^{-1} = P_1 P_2 \dots P_n M P_n^T \dots P_2^T P_1^T.$$

Consequently  $P^T = P^{-1}$  for any permutation matrix and  $\|M\| = \|PMP^T\|$  for all symmetric norms. If  $H = PMP^T$ ,  $D := PX$  and  $X_i$  is some  $i \times i$  extracted submatrix of  $X^*$ , we will have the block written matrix

$$H = \begin{pmatrix} PAP^T & PX \\ X^*P^T & B \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} aI_i & O_j & \dots & O_s \\ O_i & bI_j & \dots & O_s \\ \vdots & \vdots & \ddots & \vdots \\ O_i & O_j & \dots & rI_s \end{pmatrix} & \begin{pmatrix} X_i^* & O_i & \dots & O_i \\ O_j & X_j^* & \dots & O_j \\ \vdots & \vdots & \ddots & \vdots \\ O_s & O_s & \dots & X_s^* \end{pmatrix} \\ \begin{pmatrix} X_i & O_j & \dots & O_s \\ O_i & X_j & \dots & O_s \\ \vdots & \vdots & \ddots & \vdots \\ O_i & O_j & \dots & X_s \end{pmatrix} & \begin{pmatrix} \nu_1 & 0 & \dots & 0 \\ 0 & \nu_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \nu_n \end{pmatrix} \end{pmatrix}$$

where we denoted the diagonal matrix of order  $i$  whose diagonal entries are equal to  $a$  by  $aI_i$  and the zero block of order  $i$  by  $O_i$ . Let us calculate the roots of the characteristic polynomial of  $H$ ; that is, the roots of

$$\det \left( \begin{pmatrix} (a-\lambda)I_i & O_j & \dots & O_s \\ O_i & (b-\lambda)I_j & \dots & O_s \\ \vdots & \vdots & \ddots & \vdots \\ O_i & O_j & \dots & (r-\lambda)I_s \end{pmatrix} \begin{pmatrix} \nu_1-\lambda & 0 & \dots & 0 \\ 0 & \nu_2-\lambda & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \nu_n-\lambda \end{pmatrix} - D^*D \right) = 0$$

we translate this to a system of blocks, while each eigenvalue of  $H$ , which is the same as its singular value, will verify one of the following equations:

$$\begin{aligned} 1) \quad & \det \left( (a-\lambda)I_i \left( \begin{pmatrix} \nu_1-\lambda & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \nu_i-\lambda \end{pmatrix} \right) - X_i^* X_i \right) &= 0 \\ 2) \quad & \det \left( (b-\lambda)I_j \left( \begin{pmatrix} \nu_{i+1}-\lambda & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \nu_{i+j}-\lambda \end{pmatrix} \right) - X_j^* X_j \right) &= 0 \\ \vdots & & \vdots \\ c) \quad & \det \left( (r-\lambda)I_s \left( \begin{pmatrix} \nu_{n-s}-\lambda & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \nu_n-\lambda \end{pmatrix} \right) - X_s^* X_s \right) &= 0 \end{aligned} \tag{T}$$

where  $c$  is the number of diagonal blocks we have. Let us have a closer look to any of the equations above, without loss of generality we will take the first one, the same will hold for the others, notice that all eigenvalues  $\lambda$  are nonnegative and we have

$$M_1 = \begin{pmatrix} aI_i & X_i^* \\ X_i & \begin{pmatrix} \nu_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \nu_i \end{pmatrix} \end{pmatrix} = \begin{pmatrix} C_1 & X_i^* \\ X_i & K_1 \end{pmatrix}$$

is positive semi-definite because it's eigenvalues are a subset of those of  $M$ . The key idea is that for this matrix  $\|C_1 + K_1\| = \|C_1\| + \|K_1\|$  for all symmetric norms. where  $C_1 = aI_i$  and  $K = \begin{pmatrix} \nu_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \nu_i \end{pmatrix}$ . Now back to the system  $(T)$  we associate like we did to  $M_1$  each equation whose number is  $i$  to a positive semi-definite matrix  $M_i$  to obtain by Remark 1

$$\begin{aligned} \|M_1\|_k &\leq \left\| aI_i + \begin{pmatrix} \nu_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \nu_i \end{pmatrix} \right\|_k = \|aI_i\|_k + \left\| \begin{pmatrix} \nu_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \nu_i \end{pmatrix} \right\|_k \\ \|M_2\|_k &\leq \left\| bI_j + \begin{pmatrix} \nu_{i+1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \nu_{i+j} \end{pmatrix} \right\|_k = \|bI_j\|_k + \left\| \begin{pmatrix} \nu_{i+1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \nu_{i+j} \end{pmatrix} \right\|_k \\ &\vdots \\ \|M_c\|_k &\leq \left\| rI_s + \begin{pmatrix} \nu_{n-s} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \nu_n \end{pmatrix} \right\|_k = \|rI_s\|_k + \left\| \begin{pmatrix} \nu_{n-s} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \nu_n \end{pmatrix} \right\|_k \end{aligned}$$

for all  $k$ , but the order of the entries of  $B$  are arbitrary chosen, thus from Theorem 3.5  $\|M\|_k \leq \|A + B\|_k$  for all  $k = 1, \dots, n$  and that completes the proof.

□

**Corollary 3.3.** Let  $M = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$  be a positive semi-definite matrix written by blocks. There exist a unitary  $V$  and a unitary  $U$  such that

$$\left\| \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \right\| \leq \|UAU^* + VBV^*\| := \|A\| + \|B\|$$

for all symmetric norms.

*Proof.* Let  $U$  and  $V$  be two unitary matrix such that  $UAU^* = D_o$  and  $VBV^* = G_o$  where  $D_o$  and  $G_o$  are two diagonal matrices having the same ordering  $o$ , of eigenvalues with respect to their indexes i.e., if  $\lambda_n \leq \dots \leq \lambda_1$  are the diagonal entries of  $D_o$ , and  $\nu_n \leq \dots \leq \nu_1$  are those of  $G_o$ , then if  $\lambda_i$  is in the  $(j, j)$  position then  $\nu_i$  will be also. Consequently  $\|UAU^* + VBV^*\| = \|D_o + G_o\| = \|D_o\| + \|G_o\| = \|A\| + \|B\|$ , for all the Ky-Fan  $k$ -norms and thus for all symmetric norms. To complete the proof notice that if  $T = UXV^*$  and  $Q$  is the unitary matrix  $\begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}$ , by Remark 1

$$\left\| \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \right\| = \left\| Q \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} Q^* \right\| = \left\| \begin{pmatrix} D_o & T \\ T^* & G_o \end{pmatrix} \right\| \leq \|D_o\| + \|G_o\| \quad (13)$$

for all symmetric norms.

□

**Lemma 3.2.** *Let*

$$N = \begin{pmatrix} \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix} & D \\ D^* & \begin{pmatrix} b_1 & 0 & \cdots & 0 \\ 0 & b_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_n \end{pmatrix} \end{pmatrix}$$

where  $a_1, \dots, a_n$  respectively  $b_1, \dots, b_n$  are nonnegative respectively negative real numbers,  $A = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix}$ ,  $B = \begin{pmatrix} b_1 & 0 & \cdots & 0 \\ 0 & b_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_n \end{pmatrix}$  and  $D$  is any diagonal matrix, then nor  $N$  neither  $-N$  is positive semi-definite. Set  $(d_1, \dots, d_n)$  as the diagonal entries of  $D^*D$ , if  $a_i + b_i \geq 0$  and  $a_i b_i - d_i < 0$  for all  $i \leq n$ , then  $\|N\| > \|A + B\|$ . for all symmetric norms

*Proof.* The diagonal of  $N$  has negative and positive numbers, thus nor  $N$  neither  $-N$  is positive semi-definite, now any two diagonal matrices will commute, in particular  $D^*$  and  $A$ , by applying Theorem 3.2 we get that the eigenvalues of  $N$  are the roots of

$$\det((A - \mu I_n)(B - \mu I_n) - D^*D) = 0$$

Equivalently the eigenvalues are all the solutions of the  $n$  equations:

$$\begin{aligned} 1) \quad & (a_1 - \mu)(b_1 - \mu) - d_1 = 0 \\ 2) \quad & (a_2 - \mu)(b_2 - \mu) - d_2 = 0 \\ 3) \quad & (a_3 - \mu)(b_3 - \mu) - d_3 = 0 \\ \vdots & \vdots \\ i) \quad & (a_i - \mu)(b_i - \mu) - d_n = 0 \\ \vdots & \vdots \\ n) \quad & (a_n - \mu)(b_n - \mu) - d_n = 0 \end{aligned} \quad (S)$$

Let us denote by  $x_i$  and  $y_i$  the two solutions of the  $i^{th}$  equation then:

$$\begin{aligned} x_1 + y_1 &= a_1 + b_1 \geq 0 & x_1 y_1 &= a_1 b_1 - d_1 < 0 \\ x_2 + y_2 &= a_2 + b_2 \geq 0 & x_2 y_2 &= a_2 b_2 - d_2 < 0 \\ \vdots & \vdots & \vdots & \vdots \\ x_n + y_n &= a_n + b_n \geq 0 & x_n y_n &= a_n b_n - d_n < 0 \end{aligned}$$

This implies that each equation of  $(S)$  has one negative and one positive solution, their sum is positive, thus the positive root is bigger or equal than the negative one. Since  $A + B = \begin{pmatrix} a_1 + b_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_n + b_n \end{pmatrix}$ , summing over indexes we see that  $\|N\|_k > \|A + B\|_k$  for  $k = 1, \dots, n$  which yields to  $\|N\| > \|A + B\|$  for all symmetric norms  $\square$

It seems easy to construct examples of non P.S.D matrices  $N$ , such that  $\|N\|_s > \|A + B\|_s$ , let us have a look of such inequality for P.S.D. matrices.

**Example 3.2.** *Let*

$$C = \begin{pmatrix} \frac{4}{3} & 0 & 1 & -1 \\ 0 & 1 & 0 & \frac{1}{5} \\ 1 & 0 & \frac{3}{2} & 0 \\ -1 & \frac{1}{5} & 0 & 2 \end{pmatrix} = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$$

where  $A = \begin{pmatrix} \frac{4}{3} & 0 \\ 0 & 1 \end{pmatrix}$ ,  $B = \begin{pmatrix} \frac{3}{2} & 0 \\ 0 & 2 \end{pmatrix}$ . Since the eigenvalues of  $C$  are all positive with  $\lambda_1 \approx 3.008$ ,  $\lambda_2 \approx 1.7$ ,  $\lambda_3 \approx 0.9$ ,  $\lambda_4 \approx 0.089$ ,  $C$  is positive definite and we verify that

$$3.008 \approx \|C\|_s > \|A + B\|_s = 3$$

**Example 3.3.** *Let*

$$N_y = \begin{pmatrix} 2 & 0 & 0 & 2 \\ 0 & y & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$$

where  $A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$  and  $B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ . The eigenvalues of  $N_y$  are the numbers:  $\lambda_1 = 4$ ,  $\lambda_2 = 1$ ,  $\lambda_3 = y$ ,  $\lambda_4 = 0$ , thus if  $y \geq 0$ ,  $N_y$  is positive semi-definite and for all  $y$  such that  $0 \leq y < 1$  we have

$$1. \quad 4 = \|N_y\|_s > \|A + B\|_s = 3$$

$$2. \quad 16 + y^2 + 1 = \|N\|_{(2)}^2 > \|A + B\|_{(2)}^2 = 4(3 + y) + y^2 + 1$$

## References

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