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Variational problems on product spaces

Different obstacle constraints

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Abstract

We study two principle minimizing problems, subject of different constraints. Our open sets are assumed bounded, except mentioning otherwise; precisely $\Omega =]0, 1[^n \in \mathbb{R}^n$, $n = 1$ or $n = 2$.

Keywords: Variational problem, Optimization, Convexity, Euler Lagrange, Sobolev spaces, Weak topology.

1 Introduction

Theorem 1.1 (Rellich-Kondrachov theorem). *Suppose Ω is bounded and of class C^1 then, $W^{1,p} \subset L^p$ with compact injection for all p (and all n).*

Let $p \geq 2$ and $W^{1,p}(]0, 1[; \mathbb{R}^2) = \{u = (u_1, u_2); u_1 \in W^{1,p}(]0, 1[; \mathbb{R}), u_2 \in W^{1,p}(]0, 1[; \mathbb{R})\}$

Define the functionals

1. $W^{1,p}(]0, 1[; \mathbb{R}^2) \rightarrow \mathbb{R}_+$

$$u \rightarrow F(u) = \int_0^1 |u'(x)|_2^p = \int_0^1 \left(|u'_1(x)|^2 + |u'_2(x)|^2 \right)^{\frac{p}{2}}$$

2. $W_0^{1,p}(\Omega; \mathbb{R}^2) \rightarrow \mathbb{R}_+$

$$u \rightarrow K(u) = \int_{\Omega} |\nabla u(x)|_2^p = \int_{\Omega} \left(|\nabla u_1(x)|_2^2 + |\nabla u_2(x)|_2^2 \right)^{\frac{p}{2}}$$

Mainly, our goals are:

- show if that there exists $u_0 \in A_i$ unique such that, $G(u_0) = \inf\{G(u); u \in A_i\}$
- write the Euler-Lagrange equation satisfied by a 'smooth' u_0

Let us define the constraint sets:

1. $A_1 = \{u \in W^{1,p}(\]0, 1[; \mathbb{R}^2) : |u|_2^2 = (|u_1|^2 + |u_2|^2) = 1 \text{ a.e. so } |u_i|_\infty \leq 1, u_1^2 = 1 - u_2^2; u(0) = (0, 1), u(1) = (1, 0)\}$
2. $A_2 = \{u \in W^{1,p}(\Omega; \mathbb{R}^2); u_1 = 0 \text{ \& } u_2 = 1 \text{ on } \partial\Omega; u_1 \in W_0^{1,p}(\Omega), |u|_2^2 = (|u_1|^2 + |u_2|^2) = 1 \text{ a.e. so } |u_i|_\infty \leq 1, u_1^2 = 1 - u_2^2\}$

Note that the condition **a.e.** is implicitly important .

One can notice that it could be written directly into equation $u_2 = \sqrt{1 - (u_1)^2}$ without loss of generality we didn't do so. Clearly, boundary condition does not define a vector space, if $u_1(0) = 0, u_1(1) = 1$, we write $u_1 = g$ and $u_2 = 1 - g$ on $\partial\Omega$ and g may be a function defined on the open set Ω as well.

2 Solutions

Lemma 2.1. $A_i \neq \emptyset$ for all i .

Proof. For $i = 1$ consider the bounded smooth functionals

$$x \rightarrow u_1 = \begin{cases} \exp\left(\frac{x}{x^p - 1}\right) & \text{for any } p > 0 \text{ if } x \in [0, 1[\\ 0 & \text{if not} \end{cases}$$

For $i = 2$, similarly but more explicitly we use the following proposition about partition of unity which lead to the result after a regularization process. \square

Proposition 2.1. let Ω be an open set of \mathbb{R}^d and K a compact $\subset \Omega$. Then $\exists \Phi \in C_c(\mathbb{R}^d)$, such that

$$0 \leq \Phi \leq 1, \quad \text{supp}(\Phi) \subset \Omega.$$

Definition 2.1. The p -norm on \mathbb{R}^n is defined as:

$$x = (x_1, \dots, x_n) \in \mathbb{R}^n, p \in]0; +\infty[: x \rightarrow |x|_p = \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}}$$

and it is denoted by $|\cdot|_p$.

Lemma 2.2. *If $u_1 \in L^p(\Omega)$, and $u_2 \in L^p(\Omega)$ then $(|u_1(x)|^2 + |u_2(x)|^2)^{\frac{p}{2}} \in L^1(\Omega)$*

Proof. Write $|u|_2 \leq C|u|_p$ for some $C > 0$. □

2.1 Existence and uniqueness

Note that product spaces such $V \times V$ are equipped with the sum norm that is $\|u\| + \|v\|$.

Usually we will study $K(u)^{\frac{1}{p}}$, and $F(u)^{\frac{1}{p}}$ as the norm L^p will appear explicitly. Before we state the main theorem, we have:

Proposition 2.2. $|v(\partial\Omega)| \leq C\|v\|_{W^{1,p}} \quad \forall v \in W^{1,p}[0, 1[$
where $\partial\Omega := \{0, 1\}$

Remark 2.1. *A minimizer of a positive valued function f is also a minimizer of f^p and conversely, $\forall p > 0$.*

Theorem 2.1.

- *There exists at least one function $u = (u_1, u_2) \in A_1$ solving $F(u) = \min_{w \in A_1} F(w)$.*
- *There exists at least one function $u = (u_1, u_2) \in A_2$ solving $K(u) = \min_{w \in A_2} K(w)$.*

Proof. First $F(u)^{\frac{1}{p}}$ and $K(u)^{\frac{1}{p}}$ are both continuous convex functions thus weakly lower semi continuous. Also the constraints sets are weakly closed, in the sense that if

$u_n \rightharpoonup u$, and u_n is satisfying any of the constraint, u will be as well. For the boundary condition that is $u = g$ on the boundary, choose any h satisfying same constraints, $u_n - h$ is a sequence $\in W_0^{1,p} \times W_0^{1,p}$, a convex closed subspace of $W^{1,p} \times W^{1,p}$, hence weakly closed.

For the condition of $|u_i|_\infty \leq 1$ a.e. it suffices to show that $|u_1|_\infty \leq 1$ a.e. Take a sequence weakly convergent to u in $W^{1,p}$ by Rellich-K. Theorem we

have a strong convergence at least in one of the L^p 's. Thus we can extract a subsequence that converges a.e. to u . Giving that $|\Omega| < \infty$, by Egoroff theorem the a.e convergence is equivalent to uniform convergence, up to arbitrarily negligible sets. Since the set is closed for the uniform convergence, we conclude that $|u_i|_\infty \leq 1, i = 1, 2$ **a.e**

It could be said directly after the extraction of a subsequence a.e. convergent, that we have

$$|u_{k_{j_1}}|^2 + |u_{k_{j_2}}|^2 \rightarrow |u_1|^2 + |u_2|^2 = 1 \text{ a.e}$$

Remaining to show that the functionals verify a coercivity condition over the product space.

1. Set $f := \inf_{u \in A_1} F(u)$. If $f = +\infty$ we are done, suppose f is finite. Select a minimizing sequence $\{u_k\}$, then $F(u_k) \rightarrow f$ cause we are in \mathbb{R}

$$F(u) \geq C \left(\|u_1\|_{W_0^{1,p}}^p + \|u_2\|_{W_0^{1,p}}^p \right) \geq CC' \left(\|u_1\|_{W_0^{1,p}} + \|u_2\|_{W_0^{1,p}} \right)^p$$

One can verify because of boundary conditions (on A_1) that we have equivalence between the two norms $\|\cdot\|_{W_0^{1,p}}$ and $\|\cdot\|_{W^{1,p}}$ i.e.

$$F(u_k) \geq \alpha \|u_k\| := \alpha (\|u_{k_1}\|_{W^{1,p}} + \|u_{k_2}\|_{W^{1,p}})^p.$$

This estimate implies that $\{u_k\}$ is bounded in $W^{1,p} \times W^{1,p}$. Consequently there exist a subsequence $\{u_{k_j}\}$ and a function $u \in W^{1,p} \times W^{1,p}$ such that; $u_{k_j} \rightharpoonup u$ weakly in $W^{1,p} \times W^{1,p}$, thus $F(u)$ is weakly lower semi continuous. $F(u) \leq \liminf_{j \rightarrow \infty} F(u_{k_j}) = f$, since $u \in A_1$ it follows that

$$F(u) = f = \min_{u \in A_1} F(u).$$

2. Similarly, set $m := \inf_{u \in A_2} K(u)$. If $m = +\infty$ we are done, suppose m is finite, select a minimizing sequence $\{u_k\}$, then $K(u_k) \rightarrow m$ cause we are in \mathbb{R} .

$$\begin{aligned} K(u) &= \int_{\Omega} |\nabla u(x)|_2^p \\ &= \int_{\Omega} (|\nabla u_1(x)|_2^2 + |\nabla u_2(x)|_2^2)^{\frac{p}{2}} = \int_{\Omega} \left(\left(\frac{\partial u_1}{\partial x_1} \right)^2 + \left(\frac{\partial u_1}{\partial x_2} \right)^2 + \left(\frac{\partial u_2}{\partial x_1} \right)^2 + \left(\frac{\partial u_2}{\partial x_2} \right)^2 \right)^{\frac{p}{2}} \end{aligned}$$

$$\begin{aligned} &\geq C \int_{\Omega} \left(\left(\frac{\partial u_1}{\partial x_1} \right)^p + \left(\frac{\partial u_1}{\partial x_2} \right)^p + \left(\frac{\partial u_2}{\partial x_1} \right)^p + \left(\frac{\partial u_2}{\partial x_2} \right)^p \right) \\ K(u_k) &\geq \min(C, 1) (\|u_k\|_{W^{1,p}}^p + \|\nabla \sqrt{1 - u_k^2}\|_{L^p}^p) \end{aligned} \quad (1)$$

since $u_1 \in W_0^{1,p}$, if u_1 is bounded so is u_2 and conversely for:

$$|1 - \|u_1\|^2| \leq \|1 - u_1\|^2 \leq \|u_2\|^2$$

we conclude that the sequence $\{u_k\}$ is bounded in $W^{1,p} \times W^{1,p}$ and the proof is similar to that of $F(u)$. □

Theorem 2.2. *The minimizing problem: $F(u) = \min_{w \in A_1} F(w)$ has a unique solution*

Proof. Suppose not, if another distinct minimizer exists $v := (v_1, v_2)$ write $w = (w_1, w_2) = \left(\frac{u_1+v_1}{2}, \frac{u_2+v_2}{2}\right)$ now recall that L^p is strictly convex for $p \geq 2$, that's means as long as

$$v' \neq \alpha u' \text{ a.e.} \quad (2)$$

we have the strict inequality:

$$\begin{aligned} G(u)^{\frac{1}{p}} &= \left[\int_0^1 |w'(x)|_2^p \right]^{\frac{1}{p}} = \left[\int_0^1 \left(\left(\frac{u'_1 + v'_1}{2} \right)^2 + \left(\frac{u'_2 + v'_2}{2} \right)^2 \right)^{\frac{p}{2}} \right]^{\frac{1}{p}} \\ &< \frac{1}{2} G(u)^{\frac{1}{p}} + \frac{1}{2} G(v)^{\frac{1}{p}} \end{aligned}$$

which is of course a contradiction. That completes the proof if we showed that $v' \neq \alpha u'$ a.e.

Suppose the converse so $u = \beta v + cte$ say $u_1 = \beta_1 v_1 + cte_1$ applying boundary constraints and using Proposition 2.2 we conclude that $u_1 \neq \beta_1 v_1 + cte_1$ a.e. for any β_1 and any cte_1 □

3 Euler-Lagrange

Lemma 3.1. $F(u)$ and $K(u)$ are both differentiable (C^1) on the product space except at $(0, 0)$

Proof. This follows by the regularity of the $|\cdot|_2$ norm and derivation under integral sign. □

From this, we can compute the Euler-Lagrange equations giving the existence of a minimizer $(u_{0_1}, u_{0_2}) \neq 0$. Bearing in mind that C^1 Gateaux differentiable is the same as C^1 Frechet -differentiable. We will give the 'equation' satisfied by the 'minimizer' of $K(u)$ as it is the most general case. Fix $v \in W_0^{1,p}(\Omega, \mathbb{R}^2) \cap L^\infty(\Omega, \mathbb{R}^2)$. Since $|u|_2 = 1$ a.e, we have

$$|u + \tau v|_2 \neq 0 \text{ a.e.}$$

for each sufficiently small τ by continuity. Consequently

$$v(\tau) := \frac{u + \tau v}{|u + \tau v|_2} \in A_2$$

Thus

$$k(\tau) := K(v(\tau))$$

has a minimum at $\tau = 0$, and so

$$k'(0) = 0.$$

Norms on product spaces are of course Eucliden norms, that is $|\cdot|_2$. Matrices such the gradient matrix (here it's a 2×2 matrix) can be identified to a vector $\in \mathbb{R}^4$, and let (\cdot) denotes the usual scalar product on \mathbb{R}^n , by a direct computation we have:

Proposition 3.1. $v'(\tau) = \frac{v}{|u + \tau v|} - \frac{[(u + \tau v) \cdot v](u + \tau v)}{|u + \tau v|^3}$

Theorem 3.1. Let $u \in A_2$ satisfy

$$K(u) = \min_{w \in A_2} K(w).$$

Then

$$\int_{\Omega} p|Du|^{\frac{p}{2}-1}[(Du.Dv) - |Du|^2(u.v)] \quad (3)$$

for each $v \in W_0^{1,p}(\Omega, \mathbb{R}^2) \cap L^\infty(\Omega, \mathbb{R}^2)$.

Proof. In fact

$$K(u) = \int_{\Omega} |Du|^p$$

where Du is the gradient matrix associated to u and the norm as said is the one associated to the scalar product: $\langle A, B \rangle = \text{Tr}(B^t.A)$.

$$\begin{aligned} k'(0) = 0 &= \int_{\Omega} p|Du|^{\frac{p}{2}-1} Du.Dv'(0) \\ &= \int_{\Omega} p|Du|^{\frac{p}{2}-1} Du.D(v - (u.v)u) \end{aligned} \quad (4)$$

Upon differentiating $|u|^2 = 1$ a.e., we have

$$(Du)^T u = 0$$

Using this fact, we then verify

$$Du.(D(u.v)u) = |Du|^2(u.v) \quad \text{a.e. in } \Omega$$

This identity employed in (4) gives (3). We leave details to the interested reader. [2]

□

References

- [1] [Haim Brezis], *Functional Analysis, Sobolev Spaces and Partial Differential Equation*, Springer.
- [2] [Laurence C.Evans], *Partial Differential Equation, Graduate Studies in Mathematics*, American Mathematical Society.
- [3] [Robert R.Phelps], *Convex Functions, Monotone Operators and Differentiability*, Lecture Notes in Mathematics Springer-Verlag.