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Computing the number of h -edge spanning forests in complete bipartite graphs

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Let $f_{m,n,h}$ be the number of spanning forests with h edges in the complete bipartite graph $K_{m,n}$. Kirchhoff's Matrix Tree Theorem implies $f_{m,n,m+n-1} = m^{n-1}n^{m-1}$ when $m \geq 1$ and $n \geq 1$, since $f_{m,n,m+n-1}$ is the number of spanning trees in $K_{m,n}$. In this paper, we give an algorithm for computing $f_{m,n,h}$ for general m, n, h . We implement this algorithm and use it to compute all non-zero $f_{m,n,h}$ when $m \leq 50$ and $n \leq 50$ in under 2 days.

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1 Introduction

A consequence of Kirchhoff's Matrix Tree Theorem is that the number of spanning trees in the complete bipartite graph $K_{m,n}$ is $m^{n-1}n^{m-1}$ when $m \geq 1$ and $n \geq 1$. In this paper, we consider a generalisation of this counting problem: the number $f_{m,n,h}$ of h -edge spanning forests in $K_{m,n}$. The number of spanning trees in $K_{m,n}$ is $f_{m,n,m+n-1}$ when $m \geq 1$ and $n \geq 1$, since spanning trees of $K_{m,n}$ have $m + n - 1$ edges.

The numbers $f_{m,n,h}$ arise in relation to the Tutte polynomial of $K_{m,n}$. For a general graph $G = (V, E)$, the Tutte polynomial is

$$T_G(x, y) = \sum_{A \subseteq E} (x - 1)^{k(A) - k(E)} (y - 1)^{k(A) + |A| - |V|}$$

where $k(A)$ is the number of connected components in the graph (V, A) . Importantly, for the Tutte polynomial, we take $0^0 = 1$. Thus, the number of spanning forests of G is given by

$$\begin{aligned} T_G(2, 1) &= \sum_{A \subseteq E} 0^{k(A) + |A| - |V|} \\ &= \#\{A \subseteq E : k(A) + |A| = |V|\} \\ &= \text{number of spanning forests of } G, \end{aligned} \tag{1}$$

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since $k(A) + |A| = |V|$ if and only if A is a spanning forest.

In the present paper, we are interested in the case of when $G = K_{m,n}$, and will look for h -edge spanning forests. In this situation,

$$T_{K_{m,n}}(2, 1) = \sum_{h \geq 0} f_{m,n,h}.$$

For general graphs $G = (V, E)$, Myrvold [10] gave an algorithm for computing the number of k -component spanning forests of G . Bjöklund et al. [4] described an algorithm that could compute $T_G(2, 1)$ in time $2^{|V|}|V|^{O(1)}$ by using Kirchhoff's Matrix Tree Theorem for each of the $2^{|V|}$ subsets of V , then combining the results together with Inclusion-Exclusion. Other relevant work for the enumeration of spanning forests includes [9]. Porter [11] gave an algorithm for generating the spanning trees of $K_{m,n}$. Farr and McDiarmid [5] showed that computing the number of circuits in G is #P-complete⁽ⁱ⁾.

Bounds for the number of spanning forests in graphs were given by Teranishi [13], from which we can deduce

$$T_{K_{m,n}}(2, 1) \geq \sum_{k \geq 0} \left(\frac{\min(m, n)}{2} \right)^{m+n-k} \binom{m+n}{k}$$

for all $m \geq 0$ and $n \geq 0$. Jin and Liu [8] gave a simple formula for the number of rooted spanning forests in $K_{m,n}$ (see also [7] and [12]).

This paper instead heads in a different direction to previous work: we derive algorithms for enumerating h -edge spanning forests in $K_{m,n}$.

2 Basic results

To be clear, all forests in this paper will be labelled. Let $\mathbb{N} = \{0, 1, 2, \dots\}$. The following lemma gives the boundary conditions on $f_{m,n,h}$.

Lemma 1 *Suppose $m, n, h \in \mathbb{N}$. Then*

- $f_{m,n,0} = 1$,
- if $h \geq 1$, then $f_{0,n,h} = f_{m,0,h} = 0$,
- if $m \geq 1$ and $n \geq 1$, then $f_{m,n,h} > 0$ if and only if $0 \leq h \leq m + n - 1$,
- if $m \geq 1$ and $n \geq 1$, then $f_{m,n,m+n-1} = m^{n-1}n^{m-1}$.

Proof: If $h = 0$, then there is exactly one subgraph of $K_{m,n}$ with no edges (and it is a spanning forest), so $f_{m,n,0} = 1$. If $m = 0$ or $n = 0$, then $K_{m,n}$ has no edges, and thus we obtain the second bulleted item.

Now assume $m \geq 1$ and $n \geq 1$. Since $K_{m,n}$ is connected, it has a spanning tree, which must have exactly $m + n - 1$ edges. By deleting edges from this spanning tree, we find h -edge forests in $K_{m,n}$ for all $0 \leq h \leq m + n - 1$. Hence $f_{m,n,h} > 0$ when $0 \leq h \leq m + n - 1$.

Now assume A is an h -subset of the edges in $K_{m,n}$, where $m \geq 1$ and $n \geq 1$ and $h \geq m + n$. To be a spanning forest, we need $k(A) + h = m + n$ as per (1), but if $h \geq m + n$, then $k(A) \leq 0$, giving a contradiction.

⁽ⁱ⁾ #P is the set of counting problems which ask for the number of "yes" instances for decision problems in NP. A #P problem is in #P-complete whenever any other problem in #P can be reduced to it by a polynomial-time counting reduction.

m	n	$h=0$	1	2	3	4	5	6	7	8	9	10	11
1	1	1	1										
1	2	1	2	1									
1	3	1	3	3	1								
1	4	1	4	6	4	1							
1	5	1	5	10	10	5	1						
1	6	1	6	15	20	15	6	1					
2	2	1	4	6	4								
2	3	1	6	15	20	12							
2	4	1	8	28	56	64	32						
2	5	1	10	45	120	200	192	80					
2	6	1	12	66	220	480	672	544	192				
3	3	1	9	36	84	117	81						
3	4	1	12	66	220	477	648	432					
3	5	1	15	105	455	1335	2673	3375	2025				
3	6	1	18	153	816	3015	7938	14499	16524	8748			
4	4	1	16	120	560	1784	3936	5632	4096				
4	5	1	20	190	1140	4785	14544	31520	44800	32000			
4	6	1	24	276	2024	10536	40704	117376	244224	331776	221184		
5	5	1	25	300	2300	12550	51030	155900	347500	515625	390625		
5	6	1	30	435	4060	27255	138606	544525	1641000	3645000	5400000	4050000	
6	6	1	36	630	7140	58680	369792	1834992	7210080	22083840	50388480	77262336	60466176

Fig. 1: Small non-zero values of $f_{m,n,h}$.

The fourth bulleted item is from Kirchhoff’s Matrix Tree Theorem (already mentioned). □

The following lemma is an important (but basic) consequence of the symmetry of the problem.

Lemma 2 For all $m, n, h \in \mathbb{N}$, we have $f_{m,n,h} = f_{n,m,h}$.

In Figure 1 we list the non-zero values of $f_{m,n,h}$ when $0 \leq m \leq n \leq 6$. These values were generated using a straightforward backtracking algorithm.

3 Combinatorial equivalence

In this section, we will describe some combinatorial objects that are equivalent to h -edge spanning forests of $K_{m,n}$.

Let M be an $m \times n$ $(0, 1)$ -matrix. We define a *cycle* in M to be a set $\{e_1, e_2, \dots, e_{2t}\}$ of entries of M , such that:

- each e_i contains the symbol 1, and
- for $i \in \{1, 2, \dots, t\}$, we have that (a) e_{2i-1} belongs to the same row as e_{2i} and (b) e_{2i} belongs to the same column as e_{2i+1} (where we take $e_{2t+1} = e_1$).

For example, the matrix

0	Ⓛ	0	1	Ⓛ
0	0	1	0	1
Ⓛ	0	0	0	Ⓛ
Ⓛ	Ⓛ	0	0	0

has a cycle (consisting of the circled 1's), whereas the matrices

$$\begin{array}{|c|c|c|} \hline 0 & 1 & 1 \\ \hline 0 & 0 & 0 \\ \hline \end{array} \quad \text{and} \quad \begin{array}{|c|c|c|c|c|} \hline 0 & 1 & 1 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 & 0 \\ \hline \end{array}$$

do not have cycles.

Let $A_{m,n,h}$ be the set of $(0,1)$ -matrices with exactly h elements equal to 1. Let $p_{m,n,h}$ be the probability that an element of $A_{m,n,h}$ chosen uniformly at random contains a cycle. An anonymous user of math.stackexchange.com asked⁽ⁱⁱ⁾ for a formula for $p_{m,n,h}$. In fact, this was the original motivation for the author to study this problem.

Lemma 3 For all $m, n, h \in \mathbb{N}$, we have $f_{m,n,h}$ is the number of $m \times n$ $(0,1)$ -matrices with h 1's that do not contain a cycle, and hence

$$f_{m,n,h} = (1 - p_{m,n,h}) \binom{mn}{h}.$$

Proof: A matrix $M \in A_{m,n,h}$ can be interpreted as the biadjacency matrix⁽ⁱⁱⁱ⁾ of an h -edge subgraph $G = G(M)$ of $K_{m,n}$. A cycle in $M \in A_{m,n,h}$ corresponds to a cycle in G . Hence

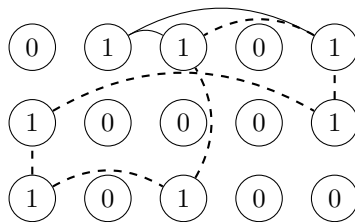
$$\begin{aligned} f_{m,n,h} &= (1 - p_{m,n,h}) |A_{m,n,h}| \\ &= (1 - p_{m,n,h}) \binom{mn}{h}. \end{aligned}$$

□

A problem related to the $(0,1)$ -matrix problem above comes from the study of $(0,1)$ -matrices that do not have a submatrix which is the incidence matrix of any cycle of length at least 3; these are called “totally balanced matrices” (see e.g. [2]).

Another interpretation of matrices in $A_{m,n,h}$ is as induced subgraphs of $K_m \square K_n$, where \square represents the Cartesian product of graphs. The graph $K_m \square K_n$ is sometimes called the “rook’s graph”, since the edges represent the legal moves of a rook on an $m \times n$ chess board. The vertices in $K_m \square K_n$ are $\{(i, j) : 1 \leq i \leq m \text{ and } 1 \leq j \leq n\}$ and there is an edge between distinct vertices (i, j) and (i', j') whenever $i = i'$ or $j = j'$. The graph $K_m \square K_n$ is also the line graph of $K_{m,n}$.

There is a bijection between induced subgraphs $H = H(M)$ of $K_m \square K_n$ and $(0,1)$ -matrices $M = (m_{ij}) \in A_{m,n,h}$: we include the vertex (i, j) in H if and only if $m_{ij} = 1$. For example, if we ignore the vertices marked 0 in



⁽ⁱⁱ⁾ Full URL: <http://math.stackexchange.com/q/24800>

⁽ⁱⁱⁱ⁾ The biadjacency matrix of a bipartite graph G on the vertex set $\{u_i\}_{1 \leq i \leq m} \cup \{v_j\}_{1 \leq j \leq n}$ is the $m \times n$ $(0,1)$ -matrix $M = (m_{ij})$ with $m_{ij} = 1$ if and only if $u_i v_j$ is an edge in G .

we obtain an induced subgraph $H(M)$ of $K_3 \square K_5$, corresponding to the matrix

$$M = \begin{array}{|ccccc|} \hline 0 & 1 & \textcircled{1} & 0 & \textcircled{1} \\ \hline \textcircled{1} & 0 & 0 & 0 & \textcircled{1} \\ \hline \textcircled{1} & 0 & \textcircled{1} & 0 & 0 \\ \hline \end{array}$$

in $A_{3,5,6}$. We see that cycles in M map from cycles C_k in H for some $k \in \{4, 6, 8, \dots\}$ (an example of a 6-cycle is highlighted). There are other cycles in H (e.g. C_3 in the above example; in fact, if M above had a row of 1's, then H would have a K_5 subgraph). However, induced cycles C_k in $H(M)$ of length $k \in \{4, 6, 8, \dots\}$ are in one-to-one correspondence with cycles in M via the above bijection.

Lemma 4 For all $m, n, h \in \mathbb{N}$, we have that $f_{m,n,h}$ is the number of h -vertex induced subgraphs of $K_m \square K_n$ that do not contain an induced C_k for any $k \in \{4, 6, 8, \dots\}$.

4 Simplifying the equation

4.1 A formula for $f_{m,n,h}$

We will look at two related ways of simplifying $f_{m,n,h}$. Let $K_{m,n}$ have the vertex bipartition $M \cup N$, where $M = \{u_1, u_2, \dots, u_m\}$ and $N = \{v_1, v_2, \dots, v_n\}$. Let $B_{m,n,h}$ be the set of h -edge spanning forests of $K_{m,n}$. Hence $f_{m,n,h} = |B_{m,n,h}|$. Define

$$W_{m,n,h} = \{G \in B_{m,n,h} : G \text{ has no isolated vertices}\}$$

and $w_{m,n,h} = |W_{m,n,h}|$. We observe the following:

- Given any spanning forest in $B_{m,n,h}$ we can delete the isolated vertices to obtain a spanning forest in $W_{i,j,h}$ for some $i \leq m$ and $j \leq n$.
- Conversely, given any spanning forest in $W_{i,j,h}$, we can add isolated vertices to it in $\binom{m}{m-i} \binom{n}{n-j}$ ways to obtain a spanning forest in $B_{m,n,h}$.

In both of the operations above, we need to relabel the vertices, but we preserve the order of the indices of the non-isolated vertices within M and N .

Hence

$$f_{m,n,h} = \sum_{i=0}^m \sum_{j=0}^n \binom{m}{i} \binom{n}{j} w_{i,j,h}. \tag{2}$$

This formula eliminates the need to further account for isolated vertices.

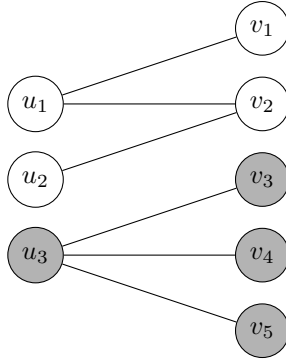
We can decompose any $G \in W_{m,n,h}$ into disjoint components, each of which belong to $W_{i,j,k}$ for some $i \leq m$ and $j \leq n$ and $k \leq h$. Hence any $G \in W_{m,n,h}$ decomposes into the following.

- A partition P of M where $x, y \in M$ belong to the same part if there is a path from x to y in G .
- A partition Q of N where $x', y' \in N$ belong to the same part if there is a path from x' to y' in G .
- A bijection $\alpha : P \rightarrow Q$ such that every edge ab in G has $a \in p$ and $b \in \alpha(p)$ for some $p \in P$. Hence we have $|P| = |Q|$.

- For each $p \in P$, a subgraph in $W_{|p|,|\alpha(p)|,|p|+|\alpha(p)|-1}$ induced by the vertices in $p \cup \alpha(p)$.

Regarding the final bulleted item, we observe that the subgraph induced by $p \cup \alpha(p)$ for some $p \in P$ must be a spanning tree on $K_{|p|,|\alpha(p)|}$. Any more edges would cause a cycle, while fewer edges would cause disjoint components (in which case G could be further decomposed).

We give an example of a graph G in $W_{3,5,6}$ below.



Here, the partitions P and Q are given by $P = \{\{u_1, u_2\}, \{u_3\}\}$ and $Q = \{\{v_1, v_2\}, \{v_3, v_4, v_5\}\}$ and we have the bijection α such that

$$\begin{aligned} \{u_1, u_2\} &\mapsto \{v_1, v_2\} \\ \{u_3\} &\mapsto \{v_3, v_4, v_5\}. \end{aligned}$$

We see that G decomposes into a graph in $W_{2,2,3}$ and a graph in $W_{1,3,3}$.

In general, for any $p \in P$, we must have exactly $|p| + |\alpha(p)| - 1$ edges in $p \cup \alpha(p)$. Summing this over all $p \in P$ gives $h = m + n - |P|$. We conclude that

$$|P| = m + n - h$$

(and thus $|Q| = m + n - h$).

We will now describe how to construct any graph in $W_{m,n,h}$ via its decomposition. Let \mathcal{P} be the set of partitions of M and let \mathcal{Q} be the set of partitions of N . Given (a) a partition $P \in \mathcal{P}$ of size $|P| = m + n - h$, (b) a partition $Q \in \mathcal{Q}$ and (c) a bijection $\alpha : P \rightarrow Q$, we can construct $\prod_{p \in P} w_{|p|,|\alpha(p)|,|p|+|\alpha(p)|-1}$ distinct graphs $G \in W_{m,n,h}$ by the following steps:

1. Start with G as the graph with vertex set $M \cup N$ and no edges.
2. For each $p \in P$ add one of the subgraphs in $W_{|p|,|\alpha(p)|,|p|+|\alpha(p)|-1}$ on the vertices $p \cup \alpha(p)$.

Since all graphs in $W_{m,n,h}$ can be constructed uniquely by the above steps we have the following theorem.

Theorem 1 For $m, n, h \in \mathbb{N}$,

$$f_{m,n,h} = \sum_{i=0}^m \sum_{j=0}^n \binom{m}{i} \binom{n}{j} w_{i,j,h}$$

where

$$w_{m,n,h} = \sum_{\substack{P \in \mathcal{P} \\ |P|=m+n-h}} \sum_{\substack{Q \in \mathcal{Q} \\ |Q|=m+n-h}} \sum_{\substack{\alpha: P \rightarrow Q \\ \alpha \text{ is a bijection}}} \prod_{p \in P} w_{|p|, |\alpha(p)|, |p|+|\alpha(p)|-1} \quad (3)$$

where

$$w_{|p|, |\alpha(p)|, |p|+|\alpha(p)|-1} = |p|^{|\alpha(p)|-1} |\alpha(p)|^{|p|-1}.$$

4.2 An improved formula for $f_{m,n,h}$

We will derive two further equations for $f_{m,n,h}$ (in Theorems 2 and 3), which are related to Theorem 1. They result in a slightly more complicated algorithm, but will allow us to compute $f_{m,n,h}$ faster than (2).

For all $m, n, h \in \mathbb{N}$, define

$$W'_{m,n,h} = \{G \in W_{m,n,h} : \text{the vertices in } N \text{ have degree } \geq 2\}.$$

Given any $G \in B_{m,n,h}$ we can delete the isolated vertices and the leaves (vertices of degree 1) in N to obtain a spanning forest in $W'_{i,j,h-k}$ for some $i \leq m$ and $j \leq n$ and $k \in \mathbb{N}$. Conversely, given any $G \in W'_{i,j,h-k}$, we can add:

- $m - i$ isolated vertices to M in $\binom{m}{m-i}$ ways, so as to increase $|M|$ to m , then
- $n - j$ isolated vertices to N in $\binom{n}{n-j}$ ways, so as to increase $|N|$ to n , then
- k edges to G in $\binom{n-j}{k} m^k$ ways, so as to increase the number of leaves in N by k ,

thereby obtaining a spanning forest in $B_{m,n,h}$. Similar to the derivation of (2), we need to relabel the vertices so as to preserve the order of the indices of the non-isolated vertices within M and the non-isolated non-leaf vertices within N . Hence

$$f_{m,n,h} = \sum_{i=0}^m \sum_{j=0}^n \sum_{k=0}^{\min(n-j,h)} \binom{m}{i} \binom{n}{j, k, n-j-k} m^k w'_{i,j,h-k}.$$

All of the steps involved for finding the formula (3) for $w_{m,n,h}$ are still valid for $w'_{m,n,h}$. Hence (3) remains true if we replace w with w' . However, we can no longer make use of the simple formula for $w_{|p|, |\alpha(p)|, |p|+|\alpha(p)|-1}$. Nevertheless, we will be able to find a formula for $w'_{|p|, |\alpha(p)|, |p|+|\alpha(p)|-1}$ via Inclusion-Exclusion.

For $I \subseteq N$, define

$$A_I = \{G \in W_{m,n,m+n-1} : \text{vertices in } I \text{ are leaves}\}.$$

Note that spanning forests in A_I might also have leaves outside of I . Hence $|A_\emptyset| = |W_{m,n,m+n-1}| = m^{n-1} n^{m-1}$. By symmetry, if $I, J \subseteq N$ and $|I| = |J|$ then $|A_I| = |A_J|$. So we will assume $I = \{n - i + 1, n - i + 2, \dots, n\}$. We can construct the graphs in A_I by adding i leaves to the graphs in $W_{m,n-|I|, m+n-|I|-1}$ (i.e., the set of spanning trees of $K_{m,n-|I|}$), and these leaves can be added in m^i ways. Hence

$$\begin{aligned} |A_I| &= m^i \cdot \text{number of spanning trees of } K_{m,n-i} \\ &= m^i m^{n-i-1} (n-i)^{m-1} \\ &= m^{n-1} (n-i)^{m-1} \end{aligned}$$

when $m \geq 1$ and $n \geq 1$. Here we take $0^0 = 1$, since we want to account for $K_{1,0} \in W_{1,0,0}$. Hence, by Inclusion-Exclusion, we find

$$\begin{aligned} w'_{m,n,m+n-1} &= |A_\emptyset| - \left| \bigcup_{\substack{I \subseteq N \\ I \neq \emptyset}} A_I \right| \\ &= m^{n-1} n^{m-1} - \sum_{i=1}^n (-1)^{i-1} \binom{n}{i} m^{n-1} (n-i)^{m-1} \\ &= m^{n-1} \sum_{i=0}^n (-1)^i \binom{n}{i} (n-i)^{m-1} \end{aligned}$$

when $m \geq 1$ and $n \geq 1$. By definition, we also have $w'_{m,n,m+n-1} = 0$ when either $m = 0$ or $n = 0$ (and $w'_{m,n,m+n-1}$ is not defined when $m = 0$ and $n = 0$).

Before identifying formulae for $w_{m,n,h}$ and $w'_{m,n,h}$ in the next section, we make the following observation.

Lemma 5 *For all $m, n, h \in \mathbb{N}$, we have $n!$ divides $w'_{m,n,h}$.*

Proof: The symmetric group on N acts on $W'_{m,n,h}$ by permuting the vertices in N . Suppose a graph $G \in W'_{m,n,h}$ has a non-trivial stabiliser subgroup under this action. Then there exists a non-identity permutation α such that $\alpha G = G$, and distinct vertices $v, v' \in N$ for which $\alpha(v) = v'$. Since $G \in W'_{m,n,h}$, the degree of v is 2 or more, so assume v has distinct neighbours $a, b \in M$. Then, since $\alpha G = G$, we find that $v' = \alpha(v)$ has the neighbours a and b too. Thus, $\{v, v', a, b\}$ induces a 4-cycle, giving a contradiction. Hence, all stabiliser subgroups are trivial, and by the Orbit-Stabiliser Theorem, all orbits have size $n!$. Hence $n!$ divides $w'_{m,n,h}$. \square

4.3 Formulae for $w_{m,n,h}$ and $w'_{m,n,h}$

Now we will simplify (3), noting that our simplifications remain valid when we replace w with w' .

For $a \geq 1$ and $t \geq 1$, let $S_{a,t}$ be the set of (number) partitions of a into t non-zero parts. We will interpret elements of $S_{a,t}$ as multisets; for example $\{4, 3, 3, 1, 1\} \in S_{12,5}$. We will require $S_{a,t} = \emptyset$ when $t < 0$, and $S_{0,0} = \{\emptyset\}$ and $S_{0,t} = \emptyset$ when $t > 0$. For $z \in S_{a,t}$, let $T[z]$ be the set of partitions of $\{1, 2, \dots, a\}$ whose part sizes induce the number partition z . For $z \in S_{a,t}$, let \hat{z} denote an arbitrary element of $T[z]$. If $P \in T[x]$ and $Q \in T[z]$, then any bijection $\alpha : P \rightarrow Q$ induces a bijection $\beta : x \rightarrow z$.

Hence

$$\begin{aligned}
 w_{m,n,h} &= \sum_{x,z \in S_{m,m+n-h}} \sum_{P \in T[x]} \sum_{Q \in T[z]} \sum_{\substack{\alpha: P \rightarrow Q \\ \alpha \text{ is a bijection}}} \prod_{p \in P} w_{|p|, |\alpha(p)|, |p| + |\alpha(p)| - 1} \\
 &= \sum_{x,z \in S_{m,m+n-h}} \sum_{P \in T[x]} \sum_{Q \in T[z]} \sum_{\substack{\alpha: \hat{x} \rightarrow \hat{z} \\ \alpha \text{ is a bijection}}} \prod_{p \in \hat{x}} w_{|p|, |\alpha(p)|, |p| + |\alpha(p)| - 1} \\
 &= \sum_{x,z \in S_{m,m+n-h}} |T[x]| \cdot |T[z]| \sum_{\substack{\alpha: \hat{x} \rightarrow \hat{z} \\ \alpha \text{ is a bijection}}} \prod_{p \in \hat{x}} w_{|p|, |\alpha(p)|, |p| + |\alpha(p)| - 1} \\
 &= \sum_{x,z \in S_{m,m+n-h}} |T[x]| \cdot |T[z]| \sum_{\substack{\beta: x \rightarrow z \\ \beta \text{ is a bijection}}} \prod_{r \in x} w_{r, \beta(r), r + \beta(r) - 1}.
 \end{aligned}$$

If $z \in S_{a,t}$, then

$$|T[z]| = \frac{a!}{\prod_{i \geq 1} i! s_i(z) s_i(z)!}$$

where $s_i(z)$ denotes the number of parts i in z (see e.g. [1, Theorem 13.2]). Hence we have the following theorems.

Theorem 2 For $m, n, h \in \mathbb{N}$,

$$f_{m,n,h} = \sum_{i=0}^m \sum_{j=0}^n \binom{m}{i} \binom{n}{j} w_{i,j,h}$$

where

$$w_{m,n,h} = \sum_{x,z \in S_{m,m+n-h}} \sum_{\substack{\beta: x \rightarrow z \\ \beta \text{ is a bijection}}} \frac{m!}{\prod_{i \geq 1} i! s_i(x) s_i(x)!} \frac{n!}{\prod_{i \geq 1} i! s_i(z) s_i(z)!} \prod_{r \in x} r^{\beta(r)-1} \beta(r)^{r-1}. \quad (4)$$

Theorem 3 For $m, n, h \in \mathbb{N}$,

$$f_{m,n,h} = \sum_{i=0}^m \sum_{j=0}^n \sum_{k=0}^{\min(n-j,h)} \binom{m}{i} \binom{n}{j, k, n-j-k} m^k w'_{i,j,h-k}.$$

where

$$w'_{m,n,h} = \sum_{x,z \in S_{m,m+n-h}} \sum_{\substack{\beta: x \rightarrow z \\ \beta \text{ is a bijection}}} \frac{m!}{\prod_{i \geq 1} i! s_i(x) s_i(x)!} \frac{n!}{\prod_{i \geq 1} i! s_i(z) s_i(z)!} \prod_{r \in x} w'_{r, \beta(r), r + \beta(r) - 1}$$

where

$$w'_{m,n,m+n-1} = m^{n-1} \sum_{i=0}^n (-1)^i \binom{n}{i} (n-i)^{m-1}.$$

The main advantage of Theorems 2 and 3 over Theorem 1 is the summation is over number partitions (rather than set partitions).

5 Implementation

The author has implemented the two formulae for $f_{m,n,h}$ described in the preceding section, along with a simple backtracking algorithm. The results between all three implementations concur, giving confidence in the accuracy of the code. The C source code is available as supplementary material to this document. The GMP library was used for arbitrary precision arithmetic [6].

The formula involving $w'_{m,n,h}$ instead of $w_{m,n,h}$ was (unsurprisingly) much faster, largely because many values of $w'_{m,n,h}$ equal 0, such as when $m \leq n$, which can be used to drastically reduce the search tree.

5.1 Symmetry breaking

The run-time of the program was also improved through the use of symmetry breaking, which we will now describe. For $x \in S_{m,m+n-h}$ and $z \in S_{n,m+n-h}$, there are often many bijections $\beta : x \rightarrow z$ which map the same elements to the same elements (since both x and z are multisets). In these instances, a naïve algorithm would repeat the same computation unnecessarily. Let x and z be the multisets $x = \{x_1, x_2, \dots, x_{m+n-h}\}$ and $z = \{z_1, z_2, \dots, z_{m+n-h}\}$. We define the condition:

Symmetry breaking condition 1: We say $\beta : x \rightarrow z$ is *half-canonical* if $\beta^{-1}(z_{i-1}) < \beta^{-1}(z_i)$ whenever $z_{i-1} = z_i$.

We need to add a multiplicative factor to adjust for the restriction to half-canonical bijections. Hence

$$w_{m,n,h} = \sum_{x,z \in S_{m,m+n-h}} \sum_{\substack{\beta: x \rightarrow z \\ \beta \text{ is a bijection} \\ \beta \text{ is half-canonical}}} \left(\prod_{i \geq 1} i^{s_i(z)} \right) \frac{m!}{\prod_{i \geq 1} i^{s_i(x)} s_i(x)!} \frac{n!}{\prod_{i \geq 1} i^{s_i(z)} s_i(z)!} \prod_{r \in x} w_{r, \beta(r), r + \beta(r) - 1}$$

and similarly with w' in place of w . Using this assumption, we reduce the number of bijections by a factor of $\prod_{i \geq 1} i^{s_i(z)}$, which results in a substantial time saving.

Instead of the single symmetry breaking assumption, it is possible to utilise symmetry breaking using an additional condition:

Symmetry breaking condition 2: We say $\beta : x \rightarrow z$ is *canonical* if it is half-canonical and $\beta(x_{i-1}) < \beta(x_i)$ whenever $x_i = x_{i-1}$.

Again, we find

$$w_{m,n,h} = \sum_{x,z \in S_{m,m+n-h}} \sum_{\substack{\beta: x \rightarrow z \\ \beta \text{ is a bijection} \\ \beta \text{ is canonical}}} \Gamma_{x,z,\beta} \frac{m!}{\prod_{i \geq 1} i^{s_i(x)} s_i(x)!} \frac{n!}{\prod_{i \geq 1} i^{s_i(z)} s_i(z)!} \prod_{r \in x} w_{r, \beta(r), r + \beta(r) - 1},$$

and similarly with w' in place of w , where $\Gamma_{x,z,\beta}$ is the number of bijections between the multisets x and z which map the same elements to the same elements (as β). A formula for $\Gamma_{x,z,\beta}$ was given by e.g. [3], namely

$$\Gamma_{x,z,\beta} = \frac{\prod_{i \geq 1} s_i(x)! \prod_{i \geq 1} s_i(z)!}{\prod_{i \geq 1} \prod_{j \geq 1} s_{i,j}(x, \beta)!}$$

where

- $s_i(z)$ denotes the number of parts i in z (as before), and
- $s_{i,j}(x, \beta)$ is the number of elements (i, j) in the multiset $\{(r, \beta(r)) : r \in x\}$.

The author has implemented both of these symmetry breaking schemes in order to compare their performance (see Section 5.3).

5.2 Pseudo-code

Algorithm 1 gives a pseudo-code version of the C code used to implement the algorithm described by Theorem 3 using the half-canonical symmetry breaking condition. The partitions of m and n into k parts were computed whenever needed and stored in memory. Iterating through the half-canonical bijections was performed “on the fly” using a backtracking algorithm.

While Theorems 2 and 3 are valid for all $m, n, h \in \mathbb{N}$, we need to set $f_{m,n,0} = 1$ separately in the C code.

5.3 Performance

The C code was run on a 2×2.66 GHz processor (although the code itself is not parallelised). The following table gives the run-times (in seconds) for the two algorithms under the two symmetry breaking schemes when computing all non-zero $f_{m,n,h}$ with $m, n \leq 19$.

	half-canonical	canonical
Theorem 2	27.1	6.8
Theorem 3	4.1	4.5

The fastest version is Theorem 3 using half-canonical symmetry breaking. Under these conditions, the code had the following run-times to find all non-zero values of $f_{m,n,h}$ with $m, n \leq t$:

t	time (sec)
25	20.5
26	27.7
27	35.4
28	46.0
29	62.2
30	83.0

This table indicates the scalability of the program, which is not overwhelming for these values of t . The author also ran the program to compute $f_{m,n,h}$ with $m, n \leq t$ where $t = 50$, which took under 2 days (to be precise, it took 1 day 15 hours and 17 minutes). The largest number encountered was $f_{50,50,101}$, which has 167 digits, and is equal to the number of spanning trees of $K_{50,50}$, which is $50^{50-1} \cdot 50^{50-1} = 50^{98}$.

5.4 Complexity

In the worst case, $x = z = \{1, 2, 3, \dots, t\}$, where $\min(m, n) = 1 + 2 + \dots + t = \frac{1}{2}t(t + 1)$ (so $t = O(\sqrt{\min(m, n)})$), in which case the program must iterate through all $t!$ bijections from x to z . So, for a worst case analysis, we assume $m = n$ and $m + n - h = \lfloor \sqrt{m} \rfloor$.

Algorithm 1 Implementation of Theorem 3 using half-canonical symmetry breaking

```

 $f_{m,n,m+n-1} := 0$ ; for all  $m, n, h$ 
 $w'_{m,n,m+n-1} := 0$ ; for all  $m, n, h$ 
 $w'_{0,0,0} := 1$ ;
for  $m = 0, 1, \dots, MAX(m)$  do
  for  $n = 0, 1, \dots, MAX(n)$  do
    if  $m > n$  and  $m > 0$  and  $n > 0$  then // Other values of  $w'_{m,n,h}$  are 0
      for  $i = 0, 1, \dots, n$  do
        // Using the Inclusion-Exclusion formula
         $w'_{m,n,m+n-1} := w'_{m,n,m+n-1} + (-1)^i \binom{n}{i} (n-i)^{m-1}$ ;
      end for
      for  $h = 0, 1, \dots, m+n-2$  do
         $w'_{m,n,h} := 0$ ;
        Compute all partitions of  $m$  into  $m+n-h$  parts and store in memory.
        Compute all partitions of  $n$  into  $m+n-h$  parts and store in memory.
        for partitions  $x$  of  $m$  into  $m+n-h$  parts do
          for partitions  $z$  of  $n$  into  $m+n-h$  parts do
            for half-canonical bijections  $\beta : x \rightarrow z$  do // via backtracking algorithm
               $\Gamma := \prod_{i \geq 1} i^{s_i(z)}$ ;
               $w'_{m,n,h} := w'_{m,n,h} + \Gamma \cdot \frac{m!}{\prod_{i \geq 1} i^{s_i(x)} s_i(x)!} \frac{n!}{\prod_{i \geq 1} i^{s_i(z)} s_i(z)!} \prod_{r \in x} w'_{r, \beta(r), r + \beta(r) - 1}$ ;
            end for
          end for
        end for
      end for
    end if
     $f_{m,n,0} = 1$ ;
    for  $h = 1, 2, \dots, m+n-1$  do
      for  $i = 0, 1, \dots, m$  do
        for  $j = 0, 1, \dots, n$  do
          for  $k = 0, 1, \dots, \min(n-j, h)$  do
             $f_{m,n,h} := f_{m,n,h} + \binom{m}{i} \binom{n}{j, k, n-j-k} m^k w'_{i,j,h-k}$ ;
          end for
        end for
      end for
    end for
  end for
end for

```

Hence, given $x, z \in S_{m,m+n-h}$, there can be $O(\lfloor \sqrt{m} \rfloor!)$ canonical bijections from x to z . Hardy and Ramanujan's asymptotic formula for the number of partitions of m , namely

$$\frac{1}{4m\sqrt{3}} e^{\pi\sqrt{2m/3}},$$

gives a crude asymptotic upper bound on $|S_{m,\lfloor \sqrt{m} \rfloor}|^2$, specifically

$$|S_{m,\lfloor \sqrt{m} \rfloor}|^2 = O\left(\frac{e^{\text{const}\cdot m}}{m^2}\right)$$

as the number of ways of choosing x and z . Hence, (4) has

$$O\left(e^{\text{const}\cdot\sqrt{m}} m^{\sqrt{m}/2+\text{const}}\right)$$

terms, by Stirling's Approximation. In contrast, a backtracking algorithm would need to generate and check around $\binom{m^2}{\lfloor \sqrt{m} \rfloor} \leq \frac{1}{\lfloor \sqrt{m} \rfloor!} m^{2\lfloor \sqrt{m} \rfloor}$ graphs, which, when $m = n$ and $h = 2m - \sqrt{m}$, is

$$O(e^{\sqrt{m}} m^{1.5\sqrt{m}+\text{const}})$$

iterations, by Stirling's Approximation. Of course, when implementing these algorithms, we use pruning whenever possible to reduce the search space, which makes a drastic difference not accounted for in these approximations.

5.4.1 When h is fixed

We conclude this paper with the observation that, when h is fixed, computing $f_{m,n,h}$ is asymptotically "easy". The underlying reason is that, for sufficiently large m or n , we must have isolated vertices in graphs in $B_{m,n,h}$. Thus, (2) only contains a finite number of non-zero terms.

Theorem 4 For fixed h , computing $f_{m,n,h}$ can be performed in time $O(\log(mn))$.

Proof: If $i > h$ or $j > h$, then $w_{i,j,h} = 0$ (since any graph in $B_{i,j,h}$ must have an isolated vertex). Hence, (2) is equivalent to

$$f_{m,n,h} = \sum_{i=0}^h \sum_{j=0}^h \binom{m}{i} \binom{n}{j} w_{i,j,h}.$$

For fixed h , there is a finite number of terms in this sum. Thus, for fixed h , we could write a program in which:

- we store a list of the pairs (i, j) for which $w_{i,j,h}$ is non-zero, along with the value of $w_{i,j,h}$,
- we iterate through this list, computing $\binom{m}{i} \binom{n}{j} w_{i,j,h}$, and add it to a running total.

We can compute $\binom{m}{i} = \frac{1}{i!} m(m-1)\cdots(m-i+1)$ using $O(h)$ multiplications (since $i \leq h$), each of which takes time $O(\log m)$, and one division. Hence, $\binom{m}{i}$ can be computed in time $O(\log m)$ time (since h is fixed). Similarly $\binom{n}{j}$ can be computed in $O(\log n)$ time. We conclude that the whole summation can be performed in time $O(\log(mn))$. \square

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