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# On Hamiltonian Paths and Cycles in Sufficiently Large Distance Graphs

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For a positive integer  $n \in \mathbb{N}$  and a set  $D \subseteq \mathbb{N}$ , the distance graph  $G_n^D$  has vertex set  $\{0, 1, \dots, n-1\}$  and two vertices  $i$  and  $j$  of  $G_n^D$  are adjacent exactly if  $|j-i| \in D$ . The condition  $\gcd(D) = 1$  is necessary for a distance graph  $G_n^D$  being connected. Let  $D = \{d_1, d_2\} \subseteq \mathbb{N}$  be such that  $d_1 > d_2$  and  $\gcd(d_1, d_2) = 1$ . We prove the following results.

- If  $n$  is sufficiently large in terms of  $D$ , then  $G_n^D$  has a Hamiltonian path with endvertices 0 and  $n-1$ .
- If  $d_1 d_2$  is odd,  $n$  is even and sufficiently large in terms of  $D$ , then  $G_n^D$  has a Hamiltonian cycle.
- If  $d_1 d_2$  is even and  $n$  is sufficiently large in terms of  $D$ , then  $G_n^D$  has a Hamiltonian cycle.

**Keywords:** Distance graph; Toeplitz graph; circulant graph; Hamiltonian path; Hamiltonian cycle; traceability

## 1 Introduction

For a finite set of positive integers  $D \subseteq \mathbb{N}$ , the *infinite distance graph*  $G^D$  has vertex set  $V(G^D) = \mathbb{Z}$  and two vertices  $u$  and  $v$  of  $G^D$  are adjacent exactly if  $|u-v| \in D$ . For a graph  $G$  and a subset  $U \subseteq V(G)$  of the vertex set, we denote by  $G[U]$  the subgraph of  $G$  induced by  $U$ . For  $i, j \in \mathbb{Z}$ ,  $i \leq j$ , we denote by  $[i, j] = \{k \in \mathbb{Z} \mid i \leq k \leq j\}$ . For a positive integer  $n \in \mathbb{N}$ , the *distance graph* (also called *Toeplitz graph* in many papers)  $G_n^D = G^D[[0, n-1]]$  is the subgraph of  $G^D$  induced by the vertices in  $[0, n-1]$ .

Infinite distance graphs and especially their colourings were first studied by Eggleton, Erdős, and Skilton [10, 11]. Most of the research on distance graphs focused on their colourings [6, 8, 9, 14, 18, 19, 28]. Distance graphs generalize the very well-studied class of *circulant graphs* [2, 16, 17, 26]. In fact, circulant graphs coincide exactly with the regular distance graphs [23]. Circulant graphs have been proposed for numerous network applications and many of their properties such as connectedness and diameter [4, 2, 16, 17], cycle and path structure [1, 3, 5], and isomorphism testing and recognition [12, 22] have

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been studied in great detail. Several fundamental results concerning circulant graphs were extended to the more general class of distance graphs in [7, 23, 24, 25]. The complexity of the connectedness problem for distance graphs was recently settled by Gómez et al. [13]. In [25, 27, 15] the existence of long paths and cycles in distance graphs is studied. The following main result from [21] confirmed a conjecture from Penso et al. [25]. [20] gives an overview on Hamiltonian cycles and paths in vertex-transitive graphs.

**Theorem 1 (Löwenstein et al. [21])** *For a finite set  $D \subseteq \mathbb{N}$  with  $|D| \geq 2$  and  $\gcd(D) = 1$ , there are infinitely many  $n \in \mathbb{N}$  such that  $G_n^D$  has a Hamiltonian cycle and  $G_{n+1}^D$  has a Hamiltonian path with endvertices 0 and  $n$ .*

We conjecture that the conclusion of the last theorem holds

- for all  $n$  that are sufficiently large in terms of  $D$  if not all elements of  $D$  are odd and
- for all even  $n$  that are sufficiently large in terms of  $D$  if all elements of  $D$  are odd.

The purpose of the present paper is to confirm this conjecture in the case that  $D$  contains just two elements. In Section 2 we introduce suitable terminology and collect some properties of distance graphs. In Section 3 we confirm our conjecture proving the existence of Hamiltonian paths. Finally, in Section 4 we provide similar results for Hamiltonian cycles.

## 2 The structure of $G^D$

Let  $D = \{d_1, d_2\}$  for two positive integers  $d_1$  and  $d_2$  such that  $\gcd(d_1, d_2) = 1$  and  $d_1 > d_2$ .

We define coordinates  $(x, y) \in (\mathbb{Z}/(d_1 + d_2)\mathbb{Z}) \times \mathbb{Z}$  for the vertices of the distance graph  $G^D$  by

$$(x, y) := y(d_1 + d_2) + a_x,$$

where  $a_x = xd_1 \pmod{d_1 + d_2}$ . Note that this bidimensional relabelling of the vertices of  $G^D$  is a bijection. A vertex  $(x, y)$  satisfying  $0 \leq xd_1 \pmod{d_1 + d_2} < d_2$  is called *lower*. A vertex  $(x, y)$  satisfying  $d_2 \leq xd_1 \pmod{d_1 + d_2} < d_1$  is called *middle*. A vertex  $(x, y)$  satisfying  $d_1 \leq xd_1 \pmod{d_1 + d_2} < d_1 + d_2$  is called *upper*.

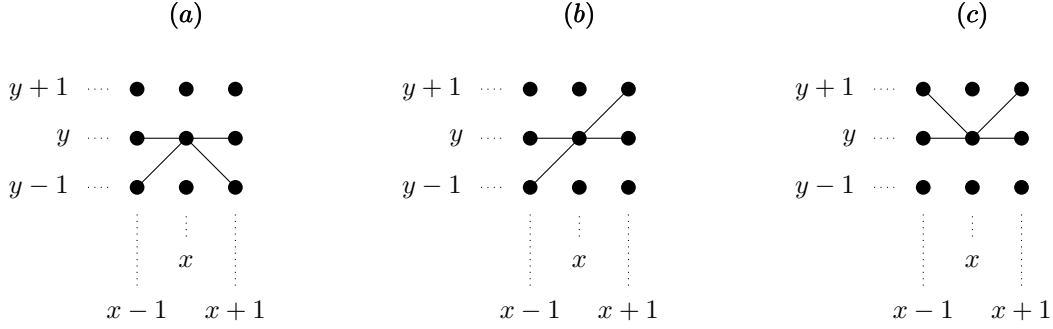
For a lower vertex  $(x, y)$ , we have

$$\begin{aligned} (x, y) + d_1 &= (x + 1, y), \\ (x, y) + d_2 &= (x - 1, y), \\ (x, y) - d_1 &= (x - 1, y - 1), \\ (x, y) - d_2 &= (x + 1, y - 1), \end{aligned}$$

which implies that a lower vertex  $(x, y)$  is adjacent to the vertices  $(x + 1, y)$ ,  $(x - 1, y)$ ,  $(x + 1, y - 1)$ , and  $(x - 1, y - 1)$ .

Similarly, for a middle vertex  $(x, y)$ , we have

$$\begin{aligned} (x, y) + d_1 &= (x + 1, y + 1), \\ (x, y) + d_2 &= (x - 1, y), \\ (x, y) - d_1 &= (x - 1, y - 1), \\ (x, y) - d_2 &= (x + 1, y), \end{aligned}$$



**Fig. 1:** Neighborhood of (a) a lower, (b) a middle, and (c) an upper vertex.

which implies that a middle vertex  $(x, y)$  is adjacent to the vertices  $(x + 1, y)$ ,  $(x - 1, y)$ ,  $(x + 1, y + 1)$ , and  $(x - 1, y - 1)$ .

Finally, for an upper vertex  $(x, y)$ , we have

$$\begin{aligned} (x, y) + d_1 &= (x + 1, y + 1), \\ (x, y) + d_2 &= (x - 1, y + 1), \\ (x, y) - d_1 &= (x - 1, y), \\ (x, y) - d_2 &= (x + 1, y), \end{aligned}$$

which implies that an upper vertex  $(x, y)$  is adjacent to the vertices  $(x + 1, y)$ ,  $(x - 1, y)$ ,  $(x + 1, y + 1)$ , and  $(x - 1, y - 1)$ .

See Figure 1 for an illustration of these observations.

For  $c \in \mathbb{Z}/(d_1 + d_2)\mathbb{Z}$ , all vertices  $(x, y)$  of  $G^D$  with  $x = c$  form the *column*  $c$ . Similarly, for  $r \in \mathbb{Z}$ , all vertices  $(x, y)$  satisfying  $y = r$  form the *row*  $r$ . Note that the vertices in a column are either all lower, or all middle, or all upper. A column that consists of lower (middle, upper) vertices is called *lower (middle, upper)*. See Figure 2 for an illustration.

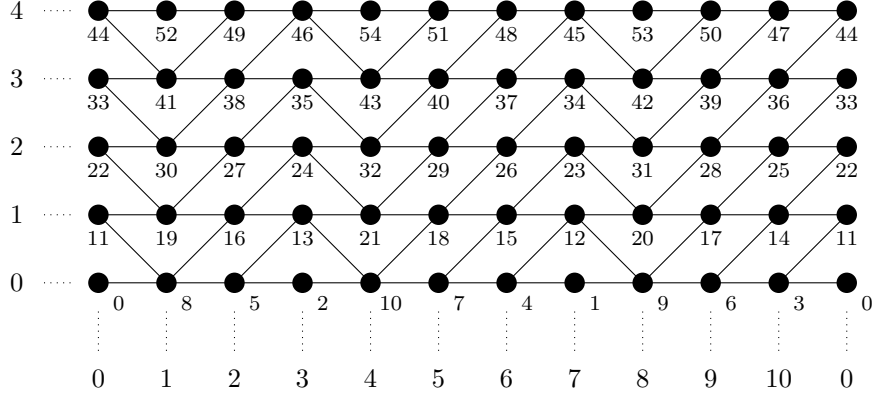
**Lemma 2** (i) For  $c \in \mathbb{Z}/(d_1 + d_2)\mathbb{Z}$ , the column  $c$  is lower if and only if the column  $c + 1$  is upper.

(ii) Column 0 is lower.

(iii) Column 1 is upper.

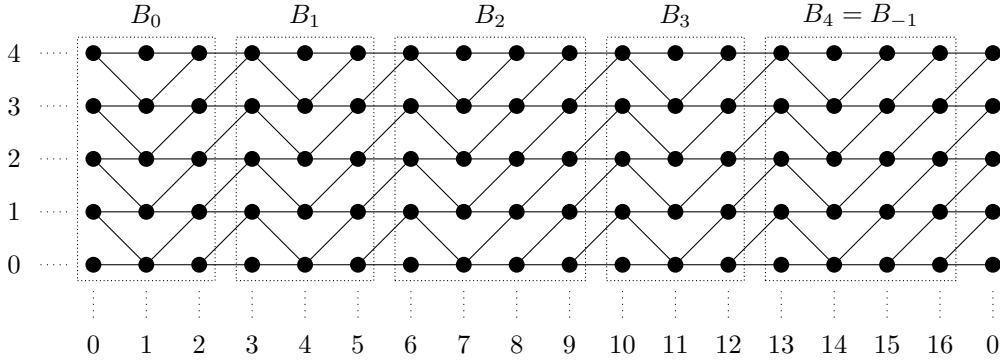
**Proof:** For  $x \in \mathbb{Z}/(d_1 + d_2)\mathbb{Z}$ , we have  $0 \leq xd_1 \pmod{d_1 + d_2} < d_2$  if and only if  $d_1 \leq (x + 1)d_1 \pmod{d_1 + d_2} < d_1 + d_2$ , which proves (i). (ii) follows, because  $0 \leq 0 = 0d_1 \pmod{d_1 + d_2} < d_2$ . Finally, (i) and (ii) imply (iii).  $\square$

The columns  $x, x + 1, \dots, x + l - 1$  form a *block of length*  $l$ , if column  $x$  is lower, column  $x + l$  is lower, and none of the columns  $x + 1, \dots, x + l - 1$  is lower. The block that contains column 0 is denoted by  $B_0$ . Let  $l$  be the length of block  $B_i$  and let column  $x$  be the unique lower column that belongs to block



**Fig. 2:** The distance graph  $G_{55}^{\{8,3\}}$ . Note that the vertices of column 0 are drawn twice. In order to simplify the drawing, we adopt the convention that such a vertex is adjacent to the union of the neighbors of the two copies, i.e. vertex 22 is adjacent to the vertices 19, 30, 14, and 25.

$B_i$ , then the block that contains column  $x + l$  is denoted by  $B_{i+1}$ . Note that the indices of the blocks are elements of  $\mathbb{Z}/d_2\mathbb{Z}$ . For  $i \in \mathbb{Z}/d_2\mathbb{Z}$ , let  $x_i$  denote the unique lower column in block  $B_i$ . Figure 3 shows the blocks of  $G_{85}^{\{12,5\}}$ .



**Fig. 3:** Blocks of  $G_{85}^{\{12,5\}}$ . Note that 4 equals  $-1$  in  $\mathbb{Z}/5\mathbb{Z}$ , that is,  $B_4 = B_{-1}$ .

**Lemma 3** (i) The length of a block is either  $\lfloor \frac{d_1}{d_2} \rfloor + 1$  or  $\lceil \frac{d_1}{d_2} \rceil + 1$ .

(ii) The length of  $B_0$  is  $\lfloor \frac{d_1}{d_2} \rfloor + 1$ .

(iii) The length of  $B_{-1}$  is  $\lceil \frac{d_1}{d_2} \rceil + 1$ .

(iv) The number of blocks is  $d_2$ .

**Proof:** Let  $x, x+1, \dots, x+l-1$  be the columns of a block  $B$  of length  $l$ . By definition and Lemma 2 (i),  $x$  is the unique lower column of block  $B$ ,  $x+1$  is the unique upper column of block  $B$ , and  $x+l$  is a lower column. Hence, for all  $y \in \mathbb{Z}$  and  $x+1 \leq k \leq x+l-1$ , we have  $(k, y) - (k+1, y) = d_2$  and therefore  $(x+1, y) - (x+l, y) = d_2(l-1)$ . Since column  $x+1$  is upper and column  $x+l$  is lower, we have  $d_1 - d_2 + 1 \leq (x+1, y) - (x+l, y) \leq d_1 + d_2 - 1$ , which implies (i).

If  $B = B_0$ , then  $x = 0$  and  $(x+1, y) \equiv d_1 \pmod{d_1+d_2}$  for all  $y \in \mathbb{Z}$ . Hence  $(x+1, y) - (x+l, y) \leq d_1$ . Together with  $(x+1, y) - (x+l, y) = d_2(l-1)$  this implies (ii).

If  $B = B_{-1}$ , then  $x+l = 0$  and  $(x+l, y) \equiv 0 \pmod{d_1+d_2}$  for all  $y \in \mathbb{Z}$ . Since column  $x+1$  is upper, we have  $(x+1, y) - (x+l, y) \geq d_1$ . Together with  $(x+1, y) - (x+l, y) = d_2(l-1)$  this implies (iii).

Since the function  $f : \{0, \dots, d_1+d_2-1\} \rightarrow \{0, \dots, d_1+d_2-1\}$  with  $f(x) = xd_1 \pmod{d_1+d_2}$  is bijective for  $\gcd(d_1, d_2) = 1$ , there are exactly  $d_2$  lower columns and therefore  $d_2$  blocks, which proves (iv).  $\square$

### 3 Hamiltonian paths of $G_n^D$

The main result of this section is the following.

**Theorem 4** For every  $D = \{d_1, d_2\} \subseteq \mathbb{N}$  with  $d_1 > d_2$  and  $\gcd(d_1, d_2) = 1$ , there is some  $n_0 \in \mathbb{N}$  such that for all integers  $n$  with  $n \geq n_0$ , the distance graph  $G_n^D$  has a Hamiltonian path with endvertices 0 and  $n-1$ .

As before let  $D = \{d_1, d_2\}$  for two positive integers  $d_1$  and  $d_2$  such that  $\gcd(d_1, d_2) = 1$  and  $d_1 > d_2$ . For two lower vertices  $(x, y)$  and  $(x', y')$  with  $x \neq x'$  and  $y < y'$  in the distance graph  $G^D$ , a path in  $G^D$  with endvertices  $(x, y)$  and  $(x', y')$  whose vertex set consists of all vertices in the rows  $y, y+1, \dots, y'-1$  and the vertex  $(x', y')$  is called an  $(x, y)$ - $(x', y')$ -climbing path of  $G^D$ . See Figure 5 for an illustration.

Before we proceed to the proof of Theorem 4, we establish a series of lemmas concerning the existence of climbing paths.

**Lemma 5** If  $B_i$  is a block of even length in  $G^D$ , then  $G^D$  has an  $(x_i, y)$ - $(x_{i+1}, y+2)$ -climbing path for all  $y$ .

**Proof:** Let

$$\begin{aligned} P & : (x_{i+1}-1, y), (x_{i+1}, y+1), (x_{i+1}-1, y+1), (x_{i+1}-2, y), \\ & (x_{i+1}-3, y), (x_{i+1}-2, y+1), (x_{i+1}-3, y+1), (x_{i+1}-4, y), \\ & \dots, (x_i+3, y), (x_i+4, y+1), (x_i+3, y+1), (x_i+2, y). \end{aligned}$$

The sequence

$$\begin{aligned} & (x_i, y), (x_i-1, y), \dots, (x_{i+1}, y), \\ & P, (x_i+1, y), \\ & (x_i+2, y+1), (x_i+1, y+1), \dots, (x_{i+1}+1, y+1), \\ & (x_{i+1}, y+2) \end{aligned}$$

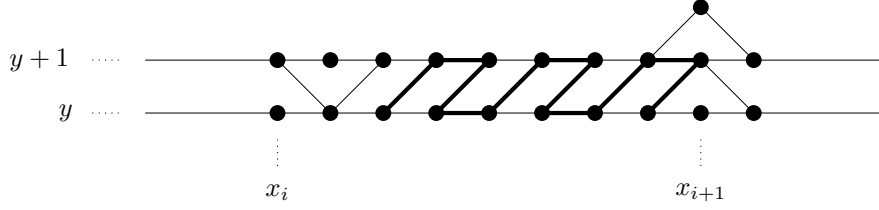


Fig. 4:  $P$  for a block  $B_i$  of length 8.

defines an  $(x_i, y)$ - $(x_{i+1}, y + 2)$ -climbing path in  $G^D$ . See Figures 4 and 5 for an illustration. □

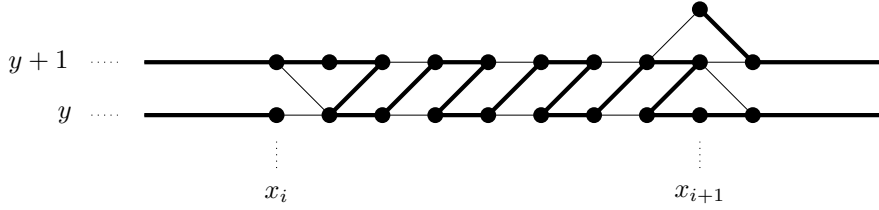


Fig. 5: An  $(x_i, y)$ - $(x_{i+1}, y + 2)$ -climbing path for block  $B_i$  of length 8.

**Lemma 6** *If  $B_{i-1}$  is a block of even length in  $G^D$ , then  $G^D$  has an  $(x_i, y)$ - $(x_{i-1}, y + 2)$ -climbing path for all  $y$ .*

**Proof:** Let

$$\begin{aligned}
 P & : (x_{i-1} + 3, y + 1), (x_{i-1} + 2, y), (x_{i-1} + 3, y), (x_{i-1} + 4, y + 1), \\
 & (x_{i-1} + 5, y + 1), (x_{i-1} + 4, y), (x_{i-1} + 5, y), (x_{i-1} + 6, y + 1), \\
 & \dots, (x_i - 1, y + 1), (x_i - 2, y), (x_i - 1, y), (x_i, y + 1).
 \end{aligned}$$

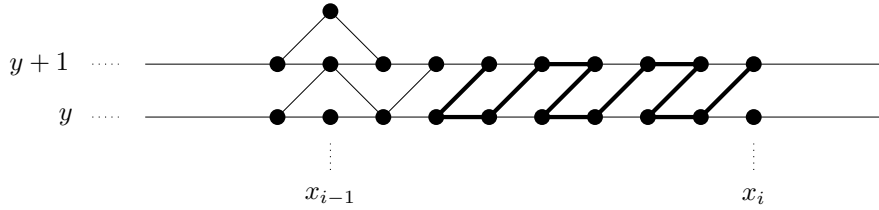
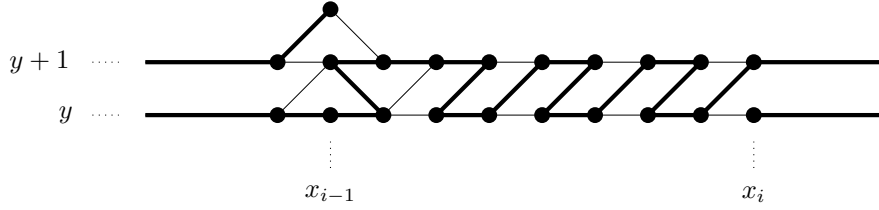


Fig. 6:  $P$  for a block  $B_{i-1}$  of length 8.

The sequence

$$\begin{aligned} & (x_i, y), (x_i + 1, y), \dots, (x_{i-1} + 1, y), \\ & (x_{i-1}, y + 1), (x_{i-1} + 1, y + 1), (x_{i-1} + 2, y + 1), P, \\ & (x_i + 1, y + 1), (x_i + 2, y + 1), \dots, (x_{i-1} - 1, y + 1), \\ & (x_{i-1}, y + 2) \end{aligned}$$

defines an  $(x_i, y)$ - $(x_{i-1}, y + 2)$ -climbing path of  $G^D$ . See Figures 6 and 7 for an illustration.  $\square$



**Fig. 7:** An  $(x_i, y)$ - $(x_{i-1}, y + 2)$ -climbing path for block  $B_{i-1}$  of length 8.

**Lemma 7** *If  $G^D$  has at least  $j + 2$  blocks for some  $j \geq 1$  and for some  $i \in \mathbb{Z}/d_2\mathbb{Z}$ , the blocks  $B_i, B_{i+1}, \dots, B_{i+j}$  of  $G^D$  are such that  $B_i$  and  $B_{i+j}$  are of odd length and  $B_{i+1}, \dots, B_{i+j-1}$  are of even length at least 4, then  $G^D$  has an  $(x_i, y)$ - $(x_{i+j+1}, y + 3)$ -climbing path for all  $y$ .*

**Proof:** By Lemma 3, the blocks  $B_i$  and  $B_{i+j}$  are of length at least 3.

Let

$$\begin{aligned} P_{i+j} & : (x_{i+j+1} - 1, y), (x_{i+j+1}, y + 1), (x_{i+j+1} - 1, y + 1), (x_{i+j+1} - 2, y), \\ & (x_{i+j+1} - 3, y), (x_{i+j+1} - 2, y + 1), (x_{i+j+1} - 3, y + 1), (x_{i+j+1} - 4, y), \\ & \dots, (x_{i+j} + 2, y), (x_{i+j} + 3, y + 1), (x_{i+j} + 2, y + 1), (x_{i+j} + 1, y). \end{aligned}$$

For  $1 \leq q \leq j - 1$ , let

$$\begin{aligned} P_{i+q} & : (x_{i+q} + 3, y + 2), (x_{i+q} + 2, y + 1), (x_{i+q} + 3, y + 1), (x_{i+q} + 4, y + 2), \\ & (x_{i+q} + 5, y + 2), (x_{i+q} + 4, y + 1), (x_{i+q} + 5, y + 1), (x_{i+q} + 6, y + 2), \\ & \dots, (x_{i+q+1} - 3, y + 2), (x_{i+q+1} - 4, y + 1), (x_{i+q+1} - 3, y + 1), (x_{i+q+1} - 2, y + 2) \end{aligned}$$

and let

$$\begin{aligned} P'_{i+q} & : P_{i+q}, (x_{i+q+1} - 1, y + 2), (x_{i+q+1} - 2, y + 1), (x_{i+q+1} - 1, y + 1), (x_{i+q+1}, y + 1), \\ & (x_{i+q+1} + 1, y + 1), (x_{i+q+1}, y + 2), (x_{i+q+1} + 1, y + 2), (x_{i+q+1} + 2, y + 2). \end{aligned}$$

Note that  $P_{i+q}$  is empty if  $B_{i+q}$  is of length 4. Furthermore, let

$$\begin{aligned} P_i & : (x_{i+1} - 2, y + 2), (x_{i+1} - 3, y + 1), (x_{i+1} - 4, y + 1), (x_{i+1} - 3, y + 2), \\ & (x_{i+1} - 4, y + 2), (x_{i+1} - 5, y + 1), (x_{i+1} - 6, y + 1), (x_{i+1} - 5, y + 2), \\ & \dots, (x_i + 3, y + 2), (x_i + 2, y + 1), (x_i + 1, y + 1), (x_i + 2, y + 2). \end{aligned}$$



Note that  $P_i$  is empty if  $B_i$  is of length 3.

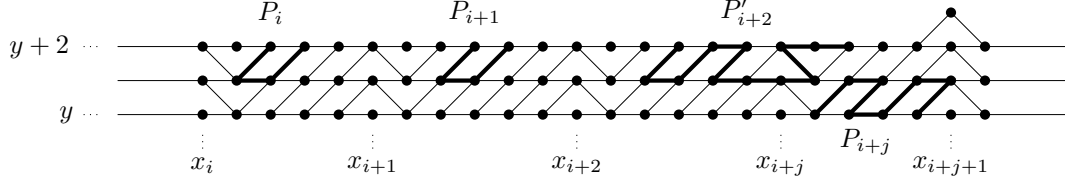


Fig. 8:  $P_i, P_{i+1}, P'_{i+2}$ , and  $P_{i+j}$  for  $j = 3$ .

Now, the sequence

$$\begin{aligned}
 & (x_i, y), (x_i - 1, y), \dots, (x_{i+j+1}, y), P_{i+j}, \\
 & (x_{i+j}, y), (x_{i+j} - 1, y), \dots, (x_{i+1} - 1, y), \\
 & (x_{i+1}, y + 1), (x_{i+1} + 1, y + 1), (x_{i+1}, y + 2), (x_{i+1} + 1, y + 2), (x_{i+1} + 2, y + 2), \\
 & P'_{i+1}, P'_{i+2}, \dots, P'_{i+j-1}, \\
 & (x_{i+j} + 3, y + 2), (x_{i+j} + 4, y + 2), \dots, (x_{i+j+1}, y + 2), \\
 & (x_{i+j+1} + 1, y + 1), (x_{i+j+1} + 2, y + 1), \dots, (x_i, y + 1), \\
 & (x_i + 1, y), (x_i + 2, y), \dots, (x_{i+1} - 2, y), \\
 & (x_{i+1} - 1, y + 1), (x_{i+1} - 2, y + 1), (x_{i+1} - 1, y + 2), P_i, \\
 & (x_i + 1, y + 2), (x_i, y + 2), \dots, (x_{i+j+1} + 1, y + 2), \\
 & (x_{i+j+1}, y + 3)
 \end{aligned}$$

defines an  $(x_i, y)$ - $(x_{i+j+1}, y + 3)$ -climbing path of  $G^D$ . See Figures 8 and 9 for an illustration.  $\square$

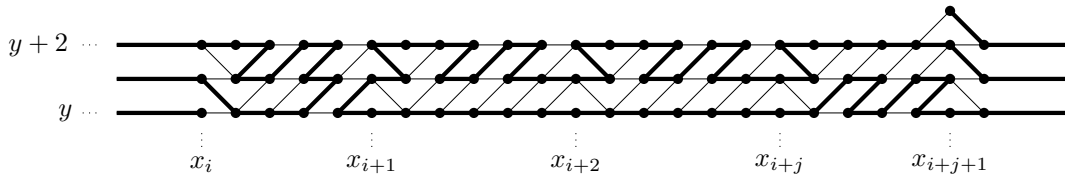


Fig. 9: An  $(x_i, y)$ - $(x_{i+j+1}, y + 3)$ -climbing path for  $j = 3$ .

**Lemma 8** *If  $G^D$  has at least  $j + 2$  blocks for some  $j \geq 1$  and for some  $i \in \mathbb{Z}/d_2\mathbb{Z}$ , the blocks  $B_i, B_{i+1}, \dots, B_{i+j}$  of  $G^D$  are such that  $B_i$  and  $B_{i+j}$  are of length 3 and  $B_{i+1}, \dots, B_{i+j-1}$  are of length 2, then  $G^D$  has an  $(x_i, y)$ - $(x_{i+j+1}, y + j + 2)$ -climbing path for all  $y$ .*

**Proof:** Note that  $x_{i+j+1} = x_i + 2j + 4$ . For  $1 \leq q \leq j - 1$ , let

$$P_q : (x_i + 2j + 2, y + q), (x_i + 2j + 3, y + q + 1), (x_i + 2j + 4, y + q + 2),$$

$$\begin{aligned}
& (x_i + 2j + 5, y + q + 1), (x_i + 2j + 6, y + q + 1), \dots, (x_i, y + q + 1), \\
& (x_i + 1, y + q), (x_i + 2, y + q), \dots, (x_i + 2j - 2q + 2, y + q), \\
& (x_i + 2j - 2q + 1, y + q + 1), (x_i + 2j - 2q + 2, y + q + 1), \dots, (x_i + 2j + 1, y + q + 1).
\end{aligned}$$

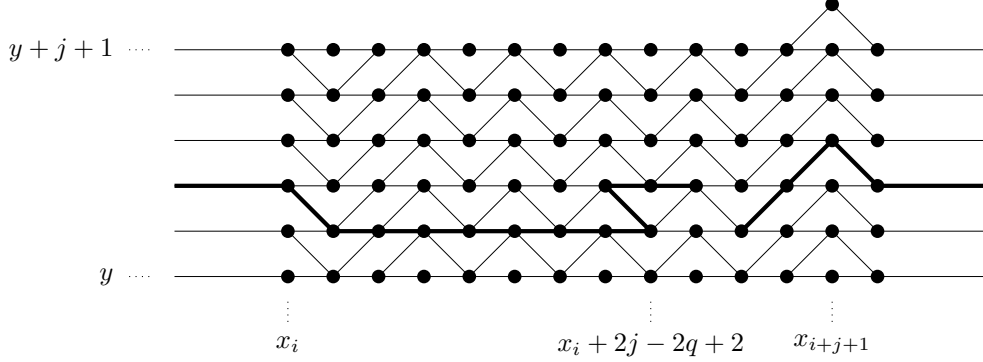


Fig. 10:  $P_q$  for  $j = 4$  and  $q = 1$ .

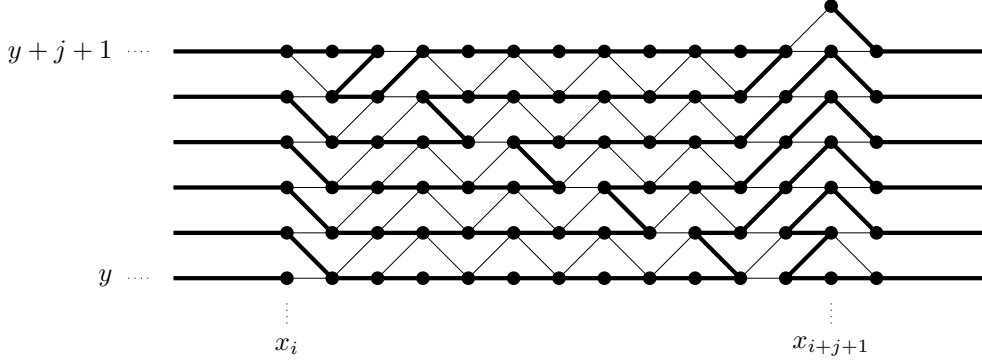
Now, the sequence

$$\begin{aligned}
& (x_i, y), (x_i - 1, y), \dots, (x_i + 2j + 4, y), \\
& (x_i + 2j + 3, y), (x_i + 2j + 4, y + 1), (x_i + 2j + 3, y + 1), (x_i + 2j + 4, y + 2), \\
& (x_i + 2j + 5, y + 1), (x_i + 2j + 6, y + 1), \dots, (x_i, y + 1), \\
& (x_i + 1, y), (x_i + 2, y), \dots, (x_i + 2j + 2, y), \\
& (x_i + 2j + 1, y + 1), \\
& P_1, P_2, \dots, P_{j-1}, \\
& (x_i + 2j + 2, y + j), \\
& (x_i + 2j + 3, y + j + 1)(x_i + 2j + 2, y + j + 1), \dots, (x_i + 3, y + j + 1), \\
& (x_i + 2, y + j), (x_i + 1, y + j), \\
& (x_i + 2, y + j + 1)(x_i + 1, y + j + 1), \dots, (x_i + 2j + 5, y + j + 1), \\
& (x_i + 2j + 4, y + j + 2)
\end{aligned}$$

defines an  $(x_i, y)$ - $(x_i+j+1, y+j+2)$ -climbing path of  $G^D$ . See Figures 10 and 11 for an illustration.  $\square$

We are now in a position to prove the main result of this section. A path  $P$  in  $G^D$  with  $V(P) = [\min(V(P)), \max(V(P))]$  is called *special*, if the endvertices of  $P$  are  $\min(V(P))$  and  $\max(V(P))$ .

**Proof of Theorem 4:** If  $d_2 = 1$ , then the statement of the theorem is trivial. Hence we assume that  $d_2 > 1$ . The idea of the proof is to show the existence of two distinct positive integers  $p_1$  and  $p_2$  with



**Fig. 11:** An  $(x_i, y)$ - $(x_{i+j+1}, y + j + 2)$ -climbing path for  $j = 4$ .

$\gcd(p_1, p_2) = 1$  such that the distance graph  $G^D$  has a special path of length  $p_1$  and a special path of length  $p_2$ . Note that we can shift special paths. If  $P : v_0, \dots, v_l$  is a special path, then also  $P + h : v_0 + h, \dots, v_l + h$  is a special path. Furthermore, we can concatenate special paths. If  $P : v_0, \dots, v_l$  is a special path of length  $l$  and  $Q : v_l, \dots, v_{l+h}$  is a special path of length  $h$ , then  $PQ : v_0, \dots, v_{l+h}$  is a special path of length  $l + h$ . Since  $\gcd(p_1, p_2) = 1$ , it follows from the extended Euclidean algorithm that every sufficiently large integer  $p$  is a positive integral linear combination of  $p_1$  and  $p_2$ . In fact, if  $p > (2p_2 - 1)p_1$  and  $p = a_1p_1 + a_2p_2$  for integers  $a_1$  and  $a_2$  such that  $a_1 \leq 0$ , then  $a_1 + sp_2 > 0$  and  $a_2 - sp_1 > 0$  for  $s = \lceil \frac{-a_1}{p_2} \rceil + 1$  and thus  $p = (a_1 + sp_2)p_1 + (a_2 - sp_1)p_2$  is a positive integral linear combination of  $p_1$  and  $p_2$ . Therefore, the desired result follows by shifting and concatenating copies of the special paths of lengths  $p_1$  and  $p_2$ , which we construct now.

It has been observed in [25, 21] that  $G^D_{d_1+d_2+1}$  has a Hamiltonian path with endvertices 0 and  $d_1 + d_2$ . Hence, for  $p_1$ , we choose  $d_1 + d_2$ .

For  $p_2$ , we show that there is a positive integer  $p_2$  with  $p_2 \equiv -1 \pmod{d_1 + d_2} \equiv -1 \pmod{p_1}$ , such that  $G^D$  has a special path of length  $p_2$ , thus  $\gcd(p_1, p_2) = 1$ .

Let  $x'$  be such that  $x'd_1 \equiv -1 \pmod{d_1 + d_2}$ . By definition and Lemma 2 (i), column  $x'$  is upper and column  $x' - 1$  is lower. In order to get a special path with endvertices  $(0, 0)$  and  $(x', y')$  for some  $y'$ , we concatenate climbing paths to form a  $(0, 0)$ - $(x' - 1, y')$ -climbing path and append the path  $(x' - 2, y'), (x' - 3, y'), \dots, (x', y')$ .

Let  $k$  be such that the block  $B_k$  contains column  $x'$ , that is,  $x_k = x' - 1$ . Since column  $x'$  is upper, column  $x' - 2$  belongs to block  $B_{k-1}$ .

Since  $\gcd(d_1, d_2) = 1$ , at least one of  $d_1$  and  $d_2$  is odd.

**Case 1** One of  $d_1$  and  $d_2$  is even and  $G^D$  has at most 2 blocks of odd length.

Since  $d_1 + d_2$  is odd, the number of blocks of odd length is odd, that is, it equals 1.

We first assume that all blocks  $B_0, B_1, \dots, B_{k-1}$  are of even length. By Lemma 5, there exists an  $(x_i, 2i)$ - $(x_{i+1}, 2i + 2)$ -climbing path  $P_i$  for  $0 \leq i \leq k - 1$ . Since  $x' - 1 = x_k$ , the concatenation of the paths  $P_0, P_1, \dots, P_{k-1}$  forms a  $(0, 0)$ - $(x' - 1, y')$ -climbing path for  $y' = 2k$ .

Next, we assume that all blocks  $B_k, B_{k+1}, \dots, B_{-1}$  are of even length. Then, by Lemma 6, there exists an  $(x_{i+1}, 2(d_1 + d_2 - i) - 2) - (x_i, 2(d_1 + d_2 - i))$ -climbing path  $P_i$  for  $k \leq i \leq d_1 + d_2 - 1$ . Since  $x' - 1 = x_k$ , the concatenation of the paths  $P_{d_1+d_2-1}, P_{d_1+d_2-2}, \dots, P_k$  forms a  $(0, 0) - (x' - 1, y')$ -climbing path for  $y' = 2(d_1 + d_2 - k)$ . This concludes the first case.

**Case 2** *One of  $d_1$  and  $d_2$  is even and  $G^D$  has at least 3 blocks of odd length.*

Since  $d_1 + d_2$  is odd, the number of blocks of odd length is odd. This implies that one of the two sequences  $B_0, B_1, \dots, B_{k-1}$  and  $B_0, B_1, \dots, B_{-1}, B_0, B_1, \dots, B_{k-1}$  has an even number of blocks with odd length. We call this sequence  $\mathcal{S}$ . We can partition  $\mathcal{S}$  into subsequences  $S_1, S_2, \dots, S_t$ , where each subsequence is either a block of even length or a sequence  $B_i, B_{i+1}, \dots, B_{i+j}$  of blocks with  $i \in \mathbb{Z}/d_2\mathbb{Z}$  and  $j \geq 1$ , such that block  $B_i$  has odd length, block  $B_{i+j}$  has odd length, and blocks  $B_{i+1}, \dots, B_{i+j-1}$  have even length. For a subsequence  $S_q$ ,  $1 \leq q \leq t$ , that consists of one block  $B_i$  with  $i \in \mathbb{Z}/d_2\mathbb{Z}$ , Lemma 5 implies that there exists an  $(x_i, y) - (x_{i+1}, y + 2)$ -climbing path  $P_{q,y}$  for every  $y$ . If  $\frac{d_1}{d_2} < 2$ , then Lemma 3 implies that the lengths of the blocks are 2 and 3. For a subsequence  $S_q$ ,  $1 \leq q \leq t$ , that consists of at least two blocks  $B_i, B_{i+1}, \dots, B_{i+j}$  with  $i \in \mathbb{Z}/d_2\mathbb{Z}$  and  $j \geq 1$ , Lemma 8 implies that there exists an  $(x_i, y) - (x_{i+j+1}, y + j + 2)$ -climbing path  $P_{q,y}$  for every  $y$ . If  $\frac{d_1}{d_2} \geq 2$ , then Lemma 3 implies that the lengths of the blocks are at least 3. For a subsequence  $S_q$ ,  $1 \leq q \leq t$ , that consists of at least two blocks  $B_i, B_{i+1}, \dots, B_{i+j}$  with  $i \in \mathbb{Z}/d_2\mathbb{Z}$  and  $j \geq 1$ , Lemma 7 implies that there exists an  $(x_i, y) - (x_{i+j+1}, y + 3)$ -climbing path  $P_{q,y}$  for every  $y$ .

The concatenation of the paths  $P_{1,y_1}, P_{2,y_2}, \dots, P_{t,y_t}$  forms a  $(0, 0) - (x' - 1, y')$ -climbing path for  $y_1 = 0$ , suitable  $y_q$ 's, where  $2 \leq q \leq t$ , and  $y' = y_t$ .

This concludes the second case.

If both  $d_1$  and  $d_2$  are odd, then  $d_1 + d_2$  is even, which implies that the number of blocks of odd length is even and exactly those vertices are even integers that are in a column with an even index. This implies that  $x'$  is odd and  $x_k = x' - 1$  is even. Since column 0 and column  $x' - 1$  are lower, the sequence  $B_0, B_1, \dots, B_{k-1}$  has an even number of blocks with odd length.

**Case 3** *Both  $d_1$  and  $d_2$  are odd and  $G^D$  has at most 2 blocks of odd length.*

Since  $d_2 \geq 2$ ,  $G$  has exactly 2 blocks of odd length. This implies that one of the two sequences  $B_0, B_1, \dots, B_{k-1}$  and  $B_k, B_{k+1}, \dots, B_{-1}$  has only blocks of even length. Now we are in the same situation as in Case 1. Arguing as in Case 1 completes this case.

**Case 4** *Both  $d_1$  and  $d_2$  are odd and  $G^D$  has at least 4 blocks of odd length.*

Since the sequence  $B_0, B_1, \dots, B_{k-1}$  has an even number of blocks of odd length, we are in the same situation as in Case 2. Arguing as in Case 2 completes this case, which concludes the proof of the theorem.  $\square$

## 4 Hamiltonian cycles of $G_n^D$

The main results of this section are the following.

**Theorem 9** *For every  $D = \{d_1, d_2\} \subseteq \mathbb{N}$  with  $d_1 > d_2$ ,  $d_1 d_2$  odd, and  $\gcd(d_1, d_2) = 1$ , there is some  $n_0 \in \mathbb{N}$  such that for all even integers  $n$  with  $n \geq n_0$ , the distance graph  $G_n^D$  has a Hamiltonian cycle.*

**Theorem 10** For every  $D = \{d_1, d_2\} \subseteq \mathbb{N}$  with  $d_1 > d_2$ ,  $d_1 d_2$  even, and  $\gcd(d_1, d_2) = 1$ , there is some  $n_0 \in \mathbb{N}$  such that for all integers  $n$  with  $n \geq n_0$ , the distance graph  $G_n^D$  has a Hamiltonian cycle.

Note that the distance graphs considered in Theorem 9 are necessarily bipartite. Therefore, they can only have a Hamiltonian cycle if their order is even.

As in Section 3 we establish several lemmas before proceeding to the proofs of Theorems 9 and 10.

For two lower vertices  $(x, y)$  and  $(x', y')$  with  $x \neq x'$ ,  $0 \notin \{x' + 1, x' + 2, \dots, x\}$ , and  $y < y'$  in the distance graph  $G^D$ , a set of vertex disjoint paths  $R_y, R_{y+1}, \dots, R_{y'-1}$  in  $G^D$  is called an  $(x, y)$ - $(x', y')$ -path-collection of  $G^D$ , if it satisfies the following conditions:

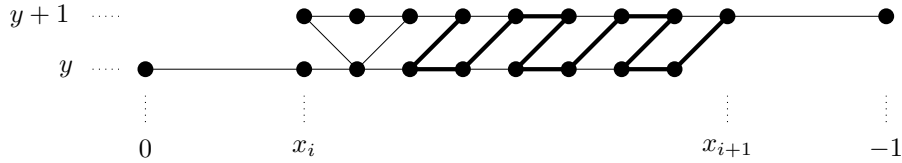
- for  $y \leq i < y'$ ,  $P_i$  has the endvertices  $(0, i)$  and  $(-1, i + 1)$ ,
- for  $y \leq i < y'$ , the path  $(0, i), (1, i), \dots, (x', i)$  is a subpath of  $P_i$ ,
- for  $y \leq i < y'$ , the path  $(x, i + 1), (x + 1, i + 1), \dots, (-1, i + 1)$  is a subpath of  $P_i$ ,
- the union of the vertex sets of the paths consists of all vertices in the rows  $y + 1, y + 2, \dots, y' - 1$ , the vertices  $\{(0, y), (1, y), \dots, (x - 1, y)\}$ , and the vertices  $\{(x', y'), (x' + 1, y'), \dots, (-1, y')\}$ , and
- no edge of the form  $\{(-1, z), (0, z')\}$  for some  $z, z' \in \mathbb{Z}$  is in the union of the edge sets of the paths.

See Figures 13, 15, and 16 for an illustration. Note, that  $(x, y)$  does not belong to any path of an  $(x, y)$ - $(x', y')$ -path-collection.

**Lemma 11** If for some  $i \neq -1$ ,  $B_i$  is a block of even length in  $G^D$ , then  $G^D$  has an  $(x_{i+1}, y)$ - $(x_i, y + 1)$  path collection for all  $y$ .

**Proof:** Let

$$\begin{aligned} P & : (x_i + 3, y + 1), (x_i + 2, y), (x_i + 3, y), (x_i + 4, y + 1), \\ & (x_i + 5, y + 1), (x_i + 4, y), (x_i + 5, y), (x_i + 6, y + 1), \\ & \dots, (x_{i+1} - 1, y + 1), (x_{i+1} - 2, y), (x_{i+1} - 1, y), (x_{i+1}, y + 1). \end{aligned}$$



**Fig. 12:**  $P$  for a block  $B_i$  of length 8.

The sequence

$$\begin{aligned} & (0, y), (1, y), \dots, (x_i + 1, y), \\ & (x_i, y + 1), (x_i + 1, y + 1), (x_i + 2, y + 1), P, \\ & (x_{i+1} + 1, y + 1), (x_{i+1} + 2, y + 1), \dots, (-1, y + 1) \end{aligned}$$

defines an  $(x_{i+1}, y)$ - $(x_i, y + 1)$ -path-collection in  $G^D$ . See Figures 12 and 13 for an illustration.  $\square$

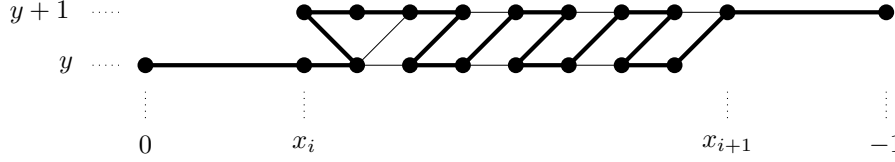


Fig. 13: An  $(x_{i+1}, y)$ - $(x_i, y + 1)$ -path-collection for a block  $B_i$  of length 8.

**Lemma 12** *If for some  $i \in \mathbb{Z}/d_2\mathbb{Z}$  and for some  $j \geq 1$ , the blocks  $B_i, B_{i+1}, \dots, B_{i+j}$  of  $G^D$  are such that  $-1 \notin \{i, i + 1, \dots, i + j\}$ ,  $B_i$  and  $B_{i+j}$  are of odd length and  $B_{i+1}, \dots, B_{i+j-1}$  are of even length at least 4, then  $G^D$  has an  $(x_{i+j+1}, y)$ - $(x_i, y + 2)$ -path-collection for all  $y$ .*

**Proof:** By Lemma 3, the blocks  $B_i$  and  $B_{i+j}$  are of length at least 3.

Let

$$\begin{aligned} P_i & : (x_i + 1, y), (x_i + 2, y + 1), (x_i + 3, y + 1), (x_i + 2, y), \\ & (x_i + 3, y), (x_i + 4, y + 1), (x_i + 5, y + 1), (x_i + 4, y), \\ & \dots, (x_{i+1} - 2, y), (x_{i+1} - 1, y + 1), (x_{i+1}, y + 1), (x_{i+1} - 1, y). \end{aligned}$$

For  $1 \leq q \leq j - 1$ , let

$$\begin{aligned} P_{i+q} & : (x_{i+q}, y), (x_{i+q} + 1, y), \\ & (x_{i+q} + 2, y), (x_{i+q} + 3, y + 1), (x_{i+q} + 4, y + 1), (x_{i+q} + 3, y), \\ & (x_{i+q} + 4, y), (x_{i+q} + 5, y + 1), (x_{i+q} + 6, y + 1), (x_{i+q} + 5, y), \\ & \dots, (x_{i+q+1} - 2, y), (x_{i+q+1} - 1, y + 1), (x_{i+q+1}, y + 1), (x_{i+q+1} - 1, y). \end{aligned}$$

Furthermore, let

$$\begin{aligned} P_{i+j} & : (x_{i+j} + 2, y), (x_{i+j} + 3, y + 1), (x_{i+j} + 4, y + 1), (x_{i+j} + 3, y), \\ & (x_{i+j} + 4, y), (x_{i+j} + 5, y + 1), (x_{i+j} + 6, y + 1), (x_{i+j} + 5, y), \\ & \dots, (x_{i+j+1} - 3, y), (x_{i+j+1} - 2, y + 1), (x_{i+j+1} - 1, y + 1), (x_{i+j+1} - 2, y). \end{aligned}$$

For  $1 \leq q \leq j$ , let

$$\begin{aligned} Q_{i+q} & : (x_{i+q-1} + 4, y + 2), (x_{i+q-1} + 5, y + 2), \dots, (x_{i+q} + 2, y + 2), \\ & (x_{i+q} + 1, y + 1), (x_{i+q} + 2, y + 1), (x_{i+q} + 3, y + 2). \end{aligned}$$

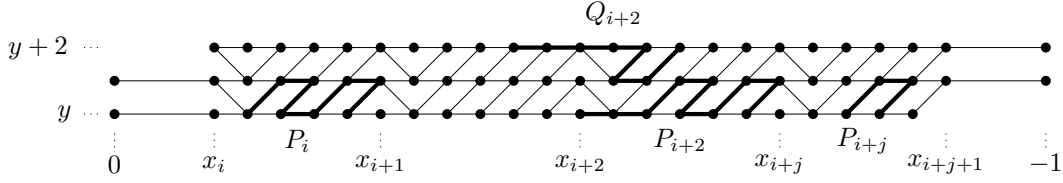


Fig. 14:  $P_i, P_{i+2}, P_{i+j}$ , and  $Q_{i+2}$  for  $j = 3$ .

Now,  $R_y$  and  $R_{y+1}$ , where

$$\begin{aligned}
 R_y & : (0, y), (1, y), \dots, (x_i, y), \\
 & P_i, \\
 & P_{i+1}, P_{i+2}, \dots, P_{i+j-1}, \\
 & (x_{i+j}, y), (x_{i+j} + 1, y), P_{i+j}, (x_{i+j+1} - 1, y), \\
 & (x_{i+j+1}, y + 1), (x_{i+j+1} + 1, y + 1), \dots, (-1, y + 1)
 \end{aligned}$$

and

$$\begin{aligned}
 R_{y+1} & : (0, y + 1), (1, y + 1), \dots, (x_i + 1, y + 1), \\
 & (x_i, y + 2), (x_i + 1, y + 2), (x_i + 2, y + 2), (x_i + 3, y + 2), \\
 & Q_{i+1}, Q_{i+2}, \dots, Q_{i+j}, \\
 & (x_{i+j+1} - 1, y + 2), (x_{i+j+1}, y + 2), \dots, (-1, y + 2)
 \end{aligned}$$

define an  $(x_{i+j+1}, y)-(x_i, y + 2)$ -path-collection of  $G^D$ . See Figures 14 and 15 for an illustration.  $\square$

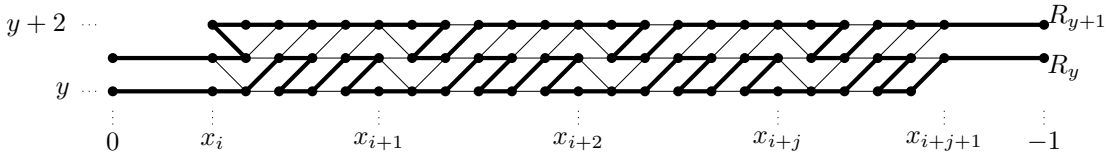


Fig. 15: An  $(x_{i+j+1}, y)-(x_i, y + 2)$ -path-collection for  $j = 3$ .

**Lemma 13** *If for some  $i \in \mathbb{Z}/d_2\mathbb{Z}$  and for some  $j \geq 1$ , the blocks  $B_i, B_{i+1}, \dots, B_{i+j}$  of  $G^D$  are such that  $-1 \notin \{i, i + 1, \dots, i + j\}$ ,  $B_i$  and  $B_{i+j}$  are of length 3 and  $B_{i+1}, \dots, B_{i+j-1}$  are of length 2, then  $G^D$  has an  $(x_{i+j+1}, y)-(x_i, y + j + 1)$ -path-collection for all  $y$ .*

**Proof:** Note that  $x_{i+j+1} = x_i + 2j + 4$ . For  $0 \leq q \leq j - 1$ , let

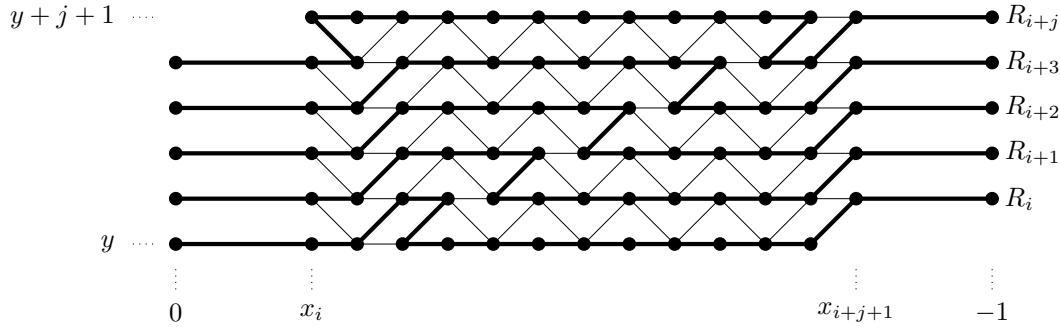
$$R_{i+q} : (0, y + q), (1, y + q), \dots, (x_i + 1, y + q),$$

$$\begin{aligned}
 & (x_i + 2, y + q + 1), (x_i + 3, y + q + 1), \dots, (x_i + 2q + 3, y + q + 1), \\
 & (x_i + 2q + 2, y + q), (x_i + 2q + 3, y + q), \dots, (x_i + 2j + 3, y + q), \\
 & (x_i + 2j + 4, y + q + 1), (x_i + 2j + 5, y + q + 1), \dots, (-1, y + q + 1).
 \end{aligned}$$

and let

$$\begin{aligned}
 R_{i+j} & : (0, y + j), (1, y + j), \dots, (x_i + 1, y + j), \\
 & (x_i, y + j + 1), (x_i + 1, y + j + 1), \dots, (x_i + 2j + 3, y + j + 1), \\
 & (x_i + 2j + 2, y + j), (x_i + 2j + 3, y + j), \\
 & (x_i + 2j + 4, y + j + 1), (x_i + 2j + 5, y + j + 1), \dots, (-1, y + j + 1).
 \end{aligned}$$

Now,  $R_i, R_{i+1}, \dots, R_{i+j}$  define an  $(x_{i+j+1}, y) - (x_i, y + j + 1)$ -path-collection of  $G^D$ . See Figure 16 for an illustration.  $\square$



**Fig. 16:** An  $(x_{i+j+1}, y) - (x_i, y + j + 1)$ -path-collection for  $j = 4$ .

**Lemma 14** *If for some  $i \in \mathbb{Z}/d_2\mathbb{Z}$  and for some  $j \geq 0$ , the sequence  $\mathcal{S} = B_i, B_{i+1}, \dots, B_{i+j}$  of blocks of  $G^D$  is such that  $-1 \notin \{i, i + 1, \dots, i + j\}$  and the number of blocks of odd length among  $B_i, B_{i+1}, \dots, B_{i+j}$  is even, then  $G^D$  has an  $(x_{i+j+1}, y) - (x_i, y + \Delta y)$ -path-collection for some  $\Delta y$  and for all  $y$ .*

**Proof:** By definition, the union of suitable path-collections is a path-collection: If for some  $x, x', x'', y, y', y''$ ,  $G^D$  has an  $(x, y) - (x', y')$ -path-collection and an  $(x', y') - (x'', y'')$ -path-collection, then  $G^D$  has an  $(x, y) - (x'', y'')$ -path-collection. We can partition  $\mathcal{S}$  into subsequences, where each subsequence is either a block of even length or a sequence  $B_k, B_{k+1}, \dots, B_{k+l}$  of blocks with  $k \in \mathbb{Z}/d_2\mathbb{Z}$  and  $l \geq 1$ , such that blocks  $B_k$  and  $B_{k+l}$  have odd length and blocks  $B_{k+1}, \dots, B_{k+l-1}$  have even length. For a subsequence that consists of one even block  $B_k$  with  $k \in \mathbb{Z}/d_2\mathbb{Z}$ , Lemma 11 implies that there exists a  $(x_{k+1}, y) - (x_k, y + 1)$  path collection for every  $y$ . If  $\frac{d_1}{d_2} < 2$ , then Lemma 3 implies that the lengths of the blocks are 2 and 3. For a subsequence that consists of at least two blocks  $B_k, B_{k+1}, \dots, B_{k+l}$  with  $k \in \mathbb{Z}/d_2\mathbb{Z}$  and  $l \geq 1$ , Lemma 13 implies that there exists an  $(x_{k+l+1}, y) - (x_k, y + l + 1)$ -path-collection for every  $y$ . If  $\frac{d_1}{d_2} \geq 2$ , then Lemma 3 implies that the lengths of the blocks are at least 3. For a subsequence that consists



of at least two blocks  $B_k, B_{k+1}, \dots, B_{k+l}$  with  $k \in \mathbb{Z}/d_2\mathbb{Z}$  and  $l \geq 1$ , Lemma 12 implies that there exists an  $(x_{k+l+1}, y)-(x_k, y+2)$ -path-collection for every  $y$ . Hence, a suitable union of path-collections forms an  $(x_{i+j+1}, y)-(x_i, y+\Delta y)$ -path-collection for a suitable  $\Delta y$  and all  $y$ .  $\square$

**Lemma 15** *If for some  $-i \in \mathbb{Z}/d_2\mathbb{Z}$ , the blocks  $B_{-i}, B_{-i+1}, \dots, B_{-1}$  of  $G^D$  are such that  $B_{-i}$  is of odd length and  $B_{-i+1}, \dots, B_{-1}$  are of even length at least 4, then for all  $y$ ,  $G^D$  has a path with endvertices  $(-1, y+1)$  and  $(-1, y+2)$  that consists of all vertices of rows  $y$  and  $y+1$  and the vertices  $(x_{-i}, y+2), (x_{-i}+1, y+2), \dots, (-1, y+2)$ .*

**Proof:** For  $1 \leq q \leq i-1$ , let

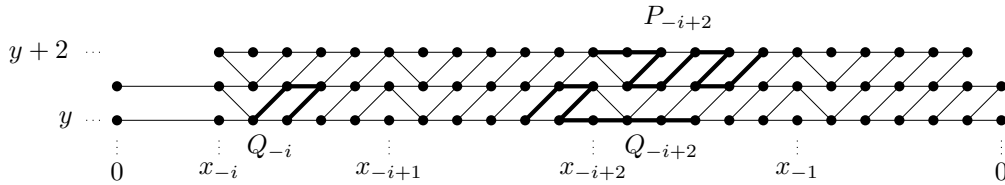
$$Q_{-q} : (x_{-q+1} - 3, y), (x_{-q+1} - 4, y), \dots, (x_{-q}, y), \\ (x_{-q} - 1, y), (x_{-q}, y+1), (x_{-q} - 1, y+1), (x_{-q} - 2, y)$$

and let

$$Q_{-i} : (x_{-i+1} - 3, y), (x_{-i+1} - 2, y+1), (x_{-i+1} - 3, y+1), (x_{-i+1} - 4, y), \\ (x_{-i+1} - 5, y), (x_{-i+1} - 4, y+1), (x_{-i+1} - 5, y+1), (x_{-i+1} - 6, y), \\ \dots, (x_{-i} + 2, y), (x_{-i} + 3, y+1), (x_{-i} + 2, y+1), (x_{-i} + 1, y).$$

Furthermore, let for  $1 \leq q \leq i-1$

$$P_{-q} : (x_{-q}, y+2), (x_{-q} + 1, y+2), \\ (x_{-q} + 2, y+2), (x_{-q} + 1, y+1), (x_{-q} + 2, y+1), (x_{-q} + 3, y+2), \\ (x_{-q} + 4, y+2), (x_{-q} + 3, y+1), (x_{-q} + 4, y+1), (x_{-q} + 5, y+2), \\ \dots, (x_{-q+1} - 2, y+2), (x_{-q+1} - 3, y+1), (x_{-q+1} - 2, y+1), (x_{-q+1} - 1, y+2).$$



**Fig. 17:**  $Q_{-i}$ ,  $Q_{-i+2}$ , and  $P_{-i+2}$  for  $i = 4$ .

Now, the sequence

$$(-1, y+1), (-2, y), \\ Q_{-1}, Q_{-2}, \dots, Q_{-i}, \\ (x_{-i}, y), (x_{-i} - 1, y), \dots, (-1, y), \\ (0, y+1), (1, y+1), \dots, (x_{-i} + 1, y+1), \\ (x_{-i}, y+2), (x_{-i} + 1, y+2), \dots, (x_{-i+1} - 1, y+2), \\ P_{-i+1}, P_{-i+2}, \dots, P_{-1}$$

defines a path that satisfies the conditions of the lemma. See Figures 17 and 18 for an illustration.  $\square$

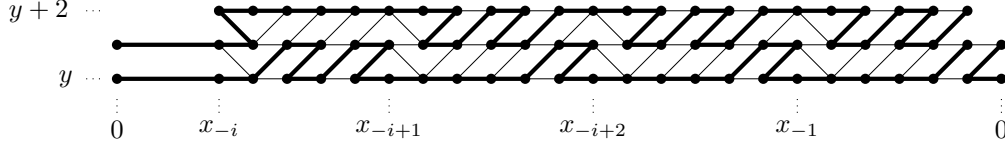


Fig. 18: A path for  $i = 4$ .

A cycle  $C$  in  $G^D$  is called *special*, if  $V(C) = [\min(V(C)), \max(V(C))]$ .

**Lemma 16** For every  $D = \{d_1, d_2\} \subseteq \mathbb{N}$  with  $d_1 > d_2$ ,  $d_1 d_2$  even, and  $\gcd(d_1, d_2) = 1$ , there is some  $n \in \mathbb{N}$  with  $n \equiv 0 \pmod{d_1 + d_2}$  such that  $G^D$  has a special cycle  $C$  of order  $n + 1$  with  $V(C) = [0, n]$ .

**Proof:** Clearly, vertex  $n$  is in column 0. Since  $d_1 d_2$  is even and  $\gcd(d_1, d_2) = 1$ , we obtain that  $d_1 + d_2$  is odd and hence the number of blocks of odd length is odd, i.e. at least 1. Let  $i \in \mathbb{Z}/d_2\mathbb{Z}$ , such that block  $B_i$  is of odd length and the blocks  $B_{i+1}, \dots, B_{-1}$  are of even length. Clearly, by Lemma 3, the length of the blocks  $B_{i+1}, \dots, B_{-1}$  are at least 4. By Lemma 15,  $G^D$  has a path  $Q$  with endvertices  $(-1, 1)$  and  $(-1, 2)$  that consists of all vertices of rows 0 and 1 and the vertices  $(x_i, 2), (x_i + 1, 2), \dots, (-1, 2)$ . Since the number of blocks of  $B_0, \dots, B_{i-1}$  of odd length is even, by Lemma 14,  $G^D$  has an  $(x_i, 2)$ - $(0, y')$ -path-collection  $\mathcal{R}$  for some  $y'$ . Note, that if  $G^D$  has only one block of odd length, then  $\mathcal{R} = \emptyset$ . In this case we define  $y' = 2$ . Let

$$P = \left( Q \cup \bigcup_{R \in \mathcal{R}} R \right) + \bigcup_{y=1}^{y'-2} \{(-1, y), (0, y+1)\}.$$

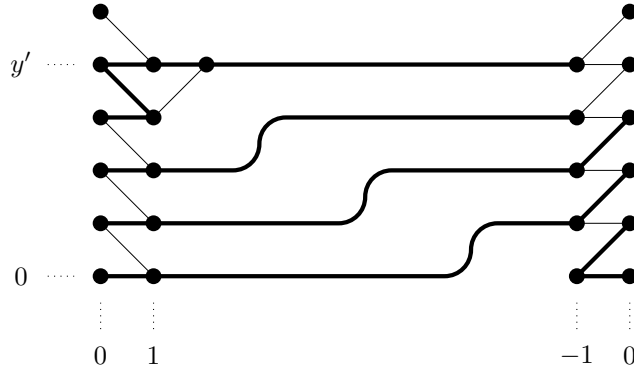
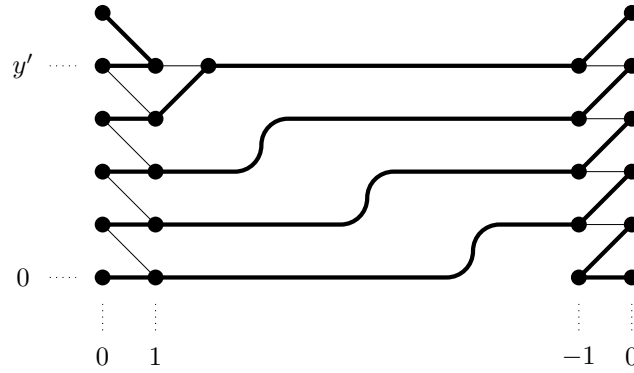


Fig. 19: The path  $P$ .

By construction,  $P$  is a path with endvertices  $(-1, y' - 1)$  and  $(-1, y')$  that consists of all vertices of rows  $0, 1, \dots, y'$ . The vertex  $(0, y')$  has the neighbors  $(1, y' - 1)$  and  $(1, y')$  in  $P$ . Since the vertex  $(1, y')$  is an upper vertex,  $(1, y')$  has the neighbors  $(0, y')$  and  $(2, y')$  in  $P$  and  $\{(1, y' - 1), (2, y')\} \in E(G^D)$ . Now,

$$\begin{aligned} C = P & \\ & + \{ \{(1, y' - 1), (2, y')\}, \{(-1, y' - 1), (0, y')\}, \{(-1, y'), (0, y' + 1)\}, \{(0, y' + 1), (1, y')\} \} \\ & - \{ \{(1, y' - 1), (0, y')\}, \{(1, y'), (2, y')\} \} \end{aligned}$$

is a special cycle of  $G^D$  of order  $n + 1$  with  $n = (y' + 1)(d_1 + d_2)$  and  $V(C) = [0, n]$ . See Figures 19 and 20 for an illustration.  $\square$



**Fig. 20:** The cycle  $C$  in the proof of Lemma 16.

**Lemma 17** For every  $D = \{d_1, d_2\} \subseteq \mathbb{N}$  with  $d_1 > d_2$ ,  $d_1 d_2$  odd, and  $\gcd(d_1, d_2) = 1$ , there is some  $n \in \mathbb{N}$  with  $n \equiv 0 \pmod{d_1 + d_2}$  such that  $G^D$  has a special cycle  $C$  of order  $n + 2$  with  $V(C) = [0, n + 1]$ .

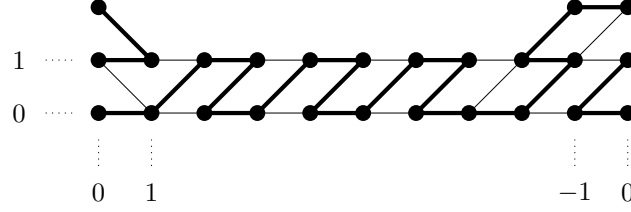
**Proof:** Clearly, vertex  $n$  is in column 0. First we assume that  $d_2 = 1$ . In that case,  $G^D$  has only one block and the vertex  $n + 1$  is in column  $-1$ . Let  $P = \emptyset$  for  $d_1 = 3$ , otherwise let

$$\begin{aligned} P : & (1, 0), (2, 1), (3, 1), (2, 0), \\ & (3, 0), (4, 1), (5, 1), (4, 0), \\ & \dots, (-5, 0), (-4, 1), (-3, 1), (-4, 0). \end{aligned}$$

The sequence

$$C : (0, 0), P, (-3, 0), (-2, 0), (-1, 1), (-2, 1), (-1, 2), (0, 2), (1, 1), (0, 1), (-1, 0), (0, 0)$$

defines a special cycle of  $G^D$  of order  $2(d_1 + d_2) + 2$  with  $V(C) = [0, 2(d_1 + d_2) + 1]$ . See Figure 21 for an illustration.



**Fig. 21:** The special cycle  $C$  for  $d_1 = 9$  and  $d_2 = 1$ .

Now we assume that  $d_2 > 1$ . Hence, by Lemma 3,  $G^D$  has more than one block. This implies that vertex  $n+1$  is lower. Let  $k \in \mathbb{Z}/d_2\mathbb{Z}$ , such that vertex  $n+1$  belongs to block  $B_k$ . Since  $d_1 + d_2$  is even, exactly those vertices are even integers that are in a column with an even index. Since vertex  $n+1$  is lower and an odd integer, the number of blocks among  $B_k, B_{k+1}, \dots, B_{-1}$  of odd length is odd, i.e. at least one. Let  $i \in \mathbb{Z}/d_2\mathbb{Z}$  be such that block  $B_i$  is of odd length and the blocks  $B_{i+1}, B_{i+2}, \dots, B_{-1}$  are of even length. Clearly, by Lemma 3, the length of the blocks  $B_{i+1}, B_{i+2}, \dots, B_{-1}$  are at least 4. By Lemma 15,  $G^D$  has a path  $Q_1$  with endvertices  $(-1, 1)$  and  $(-1, 2)$  that consists of all vertices of rows 0 and 1 and the vertices  $(x_i, 2), (x_{i+1}, 2), \dots, (-1, 2)$ . Since the number of blocks of  $B_k, B_{k+1}, \dots, B_{i-1}$  of odd length is even, by Lemma 14,  $G^D$  has an  $(x_i, 2)$ - $(x_k, y')$ -path-collection  $\mathcal{R}_1$  for some  $y'$ . Note, that if  $i = k$ , then  $\mathcal{R}_1 = \emptyset$ . In this case we define  $y' = 2$ . By the same arguments,  $G^D$  has a path  $Q_2$  with endvertices  $(-1, y' + 2)$  and  $(-1, y' + 3)$  that consists of all vertices of rows  $y' + 1$  and  $y' + 2$  and the vertices  $(x_i, y' + 3), (x_{i+1}, y' + 3), \dots, (-1, y' + 3)$  and  $G^D$  has an  $(x_i, y' + 3)$ - $(x_k, 2y' + 1)$ -path-collection  $\mathcal{R}_2$ .

By definition, for every  $y' + 1 \leq y \leq 2y'$ , the edges  $\{(0, y), (1, y)\}$  and  $\{(x_k - 1, y), (x_k, y)\}$  belong to  $Q_2$  or a path in  $\mathcal{R}_2$ . Furthermore, the path

$$P_0 : (x_k + 1, 2y'), (x_k, 2y' + 1), (x_k + 1, 2y' + 1), (x_k + 2, 2y' + 1)$$

is a subpath of a path in  $\{Q_2\} \cup \mathcal{R}_2$ . Let

$$P_1 : (0, y'), (1, y'), \dots, (x_k - 1, y')$$

and let

$$P_2 : (x_k, 2y' + 1), (x_k + 1, 2y' + 1), (x_k, 2y' + 2), \\ (x_k - 1, 2y' + 1), (x_k - 2, 2y' + 1), \dots, (1, 2y' + 1).$$

Now,

$$C = (Q_1 \cup Q_2 \cup \mathcal{R}_1 \cup \mathcal{R}_2) \\ - \left( E(P_0) \cup \bigcup_{y=y'+1}^{2y'} \{ \{(0, y), (1, y)\} \} \cup \bigcup_{y=y'+1}^{2y'} \{ \{(x_k - 1, y), (x_k, y)\} \} \right) \\ + E(P_1) \cup E(P_2) \\ + \bigcup_{y=1}^{y'} \{ \{(-1, y), (0, y + 1)\} \}$$

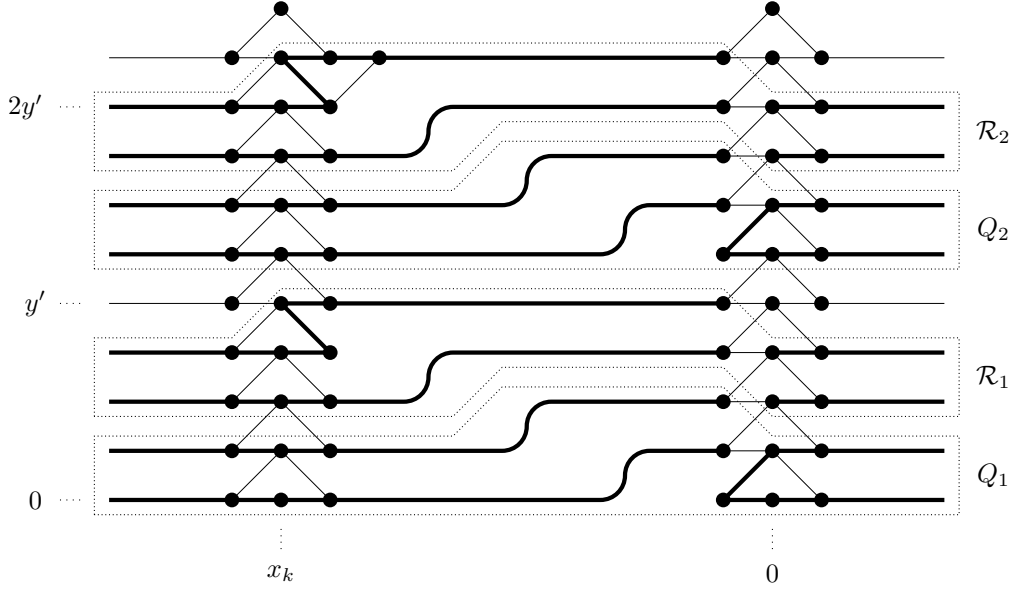


Fig. 22:  $Q_1 \cup Q_2 \cup R_1 \cup R_2$ .

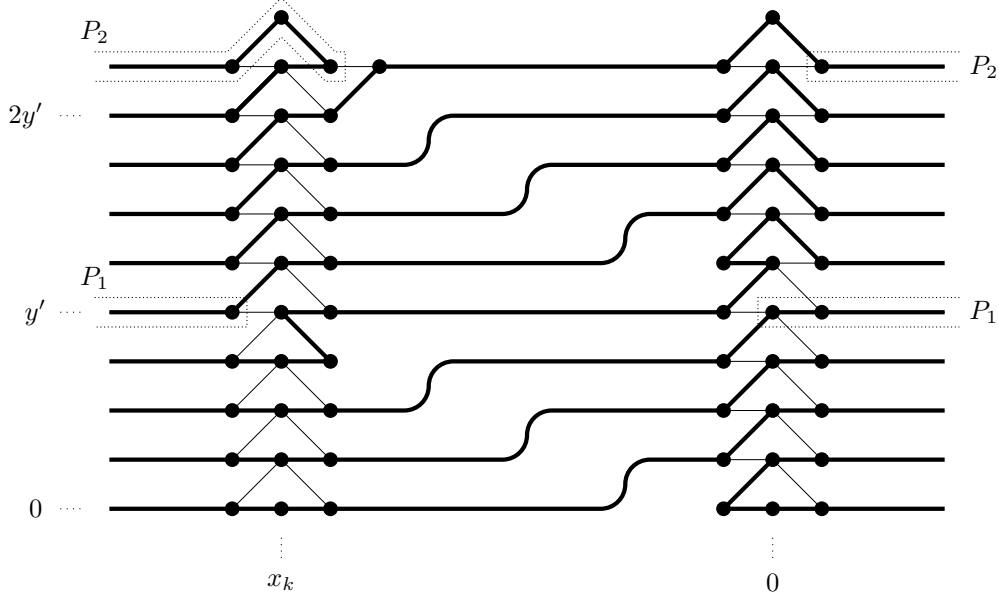
$$\begin{aligned}
& + \bigcup_{y=y'}^{2y'} \{ \{ (x_k - 1, y), (x_k, y + 1) \} \} \\
& + \bigcup_{y=y'+2}^{2y'+1} \{ \{ (-1, y), (0, y + 1) \} \} \\
& + \bigcup_{y=y'+1}^{2y'+1} \{ \{ (0, y + 1), (1, y) \} \} \\
& + \{ \{ (x_k + 1, 2y'), (x_k + 2, 2y' + 1) \} \}
\end{aligned}$$

defines a special cycle of  $G^D$  of order  $n + 2$  with  $n = (2y' + 2)(d_1 + d_2) + 2$  and  $V(C) = [0, n + 1]$ . See Figures 22 and 23 for an illustration.  $\square$

Let  $C$  be a special cycle of  $G^D$  and let  $n' = \max(V(C))$ . If for all  $a, b \in V(C)$  with  $n' - d_1 + 1 \leq a < b \leq n'$ ,  $\{a, b\} \neq \{n' - 2d_2, n' - d_2\}$ , and  $|a - b| \in D$ , we have  $\{a, b\} \in E(C)$ , then we call  $C$  good.

We are now in a position to prove the main results of this section.

**Proof of Theorem 9:** If  $D = \{1, 3\}$ , then the result follows by induction on  $n$ .  $C : 0, 1, 2, 3, 0$  is a Hamiltonian cycle of  $G_4^D$ . Let  $C_n$  be a Hamiltonian cycle of  $G_n^D$ . Since the vertex  $n - 1$  has degree 2 in



**Fig. 23:** The cycle  $C$  in the proof of Lemma 17.

$G_n^D$ ,  $\{n-2, n-1\} \in E(C_n)$ . Hence,

$$C_{n+2} = C_n + \{\{n-2, n+1\}, \{n+1, n\}, \{n, n-1\}\} - \{\{n-2, n-1\}\}$$

is a Hamiltonian cycle of  $G_{n+2}^D$ .

Hence we can assume that  $D \neq \{1, 3\}$ . Note that we can shift special cycles: If  $C : v_0, \dots, v_l, v_0$  is a special cycle in  $G^D$ , then also  $C + h : v_0 + h, \dots, v_l + h, v_0 + h$  is a special cycle in  $G^D$ . Furthermore, we can merge special cycles: If  $C_1$  and  $C_2$  are special cycles with  $\min(V(C_2)) = \max(V(C_1)) + 1$ ,  $\{a, b\} \in E(C_1)$ ,  $\{c, d\} \in E(C_2)$ , and  $\{a, c\}, \{b, d\} \in E(G^D)$ , then

$$(C_1 \cup C_2) + \{\{a, c\}, \{b, d\}\} - \{\{a, b\}, \{c, d\}\}$$

is a special cycle with vertex set  $[\min(V(C_1)), \max(V(C_2))]$ . If for  $i \leq a < b \leq j$ ,  $\{a, b\}$  is an edge of  $G^D$  and at least one of  $a, b$  has degree 2 in  $G^D[[i, j]]$ , then the edge  $\{a, b\}$  belongs to every special cycles  $C$  of  $G^D$  with  $V(C) = [i, j]$ .

**Claim 1** *If  $C_1$  and  $C_2$  are good special cycles of  $G^D$  with  $\min(V(C_2)) = \max(V(C_1)) + 1$  and  $D \neq \{1, 3\}$ , then there is a good special cycle  $C$  with  $V(C) = [\min(V(C_1)), \max(V(C_2))]$ .*

**Proof of Claim 1:** Let  $n' = \max(V(C_1))$ .

**Case 1**  $d_1 \neq 2d_2 + 1$ .

Since  $d_1 \neq 2d_2 + 1$  and  $C_1$  is good,  $e_1 = \{n' - d_1 + 1, n' - d_1 + d_2 + 1\} \in E(C_1)$ . Clearly,  $e_2 = \{n' + 1, n' + d_2 + 1\} \in E(C_2)$ . Hence,

$$C = (C_1 \cup C_2) + \{\{n' - d_1 + 1, n' + 1\}, \{n' - d_1 + d_2 + 1, n' + d_2 + 1\}\} - \{e_1, e_2\}$$

is a good special cycle with  $V(C) = [\min(V(C_1)), \max(V(C_2))]$ . This concludes the first case.

**Case 2**  $d_1 = 2d_2 + 1$ .

Since  $D \neq \{1, 3\}$ , we have  $d_2 > 1$ . Since  $d_1 = 2d_2 + 1$ , and  $C_1$  is good,  $e_1 = \{n' - d_1 + 2, n' - d_1 + d_2 + 2\} \in E(C_1)$ . Since  $d_2 > 1$ ,  $e_2 = \{n' + 2, n' + d_2 + 2\} \in E(C_2)$ . Hence,

$$C = (C_1 \cup C_2) + \{\{n' - d_1 + 2, n' + 2\}, \{n' - d_1 + d_2 + 2, n' + d_2 + 2\}\} - \{e_1, e_2\}$$

is a good special cycle with  $V(C) = [\min(V(C_1)), \max(V(C_2))]$ . This concludes the second case and the proof of Claim 1.  $\square$

**Claim 2**  $G^D$  has a good special cycle of order  $2 \pmod{d_1 + d_2}$ .

**Proof of Claim 2:** By Lemma 17,  $G^D$  has a special cycle of order  $2 \pmod{d_1 + d_2}$ . Let  $C_1$  be a special cycle of  $G^D$  of order  $2 \pmod{d_1 + d_2}$  and let  $n' = \max(V(C_1))$ . It follows from [25, 21] that  $G^D$  has a special cycle of order  $d_1 + d_2$ . Note that every vertex in  $\{j, j + 1, \dots, j + d_1 + d_2 - 1\}$  has degree 2 in  $G^D[[j, j + d_1 + d_2 - 1]]$ , for  $j \in \mathbb{Z}$  and hence a special cycle of order  $d_1 + d_2$  is good. Let  $C_2$  be a special cycle of  $G^D$  of order  $d_1 + d_2$  with  $\min(V(C_2)) = n' + 1$ . Since vertex  $n'$  has degree 2 in  $G^D[V(C_1)]$ ,  $\{n' - d_2, n'\} \in E(C_1)$  and since vertex  $n' + d_1$  has degree 2 in  $G^D[V(C_2)]$ ,  $\{n' + d_1 - d_2, n' + d_1\} \in E(C_2)$ . Hence,

$$(C_1 \cup C_2) + \{\{n' - d_2, n' + d_1 - d_2\}, \{n', n' + d_1\}\} - \{\{n' - d_2, n'\}, \{n' + d_1 - d_2, n' + d_1\}\}$$

is a good special cycle of  $G^D$ . This concludes the proof of Claim 2.  $\square$

Let  $p_1$  with  $p_1 \equiv 2 \pmod{d_1 + d_2}$ , such that  $G^D$  has a good special cycle of order  $p_1$ . By Claim 2, such a  $p_1$  exists. As said before,  $G^D$  has a good special cycle of order  $p_2 = d_1 + d_2$ . Since  $\gcd(p_1, p_2) = 2$ , it follows from the extended Euclidean algorithm that every sufficiently large even integer is a positive integral linear combination of  $p_1$  and  $p_2$ . Therefore and by Claim 1, the desired result follows by shifting and merging copies of good special cycles of order  $p_1$  and  $p_2$ .  $\square$

**Proof of Theorem 10::** The proof is analogous to the proof of Theorem 9. Instead of using Lemma 17 we use Lemma 16. Proceeding as in the proof of Theorem 9 we obtain  $p_1$  with  $p_1 \equiv 1 \pmod{d_1 + d_2}$  and hence  $\gcd(p_1, p_2) = 1$ . This clearly allows to establish the theorem for all sufficiently large  $n$  and not just for sufficiently large even  $n$ .  $\square$

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