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# Variational problems on product spaces

## Different obstacle constraints

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### Abstract

We study two principle minimizing problems, subject of different constraints. Our open sets are assumed bounded, except mentioning otherwise; precisely  $\Omega = ]0, 1[^n \in \mathbb{R}^n$ ,  $n = 1$  or  $n = 2$ .

**Keywords:** Variational problem, Optimization, Convexity, Euler Lagrange, Sobolev spaces, Weak topology.

## 1 Introduction

**Theorem 1.1** (Rellich-Kondrachov theorem). *Suppose  $\Omega$  is bounded and of class  $C^1$  then,  $W^{1,p} \subset L^p$  with compact injection for all  $p$  (and all  $n$ ).*

Let  $p \geq 2$  and  $W^{1,p}(]0, 1[; \mathbb{R}^2) = \{u = (u_1, u_2); u_1 \in W^{1,p}(]0, 1[; \mathbb{R}), u_2 \in W^{1,p}(]0, 1[; \mathbb{R})\}$

Define the functionals

1.  $W^{1,p}(]0, 1[; \mathbb{R}^2) \rightarrow \mathbb{R}_+$

$$u \rightarrow F(u) = \int_0^1 |u'(x)|_2^p = \int_0^1 \left( |u'_1(x)|^2 + |u'_2(x)|^2 \right)^{\frac{p}{2}}$$

2.  $W_0^{1,p}(\Omega; \mathbb{R}^2) \rightarrow \mathbb{R}_+$

$$u \rightarrow K(u) = \int_{\Omega} |\nabla u(x)|_2^p = \int_{\Omega} \left( |\nabla u_1(x)|_2^2 + |\nabla u_2(x)|_2^2 \right)^{\frac{p}{2}}$$

Mainly, our goals are:

- show if that there exists  $u_0 \in A_i$  unique such that,  $G(u_0) = \inf\{G(u); u \in A_i\}$
- write the Euler-Lagrange equation satisfied by a 'smooth'  $u_0$

Let us define the constraint sets:

1.  $A_1 = \{u \in W^{1,p}([0, 1[; \mathbb{R}^2) : |u|_2^2 = (|u_1|^2 + |u_2|^2) = 1 \text{ a.e. so } |u_i|_\infty \leq 1, u_1^2 = 1 - u_2^2; u(0) = (0, 1), u(1) = (1, 0)\}$
2.  $A_2 = \{u \in W^{1,p}(\Omega; \mathbb{R}^2); u_1 = 0 \text{ \& } u_2 = 1 \text{ on } \partial\Omega; u_1 \in W_0^{1,p}(\Omega), |u|_2^2 = (|u_1|^2 + |u_2|^2) = 1 \text{ a.e. so } |u_i|_\infty \leq 1, u_1^2 = 1 - u_2^2\}$

Note that the condition **a.e.** is implicitly important. One can notice that it could be written directly into equation  $u_2 = \sqrt{1 - (u_1)^2}$ ; without loss of generality we didn't do so. Clearly, boundary condition does not define a vector space, if  $u_1(0) = 0, u_1(1) = 1$ , we write  $u_1 = g$  and  $u_2 = 1 - g$  on  $\partial\Omega$  and  $g$  may be a function defined on the open set  $\Omega$  as well.

## 2 Solutions

**Lemma 2.1.**  $A_i \neq \emptyset$  for all  $i$ .

*Proof.* For  $i = 1$  consider the bounded smooth functionals

$$x \rightarrow u_1 = \begin{cases} \exp\left(\frac{x}{x^p - 1}\right) & \text{for any } p > 0 \text{ if } x \in [0, 1[ \\ 0 & \text{if not} \end{cases}$$

For  $i = 2$ , similarly but more explicitly we use the following proposition about partition of unity which lead to the result after a regularization process.  $\square$

**Proposition 2.1.** let  $\Omega$  be an open set of  $\mathbb{R}^d$  and  $K$  a compact  $\subset \Omega$ . Then  $\exists \Phi \in C_c(\mathbb{R}^d)$ , such that

$$0 \leq \Phi \leq 1, \quad \text{supp}(\Phi) \subset \Omega.$$

**Definition 2.1.** The  $p$ -norm on  $\mathbb{R}^n$  is defined as:

$$x = (x_1, \dots, x_n) \in \mathbb{R}^n, p \in ]0; +\infty[ : x \rightarrow |x|_p = \left( \sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}}$$

and it is denoted by  $|\cdot|_p$ .

**Lemma 2.2.** *If  $u_1 \in L^p(\Omega)$ , and  $u_2 \in L^p(\Omega)$  then  $(|u_1(x)|^2 + |u_2(x)|^2)^{\frac{p}{2}} \in L^1(\Omega)$*

*Proof.* Write  $|u|_2 \leq C|u|_p$  for some  $C > 0$ . □

## 2.1 Existence and uniqueness

Note that product spaces such  $V \times V$  are equipped with the sum norm that is  $\|u\| + \|v\|$ .

Usually we will study  $K(u)^{\frac{1}{p}}$ , and  $F(u)^{\frac{1}{p}}$  as the norm  $L^p$  will appear explicitly. Before we state the main theorem, we have:

**Proposition 2.2.**  $|v(\partial\Omega)| \leq C\|v\|_{W^{1,p}} \quad \forall v \in W^{1,p}[0, 1[$   
 where  $\partial\Omega := \{0, 1\}$

**Remark 2.1.** *A minimizer of a positive valued function  $f$  is also a minimizer of  $f^p$  and conversely,  $\forall p > 0$ .*

### Theorem 2.1.

- *There exists at least one function  $u = (u_1, u_2) \in A_1$  solving  $F(u) = \min_{w \in A_1} F(w)$ .*
- *There exists at least one function  $u = (u_1, u_2) \in A_2$  solving  $K(u) = \min_{w \in A_2} K(w)$ .*

*Proof.* First  $F(u)^{\frac{1}{p}}$  and  $K(u)^{\frac{1}{p}}$  are both continuous convex functions thus weakly lower semi continuous. Also the constraints sets are weakly closed, in the sense that, if  $u_n \rightharpoonup u$ , and  $u_n$  is satisfying any of the constraint,  $u$  will be as well. For the boundary condition that is  $u = g$  on the boundary, choose any  $h$  satisfying same constraints,  $u_n - h$  is a sequence  $\in W_0^{1,p} \times W_0^{1,p}$ , a convex closed subspace of  $W^{1,p} \times W^{1,p}$ , hence weakly closed.

For the condition of  $|u_i|_\infty \leq 1$  a.e. it suffices to show that  $|u_1|_\infty \leq 1$  a.e. Take a sequence weakly convergent to  $u$  in  $W^{1,p}$  by Rellich-K. Theorem we have a strong convergence at least in one of the  $L^p$ 's. Thus we can extract

a subsequence that converges a.e. to  $u$ . Given that  $|\Omega| < \infty$ , by Egoroff theorem the a.e convergence is equivalent to uniform convergence, up to arbitrarily negligible sets. Since the set is closed for the uniform convergence, we conclude that  $|u_i|_\infty \leq 1$ ,  $i = 1, 2$  **a.e**

It could be said directly after the extraction of a subsequence a.e. convergent, that we have

$$|u_{k_j1}|^2 + |u_{k_j2}|^2 \rightarrow |u_1|^2 + |u_2|^2 = 1 \text{ a.e}$$

Remaining to show that the functionals verify a coercivity condition over the product space.

1. Set  $f := \inf_{u \in A_1} F(u)$ . If  $f = +\infty$  we are done, suppose  $f$  is finite. Select a minimizing sequence  $\{u_k\}$ , then  $F(u_k) \rightarrow f$  cause we are in  $\mathbb{R}$

$$F(u) \geq C \cdot \left( \|u_1\|_{W_0^{1,p}}^p + \|u_2\|_{W_0^{1,p}}^p \right) \geq CC' \left( \|u_1\|_{W_0^{1,p}} + \|u_2\|_{W_0^{1,p}} \right)^p$$

One can verify because of boundary conditions (on  $A_1$ ) that we have equivalence between the two norms  $\|\cdot\|_{W_0^{1,p}}$  and  $\|\cdot\|_{W^{1,p}}$  i.e.

$$F(u_k) \geq \alpha \|u_k\| := \alpha (\|u_{k1}\|_{W^{1,p}} + \|u_{k2}\|_{W^{1,p}})^p.$$

This estimate implies that  $\{u_k\}$  is bounded in  $W^{1,p} \times W^{1,p}$ . Consequently there exist a subsequence  $\{u_{k_j}\}$  and a function  $u \in W^{1,p} \times W^{1,p}$  such that;  $u_{k_j} \rightharpoonup u$  weakly in  $W^{1,p} \times W^{1,p}$ , thus  $F(u)$  is weakly lower semi continuous.  $F(u) \leq \liminf_{j \rightarrow \infty} F(u_{k_j}) = f$ , since  $u \in A_1$  it follows that

$$F(u) = f = \min_{u \in A_1} F(u).$$

2. Similarly, set  $m := \inf_{u \in A_2} K(u)$ . If  $m = +\infty$  we are done, suppose  $m$  is finite, select a minimizing sequence  $\{u_k\}$ , then  $K(u_k) \rightarrow m$  because we are in  $\mathbb{R}$ .

$$\begin{aligned} K(u) &= \int_{\Omega} |\nabla u(x)|_2^p = \int_{\Omega} (|\nabla u_1(x)|_2^2 + |\nabla u_2(x)|_2^2)^{\frac{p}{2}} \\ \int_{\Omega} (|\nabla u_1(x)|_2^2 + |\nabla u_2(x)|_2^2)^{\frac{p}{2}} &= \int_{\Omega} \left( \left( \frac{\partial u_1}{\partial x_1} \right)^2 + \left( \frac{\partial u_1}{\partial x_2} \right)^2 + \left( \frac{\partial u_2}{\partial x_1} \right)^2 + \left( \frac{\partial u_2}{\partial x_2} \right)^2 \right)^{\frac{p}{2}} \\ &\geq C \int_{\Omega} \left( \left( \frac{\partial u_1}{\partial x_1} \right)^p + \left( \frac{\partial u_1}{\partial x_2} \right)^p + \left( \frac{\partial u_2}{\partial x_1} \right)^p + \left( \frac{\partial u_2}{\partial x_2} \right)^p \right) \end{aligned}$$

Consequently

$$K(u_k) \geq \min(C, 1)(\|u_{k1}\|_{W^{1,p}}^p + \|\nabla\sqrt{1-u_{k1}^2}\|_{L^p}^p) \quad (1)$$

since  $u_1 \in W_0^{1,p}$ , if  $u_1$  is bounded so is  $u_2$  and conversely for:

$$|1 - \|u_1\|^2| \leq \|1 - u_1\|^2 \leq \|u_2\|^2$$

we conclude that the sequence  $\{u_k\}$  is bounded in  $W^{1,p} \times W^{1,p}$  and the proof is similar to that of  $F(u)$ .

□

**Theorem 2.2.** *The minimizing problem:  $F(u) = \min_{w \in A_1} F(w)$  has a unique solution*

*Proof.* Suppose not, if another distinct minimizer exists  $v := (v_1, v_2)$  write  $w = (w_1, w_2) = (\frac{u_1+v_1}{2}, \frac{u_2+v_2}{2})$  now recall that  $L^p$  is strictly convex for  $p \geq 2$ , that's means as long as

$$v' \neq \alpha.u' \text{ a.e.} \quad (2)$$

we have the strict inequality:

$$\begin{aligned} G(u)^{\frac{1}{p}} &= \left[ \int_0^1 |w'(x)|_2^p \right]^{\frac{1}{p}} = \left[ \int_0^1 \left( \left( \frac{u'_1 + v'_1}{2} \right)^2 + \left( \frac{u'_2 + v'_2}{2} \right)^2 \right)^{\frac{p}{2}} \right]^{\frac{1}{p}} \\ &< \frac{1}{2}G(u)^{\frac{1}{p}} + \frac{1}{2}G(v)^{\frac{1}{p}} \end{aligned}$$

which is of course a contradiction. That completes the proof if we showed that  $v' \neq \alpha.u'$  a.e. Suppose the converse so  $u = \beta v + cte$  say  $u_1 = \beta_1 v_1 + cte_1$  applying boundary constraints and using Proposition 2.2 we conclude that  $u_1 \neq \beta_1 v_1 + cte_1$  a.e. for any  $\beta_1$  and any  $cte_1$  □

### 3 Euler-Lagrange

**Lemma 3.1.**  $F(u)$  and  $K(u)$  are both differentiable ( $C^1$ ) on the product space except at  $(0, 0)$

*Proof.* This follows by the regularity of the  $|\cdot|_2$  norm and derivation under integral sign. □

From this, we can compute the Euler-Lagrange equations giving the existence of a minimizer  $(u_{01}, u_{02}) \neq 0$ . Bearing in mind that  $C^1$  Gateaux differentiable is the same as  $C^1$  Frechet -differentiable. We will give the 'equation' satisfied by the 'minimizer' of  $K(u)$  as it is the most general case. Fix  $v \in W_0^{1,p}(\Omega, \mathbb{R}^2) \cap L^\infty(\Omega, \mathbb{R}^2)$ . Since  $|u|_2 = 1$  a.e, we have

$$|u + \tau v|_2 \neq 0 \text{ a.e.}$$

for each sufficiently small  $\tau$  by continuity. Consequently

$$v(\tau) := \frac{u + \tau v}{|u + \tau v|_2} \in A_2$$

Thus

$$k(\tau) := K(v(\tau))$$

has a minimum at  $\tau = 0$ , and so

$$k'(0) = 0.$$

Norms on product spaces are of course Eucliden norms, that is  $|\cdot|_2$ . Matrices such the gradient matrix (here it's a  $2 \times 2$  matrix) can be identified to a vector  $\in \mathbb{R}^4$ , and let  $(\cdot)$  denotes the usual scalar product on  $\mathbb{R}^n$ , by a direct computation we have:

**Proposition 3.1.**  $v'(\tau) = \frac{v}{|u + \tau v|} - \frac{[(u + \tau v) \cdot v](u + \tau v)}{|u + \tau v|^3}$

**Theorem 3.1.** Let  $u \in A_2$  satisfy

$$K(u) = \min_{w \in A_2} K(w).$$

Then

$$\int_{\Omega} p|Du|^{\frac{p}{2}-1}[(Du.Dv) - |Du|^2(u.v)] \quad (3)$$

for each  $v \in W_0^{1,p}(\Omega, \mathbb{R}^2) \cap L^\infty(\Omega, \mathbb{R}^2)$ .

*Proof.* In fact

$$K(u) = \int_{\Omega} |Du|^p$$

where  $Du$  is the gradient matrix associated to  $u$  and the norm as said is the one associated to the scalar product:  $\langle A, B \rangle = \text{Tr}(B^t.A)$ .

$$\begin{aligned} k'(0) = 0 &= \int_{\Omega} p|Du|^{\frac{p}{2}-1} Du.Dv'(0) \\ &= \int_{\Omega} p|Du|^{\frac{p}{2}-1} Du.D(v - (u.v)u) \end{aligned} \quad (4)$$

Upon differentiating  $|u|^2 = 1$  a.e., we have

$$(Du)^T u = 0$$

Using this fact, we then verify

$$Du.(D(u.v)u) = |Du|^2(u.v) \quad \text{a.e. in } \Omega$$

This identity employed in (4) gives (3). We leave details to the interested reader. [2]

□



## References

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