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Norm Inequalities of Positive Semi-Definite Matrices

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Abstract

For positive block-matrix M we write $M = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \in \mathbb{M}_{n+m}^+$, with $A \in \mathbb{M}_n^+$, $B \in \mathbb{M}_m^+$. The main result is first to study the consequences of a decomposition lemma due to C. Bourrin and second to extend the class of these P.S.D. matrices M written by blocks that satisfies the inequality: $\|M\| \leq \|A + B\|$ for all symmetric norms and to give examples whenever it is necessary.

Keywords : Matrix Analysis, Hermitian matrices, symmetric norms.

1 Introduction

Let A be an $n \times n$ matrix and F an $m \times m$ matrix, ($m > n$) written by blocks such that A is a diagonal block and all entries other than those of A are zeros, then the two matrices have the same singular values and for all unitarily invariant norms $\|A\| = \|F\| = \|A \oplus 0\|$, we say then that the symmetric norm on \mathbb{M}_m induces a symmetric norm on \mathbb{M}_n , so for square matrices we may assume that our norms are defined on all spaces \mathbb{M}_n , $n \geq 1$. The spectral norm is denoted by $\|\cdot\|_s$, the Frobenius norm by $\|\cdot\|_{(2)}$, and the Ky Fan k -norms by $\|\cdot\|_k$. Let \mathbb{M}_n^+ denote the set of positive and semi-definite part of the space of $n \times n$ complex matrices and M be any positive block-matrices; that is, $M = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \in \mathbb{M}_{n+m}^+$, with $A \in \mathbb{M}_n^+$, $B \in \mathbb{M}_m^+$.

2 Decomposition of block-matrices

Lemma 2.1. For every matrix M in \mathbb{M}_{n+m}^+ written in blocks, we have the decomposition:

$$\begin{pmatrix} A & X \\ X^* & B \end{pmatrix} = U \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} U^* + V \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix} V^*$$

for some unitaries $U, V \in \mathbb{M}_{n+m}$.

Proof. Factorize the positive matrix as a square of positive matrices:

$$\begin{pmatrix} A & X \\ X^* & B \end{pmatrix} = \begin{pmatrix} C & Y \\ Y^* & D \end{pmatrix} \cdot \begin{pmatrix} C & Y \\ Y^* & D \end{pmatrix}$$

we verify that the right hand side can be written as $T^*T + S^*S$ so :

$$\begin{pmatrix} C & Y \\ Y^* & D \end{pmatrix} \cdot \begin{pmatrix} C & Y \\ Y^* & D \end{pmatrix} = \underbrace{\begin{pmatrix} C & 0 \\ Y^* & 0 \end{pmatrix}}_{T^*} \cdot \underbrace{\begin{pmatrix} C & Y \\ 0 & 0 \end{pmatrix}}_T + \underbrace{\begin{pmatrix} 0 & Y \\ 0 & D \end{pmatrix}}_{S^*} \cdot \underbrace{\begin{pmatrix} 0 & 0 \\ Y^* & D \end{pmatrix}}_S.$$

Since $TT^* = \begin{pmatrix} CC + YY^* & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$, $SS^* = \begin{pmatrix} 0 & 0 \\ 0 & Y^*Y + DD \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}$ and AA^* is unitarily congruent to A^*A for any square matrix A , the lemma follows. \square

Remark 1. As a consequence of this lemma we have:

$$\|M\| \leq \|A\| + \|B\|$$

for all symmetric norms.

Equations involving unitary matrices are called unitary orbits representations.

Recall that if $A \in \mathbb{M}_n$, $R(A) = \frac{A + A^*}{2}$ and $I(A) = \frac{A - A^*}{2i}$.

Corollary 2.1. For every matrix in \mathbb{M}_{2n}^+ written in blocks of the same size, we have the decomposition:

$$\begin{pmatrix} A & X \\ X^* & B \end{pmatrix} = U \begin{pmatrix} \frac{A+B}{2} - R(X) & 0 \\ 0 & 0 \end{pmatrix} U^* + V \begin{pmatrix} 0 & 0 \\ 0 & \frac{A+B}{2} + R(X) \end{pmatrix} V^*$$

for some unitaries $U, V \in \mathbb{M}_{2n}$.

Proof. Let $J = \frac{1}{\sqrt{2}} \begin{pmatrix} I & -I \\ I & I \end{pmatrix}$ where I is the identity of \mathbb{M}_n , J is a unitary matrix, and we have:

$$J \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} J^* = \underbrace{\begin{pmatrix} \frac{A+B}{2} - R(X) & \frac{A-B}{2} + \frac{X^*-X}{2} \\ \frac{A-B}{2} - \frac{X-X^*}{2} & \frac{A+B}{2} + R(X) \end{pmatrix}}_N$$

Now we factorize N as a square of positive matrices:

$$\begin{pmatrix} A & X \\ X^* & B \end{pmatrix} = J^* \begin{pmatrix} L & M \\ M^* & F \end{pmatrix} \cdot \begin{pmatrix} L & M \\ M^* & F \end{pmatrix} J$$

and let:

$$\begin{aligned} \delta &= J^* \begin{pmatrix} L & M \\ M^* & F \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} L + M^* & M + F \\ M^* - L & F - M \end{pmatrix} \\ \psi &= \begin{pmatrix} L & M \\ M^* & F \end{pmatrix} J = \frac{1}{\sqrt{2}} \begin{pmatrix} L + M & M - L \\ F + M^* & F - M^* \end{pmatrix} \end{aligned}$$

A direct computation shows that:

$$\begin{aligned} \delta \cdot \psi &= \frac{1}{2} \begin{pmatrix} (L+M^*)(L+M)+(M+F)(F+M^*) & (L+M^*)(M-L)+(M+F)(F-M^*) \\ (M^*-L)(L+M)+(F-M)(F+M^*) & (M^*-L)(M-L)+(F-M)(F-M^*) \end{pmatrix} \\ &= \Gamma^* \Gamma + \Phi^* \Phi \end{aligned} \tag{1}$$

where: $\Gamma = \frac{1}{\sqrt{2}} \begin{pmatrix} L + M & M - L \\ 0 & 0 \end{pmatrix}$, and $\Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 \\ F + M^* & F - M^* \end{pmatrix}$ to finish notice that for any square matrix A , A^*A is unitarily congruent to AA^* and, $\Gamma\Gamma^*$, $\Phi\Phi^*$ have the required form. \square

The previous corollary implies that $\frac{A+B}{2} \geq R(X)$ and $\frac{A+B}{2} \geq -R(X)$.

Corollary 2.2. *For every matrix in \mathbb{M}_{2n}^+ written in blocks of the same size, we have the decomposition:*

$$\begin{pmatrix} A & X \\ X^* & B \end{pmatrix} = U \begin{pmatrix} \frac{A+B}{2} + I(X) & 0 \\ 0 & 0 \end{pmatrix} U^* + V \begin{pmatrix} 0 & 0 \\ 0 & \frac{A+B}{2} - I(X) \end{pmatrix} V^*$$

for some unitaries $U, V \in \mathbb{M}_{n+m}$.

Proof. The proof is similar to Corollary 2.1, we have: $J_1 \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} J_1^* = \begin{pmatrix} A & iX \\ -iX^* & B \end{pmatrix}$

where $J_1 = \begin{pmatrix} I & 0 \\ 0 & -iI \end{pmatrix}$, and

$$K = J J_1 \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} J_1^* J^* = \begin{pmatrix} \frac{A+B}{2} + I(X) & * \\ * & \frac{A+B}{2} - I(X) \end{pmatrix}$$

here (*) means an unspecified entry, the proof is similar to that in Corollary 2.1 but for reader's convenience we give the main headlines: first factorize K as a square of positive matrices; that is, $M = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} = J_1^* J^* L^2 J J_1$ next decompose L^2 as in Lemma 2.1 to obtain

$$M = J_1^* J^* (T^* T + S^* S) J J_1 = J_1^* J^* (T^* T) J J_1 + J_1^* J^* (S^* S) J J_1$$

where $TT^* = \begin{pmatrix} \frac{A+B}{2} + I(X) & 0 \\ 0 & 0 \end{pmatrix}$ and $SS^* = \begin{pmatrix} 0 & 0 \\ 0 & \frac{A+B}{2} - I(X) \end{pmatrix}$ finally the congruence property completes the proof. \square

The existence of unitaries U and V in the decomposition process need not to be unique as one can take the special case; that is, M any diagonal matrix with diagonal entries equals a nonnegative number k , explicitly $M = kI = U \begin{pmatrix} k \\ I \end{pmatrix} U^* + V \begin{pmatrix} k \\ I \end{pmatrix} V^*$ for any U and V unitaries.

Remark 2. Notice that from the Courant-Fischer theorem if $A, B \in \mathbb{M}_n^+$, then the eigenvalues of each matrix are the same as the singular values and $A \leq B \implies \|A\|_k \leq \|B\|_k$, for all $k = 1, \dots, n$, also $A < B \implies \|A\|_k < \|B\|_k$, for all $k = 1, \dots, n$.

Corollary 2.3. For every matrix in \mathbb{M}_{2n}^+ written in blocks of the same size, we have:

$$\begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \leq \frac{1}{2} \left\{ U \begin{pmatrix} A + B + |X - X^*| & 0 \\ 0 & 0 \end{pmatrix} U^* + V \begin{pmatrix} 0 & 0 \\ 0 & A + B + |X - X^*| \end{pmatrix} V^* \right\}$$

for some unitaries $U, V \in \mathbb{M}_{n+m}$.

Proof. This a consequence of the fact that $I(X) \leq |I(X)|$. \square

3 Symmetric Norms and Inequalities

In [1] they found that if X is hermitian then

$$\|M\| \leq \|A + B\| \tag{2}$$

for all symmetric norms. It has been given counter-examples showing that this does not necessarily holds if X is a normal but not Hermitian matrix, the main idea of this section is to give examples and counter-examples in a general way and to extend the previous inequality to a larger class of P.S.D. matrices written by blocks satisfying (2).

Theorem 3.1. If A and B are positive definite matrices of same size. Then

$$\begin{pmatrix} A & X \\ X^* & B \end{pmatrix} > 0 \iff A \geq XB^{-1}X^*$$

Proof. Write $\begin{pmatrix} A & X \\ X^* & B \end{pmatrix} = \begin{pmatrix} I & XB^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} A - XB^{-1}X^* & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} I & 0 \\ XB^{-1} & I \end{pmatrix}$ where I is the identity matrix, and that complete the proof since for any matrix A ,

$$A \geq 0 \iff X^*AX \geq 0, \forall X.$$

□

Theorem 3.2. Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be any square matrix written by blocks of same size, if $AC = CA$ then $\det(M) = \det(AD - CB)$

Proof. Suppose first that A is invertible, let us write M as

$$M = \begin{pmatrix} Z & 0 \\ V & I \end{pmatrix} \begin{pmatrix} I & E \\ 0 & F \end{pmatrix} \quad (3)$$

upon calculation we find that: $Z = A$, $V = C$, $E = A^{-1}B$, $F = D - CA^{-1}B$ taking the determinant on each side of (3) we get:

$$\det(M) = \det(A(D - CA^{-1}B)) = \det(AD - CB)$$

the result follows by a continuity argument since the Determinant function is a continuous function. □

Given the matrix $M = \begin{pmatrix} A & X \\ X^* & 0 \end{pmatrix}$ a matrix in \mathbb{M}_{2n}^+ written by blocks of same size, we know that it M is not P.S.D., to see this notice that all the 2×2 extracted principle submatrices of M are P.S.D if and only if $X = 0$ and A is positive semi-definite. Even if a proof of this exists and would take two lines, it is quite nice to see a different constructive proof, a direct consequence of Lemma 2.1.

Theorem 3.3. Given $\begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$ a matrix in \mathbb{M}_{2n}^+ written in blocks of same size:

1. If $\begin{pmatrix} A & X \\ X^* & 0 \end{pmatrix}$ is positive semi-definite, $I(X) > 0$ or $I(X) < 0$, then there exist a matrix Y such that $M = \begin{pmatrix} A & Y \\ Y^* & 0 \end{pmatrix}$ is positive semi-definite and:

$$\left\| \begin{pmatrix} A & Y \\ Y^* & 0 \end{pmatrix} \right\| > \|A\| \quad (4)$$

for all symmetric norms.

2. If $\begin{pmatrix} 0 & X \\ X^* & B \end{pmatrix}$ is positive semi-definite, $I(X) > 0$ or $I(X) < 0$ then there exist a matrix Y such that $M = \begin{pmatrix} 0 & Y \\ Y^* & B \end{pmatrix}$ is positive semi-definite and:

$$\left\| \begin{pmatrix} 0 & Y \\ Y^* & B \end{pmatrix} \right\| > \|B\| \quad (5)$$

The same result holds if we replaced $I(X)$ by $R(X)$ because $\begin{pmatrix} A & iX \\ -iX^* & B \end{pmatrix}$ is unitarily congruent to $\begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$.

Proof. Without loss of generality we can consider $I(X) > 0$ cause $\begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$ and $\begin{pmatrix} A & -X \\ -X^* & B \end{pmatrix}$ are unitarily congruent, we will show the first statement as the second one has a similar proof, from Corollary 2.2 we have:

$$\begin{pmatrix} A & X \\ X^* & 0 \end{pmatrix} \geq U \begin{pmatrix} \frac{A}{2} & 0 \\ 0 & 0 \end{pmatrix} U^* + U \begin{pmatrix} I(X) & 0 \\ 0 & 0 \end{pmatrix} U^* + V \begin{pmatrix} 0 & 0 \\ 0 & \frac{A}{2} \end{pmatrix} V^*$$

Since $\begin{pmatrix} A & X \\ X^* & 0 \end{pmatrix}$ is congruent to $L = \begin{pmatrix} A & lX \\ lX^* & 0 \end{pmatrix}$ for any $l \in \mathbb{C}$, L is P.S.D. A is a fixed matrix, we have $\left\| U \begin{pmatrix} \frac{A}{2} & 0 \\ 0 & 0 \end{pmatrix} U^* + V \begin{pmatrix} 0 & 0 \\ 0 & \frac{A}{2} \end{pmatrix} V^* \right\|_k = \beta \|A\|_k$ for some $\beta \leq 1$ finally we set $Y = lX$ where $l \in \mathbb{R}$ is large enough to have $\|M\|_k > \|A\|_k, \forall k$ thus $\|M\| > \|A\|$ for all symmetric norms.

□

Notice that there exist a permutation matrix P such that $P \begin{pmatrix} A & X \\ X^* & 0 \end{pmatrix} P^{-1} = \begin{pmatrix} 0 & X^* \\ X & A \end{pmatrix}$ and since $I(X) > 0$ if and only if $I(X^*) < 0$, the two assertions of Theorem 3.3 are equivalent up to a permutation similarity.

Corollary 3.1. *If $M = \begin{pmatrix} A & X \\ X^* & 0 \end{pmatrix}$, A a positive semi-definite matrix, and we have one of the following conditions:*

1. $R(X) > 0$
2. $R(X) < 0$
3. $I(X) > 0$

4. $I(X) < 0$

Then M can't be positive semi-definite.

Proof. By Remark 1 any positive matrix M written in blocks must satisfy $\|M\| \leq \|A\| + \|B\|$ for all symmetric norms which is not the case of the matrix M constructed in Theorem 3.3. \square

Finally we get:

Theorem 3.4. *If $X \neq 0$ and $B = 0$, $A \geq 0$, the matrix $M = \begin{pmatrix} A & X \\ X^* & 0 \end{pmatrix}$ cannot be positive semi-definite.*

Proof. Suppose the converse, so $M = \begin{pmatrix} A & X \\ X^* & 0 \end{pmatrix}$ is positive semi-definite, without loss of generality the only case we need to discuss is when $R(X)$ has positive and negative eigenvalues, by Corollary 2.1 we can write:

$$M = U \begin{pmatrix} \frac{A}{2} - R(X) & 0 \\ 0 & 0 \end{pmatrix} U^* + V \begin{pmatrix} 0 & 0 \\ 0 & \frac{A}{2} + R(X) \end{pmatrix} V^*$$

for some unitaries $U, V \in \mathbb{M}_{2n}$.

Now if $R(X)$ has $-\alpha$ the smallest negative eigenvalue $R(X) + (\alpha + \epsilon)I > 0$ consequently the matrix

$$H = U \begin{pmatrix} \frac{A}{2} - R(X) & 0 \\ 0 & 0 \end{pmatrix} U^* + V \begin{pmatrix} 0 & 0 \\ 0 & \frac{A}{2} + (\alpha + \epsilon)I + R(X) + (\alpha + \epsilon)I \end{pmatrix} V^* \quad (6)$$

$$= \begin{pmatrix} A + 2(\alpha + \epsilon)I & X + (\alpha + \epsilon)I \\ (X + (\alpha + \epsilon)I)^* & 0 \end{pmatrix} \quad (7)$$

is positive semi-definite with $R(Y) > 0$, where $Y = X + (\alpha + \epsilon)I$, by Corollary 3.1 this is a contradiction. \square

A natural question would be how many are the nontrivial P.S.D.matrices written by blocks ? The following lemma will show us how to construct some of them.

Lemma 3.1. *Let A and B be any $n \times n$ positive definite matrices, then there exist an integer $t \geq 1$ such that the matrix $F_t = \begin{pmatrix} tA & X \\ X^* & tB \end{pmatrix}$ is positive definite.*

Proof. Recall from Theorem 3.1 that F_1 is positive definite if and only if $A > XB^{-1}X^*$, which is equivalent to $x^*Ax > x^*XB^{-1}X^*x$ for all $x \in \mathbb{C}^n$. Set $f(x) := x^*Ax$ and $g(x) := x^*XB^{-1}X^*x$ and let us suppose, to the contrary, that there exist a vector z such that $f(z) \leq g(z)$ since $f(x)$ and $g(x)$ are homogeneous functions of degree $d = 2$ over \mathbb{R} if $f(x) \geq g(x)$ for all x such that $\|x\|_s = 1$ then $f(x) \geq g(x)$ for any $x \in \mathbb{C}^n$. So let us set $K = \max_{\|x\|_s=1} g(x)$, and $L = \min_{\|x\|_s=1} f(x)$ since $g(x)$ and $f(x)$ are continuous functions and $\{x; \|x\|_s = 1\}$ is compact, there exist a vector w respectively v such that $K = g(w)$, respectively $L = f(v)$. Now choose $t \geq 1$ such that $tf(v) > \frac{g(w)}{t}$, to obtain

$$x^*(tA)x \geq v^*(tA)v > w^*X(tB)^{-1}X^*w \geq x^*X(tB)^{-1}X^*x$$

for all x such that $\|x\|_s = 1$, thus $x^*(tA)x > x^*X(tB)^{-1}X^*x$ for any $x \in \mathbb{C}^n$ which completes the proof. □

Theorem 3.5. *Let $A = \text{diag}(\lambda_1, \dots, \lambda_n)$, $B = \text{diag}(\nu_1, \dots, \nu_n)$ and $M = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$ a given positive semi-definite matrix. If X^* commute with A and X^*X equals a diagonal matrix, then*

$$\|M\| \leq \|A + B\|$$

for all symmetric norms. The same inequality holds if X commute with B and XX^ is diagonal.*

Proof. It suffices to prove the inequality for the Ky Fan k -norms $k = 1, \dots, n$, let $P = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$ where I_n is the identity matrix of order n , since $\begin{pmatrix} B & X^* \\ X & A \end{pmatrix} = P \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} P^{-1}$ and $\begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$ have same singular values, we will discuss only the first case; that is, when X^* commute with A and X^*X is diagonal, as the second case will follows. Let $D := X^*X = \begin{pmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n \end{pmatrix}$, as X^* commute with A , from Theorem 3.2 we conclude that the eigenvalues of $\begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$ are the roots of

$$\det((A - \mu I_n)(B - \mu I_n) - D) = 0$$

Equivalently the eigenvalues are all the solutions of the n equations:

$$\begin{aligned}
 1) \quad & (\lambda_1 - \mu)(\nu_1 - \mu) - d_1 = 0 \\
 2) \quad & (\lambda_2 - \mu)(\nu_2 - \mu) - d_2 = 0 \\
 3) \quad & (\lambda_3 - \mu)(\nu_3 - \mu) - d_3 = 0 \\
 & \vdots \\
 i) \quad & (\lambda_i - \mu)(\nu_i - \mu) - d_i = 0 \\
 & \vdots \\
 n) \quad & (\lambda_n - \mu)(\nu_n - \mu) - d_n = 0
 \end{aligned}$$

Each equation is of 2^{nd} degree, if we denote by a_i and b_i the two solutions of the i^{th} equation we deduce that:

$$\begin{aligned}
 a_1 + b_1 &= \lambda_1 + \nu_1 \\
 a_2 + b_2 &= \lambda_2 + \nu_2 \\
 &\vdots \\
 a_n + b_n &= \lambda_n + \nu_n
 \end{aligned}$$

But

$$A + B = \begin{pmatrix} \lambda_1 + \nu_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 + \nu_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n + \nu_n \end{pmatrix}$$

and each diagonal entry of $A + B$ is equal the sum of two nonnegative eigenvalues of M , thus we have necessarily: $\|M\|_k \leq \|A + B\|_k$ for all $k = 1, \dots, n$ which completes the proof. \square

Example 3.1. *Let*

$$M_x = \begin{pmatrix} x & 0 & \frac{i}{2} & 0 \\ 0 & \frac{99}{100} & 0 & -\frac{i}{2} \\ -\frac{i}{2} & 0 & \frac{99}{100} & 0 \\ 0 & \frac{i}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

If $\frac{3}{10} \leq x \leq \frac{1}{2}$, M_x is positive definite and we have:

$$\|M_x\| \leq \|A + B\| \tag{8}$$

for all symmetric norms, where $A = \begin{pmatrix} x & 0 \\ 0 & \frac{99}{100} \end{pmatrix}$ and $B = \begin{pmatrix} \frac{99}{100} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$. If M_x is positive definite for $x = \frac{3}{10}$ then M_x is P.D. for all $x > \frac{3}{10}$. The eigenvalues of $M_{\frac{3}{10}}$ which are

the same as the singular values of $M_{\frac{3}{10}}$ are:

$$\lambda_1 = \frac{149}{200} + \frac{\sqrt{12401}}{200} \approx 1.301 \quad (9)$$

$$\lambda_2 = \frac{129}{200} + \frac{\sqrt{14761}}{200} \approx 1.25 \quad (10)$$

$$\lambda_3 = \frac{149}{200} - \frac{\sqrt{12401}}{200} \approx 0.188 \quad (11)$$

$$\lambda_4 = \frac{129}{200} - \frac{\sqrt{14761}}{200} \approx 0.0375 \quad (12)$$

And the (8) inequality follows from Theorem 3.5.

Let us study the commutation condition in Theorem 3.5. First notice that any square matrix $X = (x_{ij}) \in \mathbb{M}_n$ will commute with $A = \text{diag}(a_1, \dots, a_n)$ if and only if :

$$X = \begin{pmatrix} x_{1,1}a_1 & x_{1,2}a_2 & \cdots & x_{1,n}a_n \\ x_{2,1}a_1 & x_{2,2}a_2 & \cdots & x_{2,n}a_n \\ \vdots & \vdots & \ddots & \vdots \\ x_{n,1}a_1 & x_{n,2}a_2 & \cdots & x_{n,n}a_n \end{pmatrix} = \begin{pmatrix} x_{1,1}a_1 & x_{1,2}a_1 & \cdots & x_{1,n}a_1 \\ x_{2,1}a_2 & x_{2,2}a_2 & \cdots & x_{2,n}a_2 \\ \vdots & \vdots & \ddots & \vdots \\ x_{n,1}a_n & x_{n,2}a_n & \cdots & x_{n,n}a_n \end{pmatrix} = Y$$

An (i, j) entry of X is equal to that of Y if and only if $x_{i,j}a_j = x_{i,j}a_i$, i.e. either $a_i = a_j$ or $x_{i,j} = 0$.

Corollary 3.2. *Let $A = \text{diag}(\lambda_1, \dots, \lambda_n)$, $B = \text{diag}(\nu_1, \dots, \nu_n)$ and $M = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$ a given positive semi-definite matrix. If X^* commute with A , or X commute with B , then $\|M\| \leq \|A + B\|$ for all symmetric norms.*

Proof. As in Theorem 3.5, we will assume without loss of generality that X^* commute with A , as the other case is similar. If X^* is diagonal the result follows from Theorem 3.5, suppose there is an off diagonal entry $x_{i,j}$ of X^* different from 0, from the commutation condition we have $a_i = a_j$ and the same goes for all such entries, of course if $AX = XA$ then

$$PAXP^{-1} = PXAP^{-1} = PAP^{-1}PXP^{-1} = PXP^{-1}PAP^{-1} = PXAP^{-1}$$

Take P to be the permutation matrix that will order the same diagonal entries of A in a one diagonal block and keeps the matrix B the same, since M is Hermitian so is PMP^{-1}

because we can consider the permutation matrix as a product of transposition matrices P_1, \dots, P_n which are orthogonal; in other words

$$PMP^{-1} = P_1 P_2 \dots P_n M P_n^T \dots P_2^T P_1^T.$$

Consequently $P^T = P^{-1}$ for any permutation matrix and $\|M\| = \|PMP^T\|$ for all symmetric norms. If $H = PMP^T$, $D := PX$ and X_i is some $i \times i$ extracted submatrix of X^* , we will have the block written matrix

$$H = \begin{pmatrix} PAP^T & PX \\ X^*P^T & B \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} aI_i & O_j & \dots & O_s \\ O_i & bI_j & \dots & O_s \\ \vdots & \vdots & \ddots & \vdots \\ O_i & O_j & \dots & rI_s \end{pmatrix} & \begin{pmatrix} X_i^* & O_i & \dots & O_i \\ O_j & X_j^* & \dots & O_j \\ \vdots & \vdots & \ddots & \vdots \\ O_s & O_s & \dots & X_s^* \end{pmatrix} \\ \begin{pmatrix} X_i & O_j & \dots & O_s \\ O_i & X_j & \dots & O_s \\ \vdots & \vdots & \ddots & \vdots \\ O_i & O_j & \dots & X_s \end{pmatrix} & \begin{pmatrix} \nu_1 & 0 & \dots & 0 \\ 0 & \nu_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \nu_n \end{pmatrix} \end{pmatrix}$$

where we denoted the diagonal matrix of order i whose diagonal entries are equal to a by aI_i and the zero block of order i by O_i . Let us calculate the roots of the characteristic polynomial of H ; that is, the roots of

$$\det \left(\begin{pmatrix} (a-\lambda)I_i & O_j & \dots & O_s \\ O_i & (b-\lambda)I_j & \dots & O_s \\ \vdots & \vdots & \ddots & \vdots \\ O_i & O_j & \dots & (r-\lambda)I_s \end{pmatrix} \begin{pmatrix} \nu_1-\lambda & 0 & \dots & 0 \\ 0 & \nu_2-\lambda & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \nu_n-\lambda \end{pmatrix} - D^*D \right) = 0$$

we translate this to a system of blocks, while each eigenvalue of H , which is the same as its singular value, will verify one of the following equations:

$$\begin{aligned} 1) \quad & \det \left((a-\lambda)I_i \left(\begin{pmatrix} \nu_1-\lambda & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \nu_i-\lambda \end{pmatrix} \right) - X_i^*X_i \right) = 0 \\ 2) \quad & \det \left((b-\lambda)I_j \left(\begin{pmatrix} \nu_{i+1}-\lambda & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \nu_{i+j}-\lambda \end{pmatrix} \right) - X_j^*X_j \right) = 0 \\ & \vdots \\ c) \quad & \det \left((r-\lambda)I_s \left(\begin{pmatrix} \nu_{n-s}-\lambda & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \nu_n-\lambda \end{pmatrix} \right) - X_s^*X_s \right) = 0 \end{aligned} \tag{T}$$

where c is the number of diagonal blocks we have. Let us have a closer look to any of the equations above, without loss of generality we will take the first one, the same will hold for the others, notice that all eigenvalues λ are nonnegative and we have

$$M_1 = \begin{pmatrix} aI_i & X_i^* \\ X_i & \begin{pmatrix} \nu_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \nu_i \end{pmatrix} \end{pmatrix} = \begin{pmatrix} C_1 & X_i^* \\ X_i^* & K_1 \end{pmatrix}$$

is positive semi-definite because it's eigenvalues are a subset of those of M . The key idea is that for this matrix $\|C_1 + K_1\| = \|C_1\| + \|K_1\|$ for all symmetric norms. where $C_1 = aI_i$ and $K = \begin{pmatrix} \nu_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \nu_i \end{pmatrix}$. Now back to the system (T) we associate like we did to M_1 each equation whose number is i to a positive semi-definite matrix M_i to obtain by

Remark 1

$$\begin{aligned} \|M_1\|_k &\leq \left\| aI_i + \begin{pmatrix} \nu_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \nu_i \end{pmatrix} \right\|_k = \|aI_i\|_k + \left\| \begin{pmatrix} \nu_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \nu_i \end{pmatrix} \right\|_k \\ \|M_2\|_k &\leq \left\| bI_j + \begin{pmatrix} \nu_{i+1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \nu_{i+j} \end{pmatrix} \right\|_k = \|bI_j\|_k + \left\| \begin{pmatrix} \nu_{i+1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \nu_{i+j} \end{pmatrix} \right\|_k \\ &\vdots \\ \|M_c\|_k &\leq \left\| rI_s + \begin{pmatrix} \nu_{n-s} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \nu_n \end{pmatrix} \right\|_k = \|rI_s\|_k + \left\| \begin{pmatrix} \nu_{n-s} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \nu_n \end{pmatrix} \right\|_k \end{aligned}$$

for all k , but the order of the entries of B are arbitrary chosen, thus from Theorem 3.5

$\|M\|_k \leq \|A + B\|_k$ for all $k = 1, \dots, n$ and that completes the proof.

□

Corollary 3.3. Let $M = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$ be a positive semi-definite matrix written by blocks.

There exist a unitary V and a unitary U such that

$$\left\| \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \right\| \leq \|UAU^* + VB V^*\| := \|A\| + \|B\|$$

for all symmetric norms.

Proof. Let U and V be two unitary matrix such that $UAU^* = D_o$ and $VB V^* = G_o$ where D_o and G_o are two diagonal matrices having the same ordering o , of eigenvalues with respect to their indexes i.e., if $\lambda_n \leq \dots \leq \lambda_1$ are the diagonal entries of D_o , and $\nu_n \leq \dots \leq \nu_1$ are those of G_o , then if λ_i is in the (j, j) position then ν_i will be also. Consequently $\|UAU^* + VB V^*\| = \|D_o + G_o\| = \|D_o\| + \|G_o\| = \|A\| + \|B\|$, for all the Ky-Fan k -norms and thus for all symmetric norms. To complete the proof notice that if $T = UXV^*$ and Q is the unitary matrix $\begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}$, by Remark 1

$$\left\| \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \right\| = \left\| Q \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} Q^* \right\| = \left\| \begin{pmatrix} D_o & T \\ T^* & G_o \end{pmatrix} \right\| \leq \|D_o\| + \|G_o\| \quad (13)$$

for all symmetric norms.

□

It seems easy to construct examples of non P.S.D matrices N , such that $\|N\|_s > \|A + B\|_s$, let us have a look of such inequality for P.S.D. matrices.

Example 3.2. *Let*

$$C = \begin{pmatrix} \frac{4}{3} & 0 & 1 & -1 \\ 0 & 1 & 0 & \frac{1}{5} \\ 1 & 0 & \frac{3}{2} & 0 \\ -1 & \frac{1}{5} & 0 & 2 \end{pmatrix} = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$$

where $A = \begin{pmatrix} \frac{4}{3} & 0 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} \frac{3}{2} & 0 \\ 0 & 2 \end{pmatrix}$. Since the eigenvalues of C are all positive with $\lambda_1 \approx 3.008$, $\lambda_2 \approx 1.7$, $\lambda_3 \approx 0.9$, $\lambda_4 \approx 0.089$, C is positive definite and we verify that

$$3.008 \approx \|C\|_s > \|A + B\|_s = 3$$

Example 3.3. *Let*

$$N_y = \begin{pmatrix} 2 & 0 & 0 & 2 \\ 0 & y & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$$

where $A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. The eigenvalues of N_y are the numbers: $\lambda_1 = 4$, $\lambda_2 = 1$, $\lambda_3 = y$, $\lambda_4 = 0$, thus if $y \geq 0$, N_y is positive semi-definite and for all y such that $0 \leq y < 1$ we have

1. $4 = \|N_y\|_s > \|A + B\|_s = 3$
2. $16 + y^2 + 1 = \|N\|_{(2)}^2 > \|A + B\|_{(2)}^2 = 4(3 + y) + y^2 + 1$

References

- [1] J. C. Bourin, E. Y. Lee, and M. Lin, *On a decomposition lemma for positive semi-definite block-matrices*, Linear Algebra and its Applications 437, pp.1906–1912, (2012).
- [2] J. C. Bourin, F. Hiai, *Norm and anti-norm inequalities for positive semi-definite matrices*, Internat.J.Math.63, pp.1121-1138, (2011).