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On Hamiltonian Paths and Cycles in Sufficiently Large Distance Graphs

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For a positive integer $n \in \mathbb{N}$ and a set $D \subseteq \mathbb{N}$, the distance graph G_n^D has vertex set $\{0, 1, \dots, n-1\}$ and two vertices i and j of G_n^D are adjacent exactly if $|j-i| \in D$. The condition $\gcd(D) = 1$ is necessary for a distance graph G_n^D being connected. Let $D = \{d_1, d_2\} \subseteq \mathbb{N}$ be such that $d_1 > d_2$ and $\gcd(d_1, d_2) = 1$. We prove the following results.

- If n is sufficiently large in terms of D , then G_n^D has a Hamiltonian path with endvertices 0 and $n-1$.
- If $d_1 d_2$ is odd, n is even and sufficiently large in terms of D , then G_n^D has a Hamiltonian cycle.
- If $d_1 d_2$ is even and n is sufficiently large in terms of D , then G_n^D has a Hamiltonian cycle.

Keywords: Distance graph; Toeplitz graph; circulant graph; Hamiltonian path; Hamiltonian cycle; traceability

1 Introduction

For a finite set of positive integers $D \subseteq \mathbb{N}$, the *infinite distance graph* G^D has vertex set $V(G^D) = \mathbb{Z}$ and two vertices u and v of G^D are adjacent exactly if $|u-v| \in D$. For a graph G and a subset $U \subseteq V(G)$ of the vertex set, we denote by $G[U]$ the subgraph of G induced by U . For $i, j \in \mathbb{Z}$, $i \leq j$, we denote by $[i, j] = \{k \in \mathbb{Z} \mid i \leq k \leq j\}$. For a positive integer $n \in \mathbb{N}$, the *distance graph* (also called *Toeplitz graph* in many papers) $G_n^D = G^D[[0, n-1]]$ is the subgraph of G^D induced by the vertices in $[0, n-1]$.

Infinite distance graphs and especially their colourings were first studied by Eggleton, Erdős, and Skilton [10, 11]. Most of the research on distance graphs focused on their colourings [6, 8, 9, 14, 18, 19, 28]. Distance graphs generalize the very well-studied class of *circulant graphs* [2, 16, 17, 26]. In fact, circulant graphs coincide exactly with the regular distance graphs [23]. Circulant graphs have been proposed for numerous network applications and many of their properties such as connectedness and diameter [4, 2, 16, 17], cycle and path structure [1, 3, 5], and isomorphism testing and recognition [12, 22] have

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been studied in great detail. Several fundamental results concerning circulant graphs were extended to the more general class of distance graphs in [7, 23, 24, 25]. The complexity of the connectedness problem for distance graphs was recently settled by Gómez et al. [13]. In [25, 27, 15] the existence of long paths and cycles in distance graphs is studied. The following main result from [21] confirmed a conjecture from Penso et al. [25]. [20] gives an overview on Hamiltonian cycles and paths in vertex-transitive graphs.

Theorem 1 (Löwenstein et al. [21]) *For a finite set $D \subseteq \mathbb{N}$ with $|D| \geq 2$ and $\gcd(D) = 1$, there are infinitely many $n \in \mathbb{N}$ such that G_n^D has a Hamiltonian cycle and G_{n+1}^D has a Hamiltonian path with endvertices 0 and n .*

We conjecture that the conclusion of the last theorem holds

- for all n that are sufficiently large in terms of D if not all elements of D are odd and
- for all even n that are sufficiently large in terms of D if all elements of D are odd.

The purpose of the present paper is to confirm this conjecture in the case that D contains just two elements. In Section 2 we introduce suitable terminology and collect some properties of distance graphs. In Section 3 we confirm our conjecture proving the existence of Hamiltonian paths. Finally, in Section 4 we provide similar results for Hamiltonian cycles.

2 The structure of G^D

Let $D = \{d_1, d_2\}$ for two positive integers d_1 and d_2 such that $\gcd(d_1, d_2) = 1$ and $d_1 > d_2$.

We define coordinates $(x, y) \in (\mathbb{Z}/(d_1 + d_2)\mathbb{Z}) \times \mathbb{Z}$ for the vertices of the distance graph G^D by

$$(x, y) := y(d_1 + d_2) + a_x,$$

where $a_x = xd_1 \pmod{d_1 + d_2}$. Note that this bidimensional relabelling of the vertices of G^D is a bijection. A vertex (x, y) satisfying $0 \leq xd_1 \pmod{d_1 + d_2} < d_2$ is called *lower*. A vertex (x, y) satisfying $d_2 \leq xd_1 \pmod{d_1 + d_2} < d_1$ is called *middle*. A vertex (x, y) satisfying $d_1 \leq xd_1 \pmod{d_1 + d_2} < d_1 + d_2$ is called *upper*.

For a lower vertex (x, y) , we have

$$\begin{aligned} (x, y) + d_1 &= (x + 1, y), \\ (x, y) + d_2 &= (x - 1, y), \\ (x, y) - d_1 &= (x - 1, y - 1), \\ (x, y) - d_2 &= (x + 1, y - 1), \end{aligned}$$

which implies that a lower vertex (x, y) is adjacent to the vertices $(x + 1, y)$, $(x - 1, y)$, $(x + 1, y - 1)$, and $(x - 1, y - 1)$.

Similarly, for a middle vertex (x, y) , we have

$$\begin{aligned} (x, y) + d_1 &= (x + 1, y + 1), \\ (x, y) + d_2 &= (x - 1, y), \\ (x, y) - d_1 &= (x - 1, y - 1), \\ (x, y) - d_2 &= (x + 1, y), \end{aligned}$$

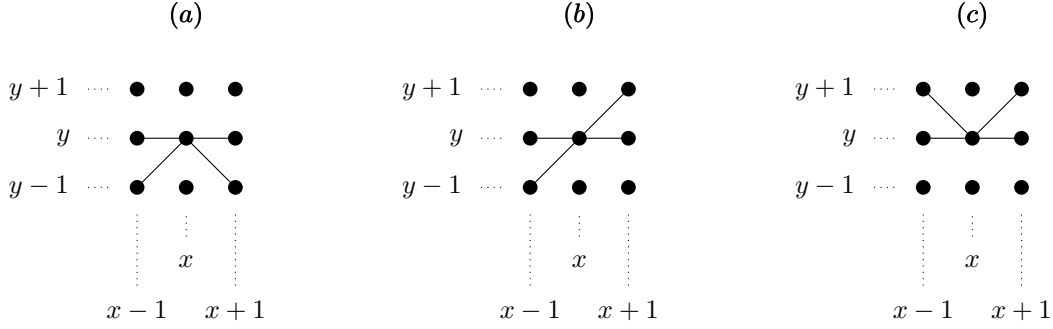


Fig. 1: Neighborhood of (a) a lower, (b) a middle, and (c) an upper vertex.

which implies that a middle vertex (x, y) is adjacent to the vertices $(x + 1, y)$, $(x - 1, y)$, $(x + 1, y + 1)$, and $(x - 1, y - 1)$.

Finally, for an upper vertex (x, y) , we have

$$\begin{aligned} (x, y) + d_1 &= (x + 1, y + 1), \\ (x, y) + d_2 &= (x - 1, y + 1), \\ (x, y) - d_1 &= (x - 1, y), \\ (x, y) - d_2 &= (x + 1, y), \end{aligned}$$

which implies that an upper vertex (x, y) is adjacent to the vertices $(x + 1, y)$, $(x - 1, y)$, $(x + 1, y + 1)$, and $(x - 1, y - 1)$.

See Figure 1 for an illustration of these observations.

For $c \in \mathbb{Z}/(d_1 + d_2)\mathbb{Z}$, all vertices (x, y) of G^D with $x = c$ form the *column* c . Similarly, for $r \in \mathbb{Z}$, all vertices (x, y) satisfying $y = r$ form the *row* r . Note that the vertices in a column are either all lower, or all middle, or all upper. A column that consists of lower (middle, upper) vertices is called *lower (middle, upper)*. See Figure 2 for an illustration.

Lemma 2 (i) For $c \in \mathbb{Z}/(d_1 + d_2)\mathbb{Z}$, the column c is lower if and only if the column $c + 1$ is upper.

(ii) Column 0 is lower.

(iii) Column 1 is upper.

Proof: For $x \in \mathbb{Z}/(d_1 + d_2)\mathbb{Z}$, we have $0 \leq xd_1 \pmod{d_1 + d_2} < d_2$ if and only if $d_1 \leq (x + 1)d_1 \pmod{d_1 + d_2} < d_1 + d_2$, which proves (i). (ii) follows, because $0 \leq 0 = 0d_1 \pmod{d_1 + d_2} < d_2$. Finally, (i) and (ii) imply (iii). \square

The columns $x, x + 1, \dots, x + l - 1$ form a *block of length* l , if column x is lower, column $x + l$ is lower, and none of the columns $x + 1, \dots, x + l - 1$ is lower. The block that contains column 0 is denoted by B_0 . Let l be the length of block B_i and let column x be the unique lower column that belongs to block

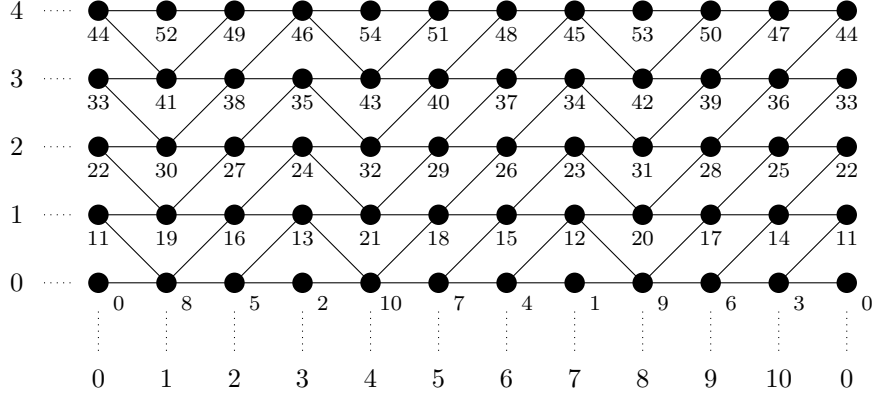


Fig. 2: The distance graph $G_{55}^{\{8,3\}}$. Note that the vertices of column 0 are drawn twice. In order to simplify the drawing, we adopt the convention that such a vertex is adjacent to the union of the neighbors of the two copies, i.e. vertex 22 is adjacent to the vertices 19, 30, 14, and 25.

B_i , then the block that contains column $x + l$ is denoted by B_{i+1} . Note that the indices of the blocks are elements of $\mathbb{Z}/d_2\mathbb{Z}$. For $i \in \mathbb{Z}/d_2\mathbb{Z}$, let x_i denote the unique lower column in block B_i . Figure 3 shows the blocks of $G_{85}^{\{12,5\}}$.

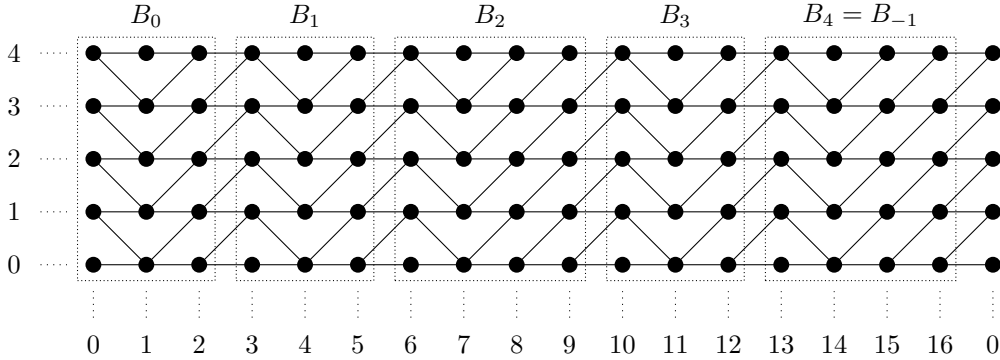


Fig. 3: Blocks of $G_{85}^{\{12,5\}}$. Note that 4 equals -1 in $\mathbb{Z}/5\mathbb{Z}$, that is, $B_4 = B_{-1}$.

Lemma 3 (i) The length of a block is either $\lfloor \frac{d_1}{d_2} \rfloor + 1$ or $\lceil \frac{d_1}{d_2} \rceil + 1$.

(ii) The length of B_0 is $\lfloor \frac{d_1}{d_2} \rfloor + 1$.

(iii) The length of B_{-1} is $\lceil \frac{d_1}{d_2} \rceil + 1$.

(iv) The number of blocks is d_2 .

Proof: Let $x, x+1, \dots, x+l-1$ be the columns of a block B of length l . By definition and Lemma 2 (i), x is the unique lower column of block B , $x+1$ is the unique upper column of block B , and $x+l$ is a lower column. Hence, for all $y \in \mathbb{Z}$ and $x+1 \leq k \leq x+l-1$, we have $(k, y) - (k+1, y) = d_2$ and therefore $(x+1, y) - (x+l, y) = d_2(l-1)$. Since column $x+1$ is upper and column $x+l$ is lower, we have $d_1 - d_2 + 1 \leq (x+1, y) - (x+l, y) \leq d_1 + d_2 - 1$, which implies (i).

If $B = B_0$, then $x = 0$ and $(x+1, y) \equiv d_1 \pmod{d_1+d_2}$ for all $y \in \mathbb{Z}$. Hence $(x+1, y) - (x+l, y) \leq d_1$. Together with $(x+1, y) - (x+l, y) = d_2(l-1)$ this implies (ii).

If $B = B_{-1}$, then $x+l = 0$ and $(x+l, y) \equiv 0 \pmod{d_1+d_2}$ for all $y \in \mathbb{Z}$. Since column $x+1$ is upper, we have $(x+1, y) - (x+l, y) \geq d_1$. Together with $(x+1, y) - (x+l, y) = d_2(l-1)$ this implies (iii).

Since the function $f : \{0, \dots, d_1+d_2-1\} \rightarrow \{0, \dots, d_1+d_2-1\}$ with $f(x) = xd_1 \pmod{d_1+d_2}$ is bijective for $\gcd(d_1, d_2) = 1$, there are exactly d_2 lower columns and therefore d_2 blocks, which proves (iv). \square

3 Hamiltonian paths of G_n^D

The main result of this section is the following.

Theorem 4 For every $D = \{d_1, d_2\} \subseteq \mathbb{N}$ with $d_1 > d_2$ and $\gcd(d_1, d_2) = 1$, there is some $n_0 \in \mathbb{N}$ such that for all integers n with $n \geq n_0$, the distance graph G_n^D has a Hamiltonian path with endvertices 0 and $n-1$.

As before let $D = \{d_1, d_2\}$ for two positive integers d_1 and d_2 such that $\gcd(d_1, d_2) = 1$ and $d_1 > d_2$. For two lower vertices (x, y) and (x', y') with $x \neq x'$ and $y < y'$ in the distance graph G^D , a path in G^D with endvertices (x, y) and (x', y') whose vertex set consists of all vertices in the rows $y, y+1, \dots, y'-1$ and the vertex (x', y') is called an (x, y) - (x', y') -climbing path of G^D . See Figure 5 for an illustration.

Before we proceed to the proof of Theorem 4, we establish a series of lemmas concerning the existence of climbing paths.

Lemma 5 If B_i is a block of even length in G^D , then G^D has an (x_i, y) - $(x_{i+1}, y+2)$ -climbing path for all y .

Proof: Let

$$\begin{aligned} P & : (x_{i+1}-1, y), (x_{i+1}, y+1), (x_{i+1}-1, y+1), (x_{i+1}-2, y), \\ & (x_{i+1}-3, y), (x_{i+1}-2, y+1), (x_{i+1}-3, y+1), (x_{i+1}-4, y), \\ & \dots, (x_i+3, y), (x_i+4, y+1), (x_i+3, y+1), (x_i+2, y). \end{aligned}$$

The sequence

$$\begin{aligned} & (x_i, y), (x_i-1, y), \dots, (x_{i+1}, y), \\ & P, (x_i+1, y), \\ & (x_i+2, y+1), (x_i+1, y+1), \dots, (x_{i+1}+1, y+1), \\ & (x_{i+1}, y+2) \end{aligned}$$

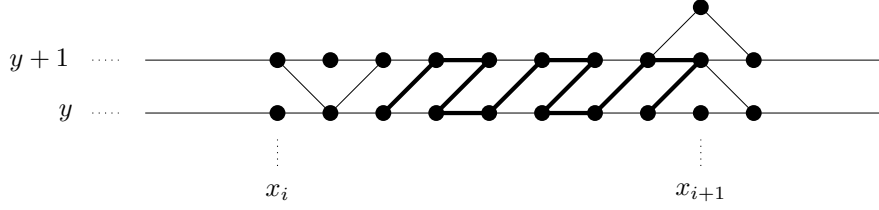


Fig. 4: P for a block B_i of length 8.

defines an (x_i, y) - $(x_{i+1}, y + 2)$ -climbing path in G^D . See Figures 4 and 5 for an illustration. □

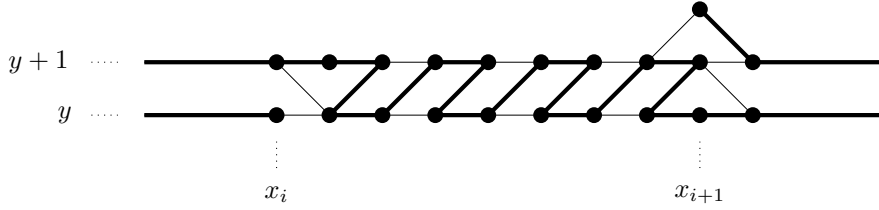


Fig. 5: An (x_i, y) - $(x_{i+1}, y + 2)$ -climbing path for block B_i of length 8.

Lemma 6 *If B_{i-1} is a block of even length in G^D , then G^D has an (x_i, y) - $(x_{i-1}, y + 2)$ -climbing path for all y .*

Proof: Let

$$\begin{aligned}
 P & : (x_{i-1} + 3, y + 1), (x_{i-1} + 2, y), (x_{i-1} + 3, y), (x_{i-1} + 4, y + 1), \\
 & (x_{i-1} + 5, y + 1), (x_{i-1} + 4, y), (x_{i-1} + 5, y), (x_{i-1} + 6, y + 1), \\
 & \dots, (x_i - 1, y + 1), (x_i - 2, y), (x_i - 1, y), (x_i, y + 1).
 \end{aligned}$$

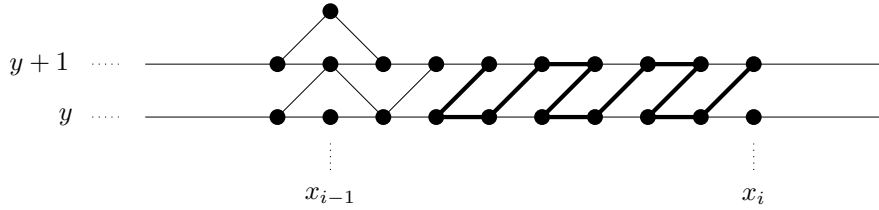


Fig. 6: P for a block B_{i-1} of length 8.

The sequence

$$\begin{aligned} & (x_i, y), (x_i + 1, y), \dots, (x_{i-1} + 1, y), \\ & (x_{i-1}, y + 1), (x_{i-1} + 1, y + 1), (x_{i-1} + 2, y + 1), P, \\ & (x_i + 1, y + 1), (x_i + 2, y + 1), \dots, (x_{i-1} - 1, y + 1), \\ & (x_{i-1}, y + 2) \end{aligned}$$

defines an (x_i, y) - $(x_{i-1}, y + 2)$ -climbing path of G^D . See Figures 6 and 7 for an illustration. \square

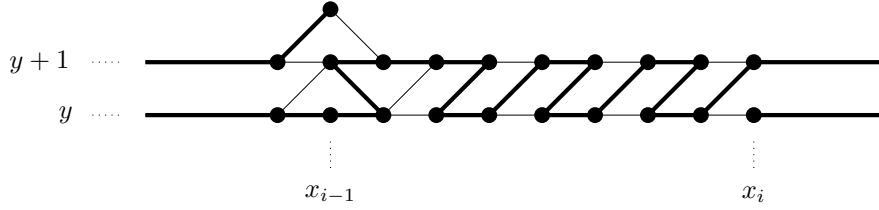


Fig. 7: An (x_i, y) - $(x_{i-1}, y + 2)$ -climbing path for block B_{i-1} of length 8.

Lemma 7 *If G^D has at least $j + 2$ blocks for some $j \geq 1$ and for some $i \in \mathbb{Z}/d_2\mathbb{Z}$, the blocks $B_i, B_{i+1}, \dots, B_{i+j}$ of G^D are such that B_i and B_{i+j} are of odd length and $B_{i+1}, \dots, B_{i+j-1}$ are of even length at least 4, then G^D has an (x_i, y) - $(x_{i+j+1}, y + 3)$ -climbing path for all y .*

Proof: By Lemma 3, the blocks B_i and B_{i+j} are of length at least 3.

Let

$$\begin{aligned} P_{i+j} : & (x_{i+j+1} - 1, y), (x_{i+j+1}, y + 1), (x_{i+j+1} - 1, y + 1), (x_{i+j+1} - 2, y), \\ & (x_{i+j+1} - 3, y), (x_{i+j+1} - 2, y + 1), (x_{i+j+1} - 3, y + 1), (x_{i+j+1} - 4, y), \\ & \dots, (x_{i+j} + 2, y), (x_{i+j} + 3, y + 1), (x_{i+j} + 2, y + 1), (x_{i+j} + 1, y). \end{aligned}$$

For $1 \leq q \leq j - 1$, let

$$\begin{aligned} P_{i+q} : & (x_{i+q} + 3, y + 2), (x_{i+q} + 2, y + 1), (x_{i+q} + 3, y + 1), (x_{i+q} + 4, y + 2), \\ & (x_{i+q} + 5, y + 2), (x_{i+q} + 4, y + 1), (x_{i+q} + 5, y + 1), (x_{i+q} + 6, y + 2), \\ & \dots, (x_{i+q+1} - 3, y + 2), (x_{i+q+1} - 4, y + 1), (x_{i+q+1} - 3, y + 1), (x_{i+q+1} - 2, y + 2) \end{aligned}$$

and let

$$\begin{aligned} P'_{i+q} : & P_{i+q}, (x_{i+q+1} - 1, y + 2), (x_{i+q+1} - 2, y + 1), (x_{i+q+1} - 1, y + 1), (x_{i+q+1}, y + 1), \\ & (x_{i+q+1} + 1, y + 1), (x_{i+q+1}, y + 2), (x_{i+q+1} + 1, y + 2), (x_{i+q+1} + 2, y + 2). \end{aligned}$$

Note that P_{i+q} is empty if B_{i+q} is of length 4. Furthermore, let

$$\begin{aligned} P_i : & (x_{i+1} - 2, y + 2), (x_{i+1} - 3, y + 1), (x_{i+1} - 4, y + 1), (x_{i+1} - 3, y + 2), \\ & (x_{i+1} - 4, y + 2), (x_{i+1} - 5, y + 1), (x_{i+1} - 6, y + 1), (x_{i+1} - 5, y + 2), \\ & \dots, (x_i + 3, y + 2), (x_i + 2, y + 1), (x_i + 1, y + 1), (x_i + 2, y + 2). \end{aligned}$$

Note that P_i is empty if B_i is of length 3.

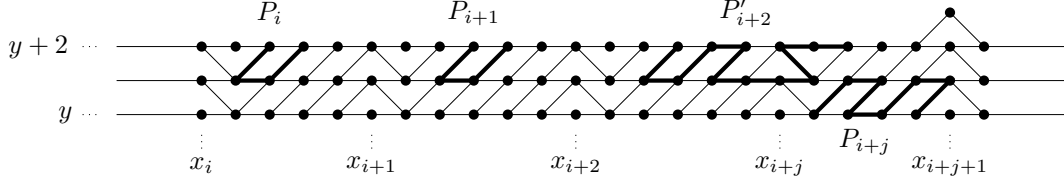


Fig. 8: $P_i, P_{i+1}, P'_{i+2},$ and P_{i+j} for $j = 3$.

Now, the sequence

$$\begin{aligned}
 & (x_i, y), (x_i - 1, y), \dots, (x_{i+j+1}, y), P_{i+j}, \\
 & (x_{i+j}, y), (x_{i+j} - 1, y), \dots, (x_{i+1} - 1, y), \\
 & (x_{i+1}, y + 1), (x_{i+1} + 1, y + 1), (x_{i+1}, y + 2), (x_{i+1} + 1, y + 2), (x_{i+1} + 2, y + 2), \\
 & P'_{i+1}, P'_{i+2}, \dots, P'_{i+j-1}, \\
 & (x_{i+j} + 3, y + 2), (x_{i+j} + 4, y + 2), \dots, (x_{i+j+1}, y + 2), \\
 & (x_{i+j+1} + 1, y + 1), (x_{i+j+1} + 2, y + 1), \dots, (x_i, y + 1), \\
 & (x_i + 1, y), (x_i + 2, y), \dots, (x_{i+1} - 2, y), \\
 & (x_{i+1} - 1, y + 1), (x_{i+1} - 2, y + 1), (x_{i+1} - 1, y + 2), P_i, \\
 & (x_i + 1, y + 2), (x_i, y + 2), \dots, (x_{i+j+1} + 1, y + 2), \\
 & (x_{i+j+1}, y + 3)
 \end{aligned}$$

defines an (x_i, y) - $(x_{i+j+1}, y + 3)$ -climbing path of G^D . See Figures 8 and 9 for an illustration. \square

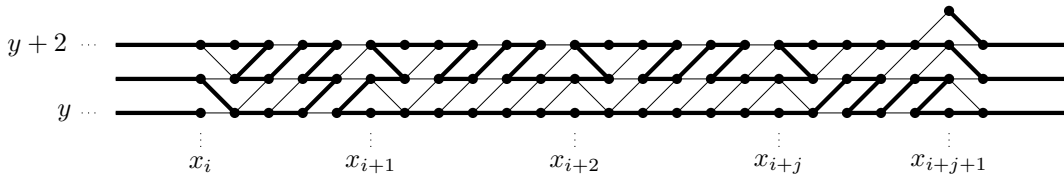


Fig. 9: An (x_i, y) - $(x_{i+j+1}, y + 3)$ -climbing path for $j = 3$.

Lemma 8 *If G^D has at least $j + 2$ blocks for some $j \geq 1$ and for some $i \in \mathbb{Z}/d_2\mathbb{Z}$, the blocks $B_i, B_{i+1}, \dots, B_{i+j}$ of G^D are such that B_i and B_{i+j} are of length 3 and $B_{i+1}, \dots, B_{i+j-1}$ are of length 2, then G^D has an (x_i, y) - $(x_{i+j+1}, y + j + 2)$ -climbing path for all y .*

Proof: Note that $x_{i+j+1} = x_i + 2j + 4$. For $1 \leq q \leq j - 1$, let

$$P_q : (x_i + 2j + 2, y + q), (x_i + 2j + 3, y + q + 1), (x_i + 2j + 4, y + q + 2),$$

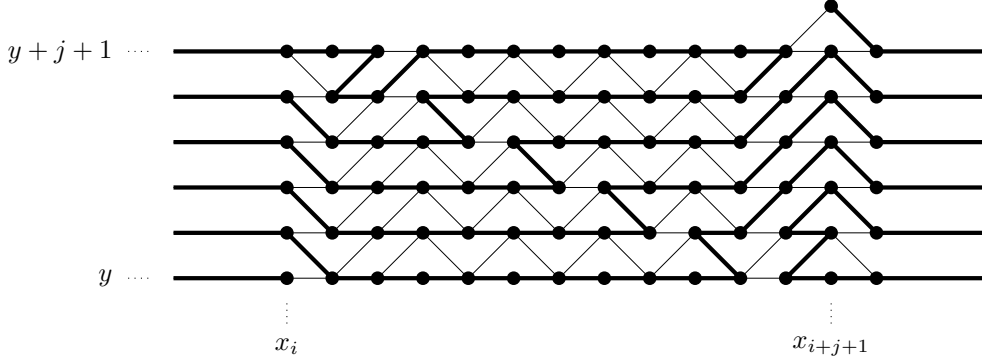


Fig. 11: An (x_i, y) - $(x_{i+j+1}, y + j + 2)$ -climbing path for $j = 4$.

$\gcd(p_1, p_2) = 1$ such that the distance graph G^D has a special path of length p_1 and a special path of length p_2 . Note that we can shift special paths. If $P : v_0, \dots, v_l$ is a special path, then also $P + h : v_0 + h, \dots, v_l + h$ is a special path. Furthermore, we can concatenate special paths. If $P : v_0, \dots, v_l$ is a special path of length l and $Q : v_l, \dots, v_{l+h}$ is a special path of length h , then $PQ : v_0, \dots, v_{l+h}$ is a special path of length $l + h$. Since $\gcd(p_1, p_2) = 1$, it follows from the extended Euclidean algorithm that every sufficiently large integer p is a positive integral linear combination of p_1 and p_2 . In fact, if $p > (2p_2 - 1)p_1$ and $p = a_1p_1 + a_2p_2$ for integers a_1 and a_2 such that $a_1 \leq 0$, then $a_1 + sp_2 > 0$ and $a_2 - sp_1 > 0$ for $s = \lceil \frac{-a_1}{p_2} \rceil + 1$ and thus $p = (a_1 + sp_2)p_1 + (a_2 - sp_1)p_2$ is a positive integral linear combination of p_1 and p_2 . Therefore, the desired result follows by shifting and concatenating copies of the special paths of lengths p_1 and p_2 , which we construct now.

It has been observed in [25, 21] that $G^D_{d_1+d_2+1}$ has a Hamiltonian path with endvertices 0 and $d_1 + d_2$. Hence, for p_1 , we choose $d_1 + d_2$.

For p_2 , we show that there is a positive integer p_2 with $p_2 \equiv -1 \pmod{d_1 + d_2} \equiv -1 \pmod{p_1}$, such that G^D has a special path of length p_2 , thus $\gcd(p_1, p_2) = 1$.

Let x' be such that $x'd_1 \equiv -1 \pmod{d_1 + d_2}$. By definition and Lemma 2 (i), column x' is upper and column $x' - 1$ is lower. In order to get a special path with endvertices $(0, 0)$ and (x', y') for some y' , we concatenate climbing paths to form a $(0, 0)$ - $(x' - 1, y')$ -climbing path and append the path $(x' - 2, y'), (x' - 3, y'), \dots, (x', y')$.

Let k be such that the block B_k contains column x' , that is, $x_k = x' - 1$. Since column x' is upper, column $x' - 2$ belongs to block B_{k-1} .

Since $\gcd(d_1, d_2) = 1$, at least one of d_1 and d_2 is odd.

Case 1 One of d_1 and d_2 is even and G^D has at most 2 blocks of odd length.

Since $d_1 + d_2$ is odd, the number of blocks of odd length is odd, that is, it equals 1.

We first assume that all blocks B_0, B_1, \dots, B_{k-1} are of even length. By Lemma 5, there exists an $(x_i, 2i)$ - $(x_{i+1}, 2i + 2)$ -climbing path P_i for $0 \leq i \leq k - 1$. Since $x' - 1 = x_k$, the concatenation of the paths P_0, P_1, \dots, P_{k-1} forms a $(0, 0)$ - $(x' - 1, y')$ -climbing path for $y' = 2k$.

Next, we assume that all blocks $B_k, B_{k+1}, \dots, B_{-1}$ are of even length. Then, by Lemma 6, there exists an $(x_{i+1}, 2(d_1 + d_2 - i) - 2) - (x_i, 2(d_1 + d_2 - i))$ -climbing path P_i for $k \leq i \leq d_1 + d_2 - 1$. Since $x' - 1 = x_k$, the concatenation of the paths $P_{d_1+d_2-1}, P_{d_1+d_2-2}, \dots, P_k$ forms a $(0, 0) - (x' - 1, y')$ -climbing path for $y' = 2(d_1 + d_2 - k)$. This concludes the first case.

Case 2 *One of d_1 and d_2 is even and G^D has at least 3 blocks of odd length.*

Since $d_1 + d_2$ is odd, the number of blocks of odd length is odd. This implies that one of the two sequences B_0, B_1, \dots, B_{k-1} and $B_0, B_1, \dots, B_{-1}, B_0, B_1, \dots, B_{k-1}$ has an even number of blocks with odd length. We call this sequence \mathcal{S} . We can partition \mathcal{S} into subsequences S_1, S_2, \dots, S_t , where each subsequence is either a block of even length or a sequence $B_i, B_{i+1}, \dots, B_{i+j}$ of blocks with $i \in \mathbb{Z}/d_2\mathbb{Z}$ and $j \geq 1$, such that block B_i has odd length, block B_{i+j} has odd length, and blocks $B_{i+1}, \dots, B_{i+j-1}$ have even length. For a subsequence S_q , $1 \leq q \leq t$, that consists of one block B_i with $i \in \mathbb{Z}/d_2\mathbb{Z}$, Lemma 5 implies that there exists an $(x_i, y) - (x_{i+1}, y + 2)$ -climbing path $P_{q,y}$ for every y . If $\frac{d_1}{d_2} < 2$, then Lemma 3 implies that the lengths of the blocks are 2 and 3. For a subsequence S_q , $1 \leq q \leq t$, that consists of at least two blocks $B_i, B_{i+1}, \dots, B_{i+j}$ with $i \in \mathbb{Z}/d_2\mathbb{Z}$ and $j \geq 1$, Lemma 8 implies that there exists an $(x_i, y) - (x_{i+j+1}, y + j + 2)$ -climbing path $P_{q,y}$ for every y . If $\frac{d_1}{d_2} \geq 2$, then Lemma 3 implies that the lengths of the blocks are at least 3. For a subsequence S_q , $1 \leq q \leq t$, that consists of at least two blocks $B_i, B_{i+1}, \dots, B_{i+j}$ with $i \in \mathbb{Z}/d_2\mathbb{Z}$ and $j \geq 1$, Lemma 7 implies that there exists an $(x_i, y) - (x_{i+j+1}, y + 3)$ -climbing path $P_{q,y}$ for every y .

The concatenation of the paths $P_{1,y_1}, P_{2,y_2}, \dots, P_{t,y_t}$ forms a $(0, 0) - (x' - 1, y')$ -climbing path for $y_1 = 0$, suitable y_q 's, where $2 \leq q \leq t$, and $y' = y_t$.

This concludes the second case.

If both d_1 and d_2 are odd, then $d_1 + d_2$ is even, which implies that the number of blocks of odd length is even and exactly those vertices are even integers that are in a column with an even index. This implies that x' is odd and $x_k = x' - 1$ is even. Since column 0 and column $x' - 1$ are lower, the sequence B_0, B_1, \dots, B_{k-1} has an even number of blocks with odd length.

Case 3 *Both d_1 and d_2 are odd and G^D has at most 2 blocks of odd length.*

Since $d_2 \geq 2$, G has exactly 2 blocks of odd length. This implies that one of the two sequences B_0, B_1, \dots, B_{k-1} and $B_k, B_{k+1}, \dots, B_{-1}$ has only blocks of even length. Now we are in the same situation as in Case 1. Arguing as in Case 1 completes this case.

Case 4 *Both d_1 and d_2 are odd and G^D has at least 4 blocks of odd length.*

Since the sequence B_0, B_1, \dots, B_{k-1} has an even number of blocks of odd length, we are in the same situation as in Case 2. Arguing as in Case 2 completes this case, which concludes the proof of the theorem. \square

4 Hamiltonian cycles of G_n^D

The main results of this section are the following.

Theorem 9 *For every $D = \{d_1, d_2\} \subseteq \mathbb{N}$ with $d_1 > d_2$, $d_1 d_2$ odd, and $\gcd(d_1, d_2) = 1$, there is some $n_0 \in \mathbb{N}$ such that for all even integers n with $n \geq n_0$, the distance graph G_n^D has a Hamiltonian cycle.*

The sequence

$$\begin{aligned} & (0, y), (1, y), \dots, (x_i + 1, y), \\ & (x_i, y + 1), (x_i + 1, y + 1), (x_i + 2, y + 1), P, \\ & (x_{i+1} + 1, y + 1), (x_{i+1} + 2, y + 1), \dots, (-1, y + 1) \end{aligned}$$

defines an (x_{i+1}, y) - $(x_i, y + 1)$ -path-collection in G^D . See Figures 12 and 13 for an illustration. \square

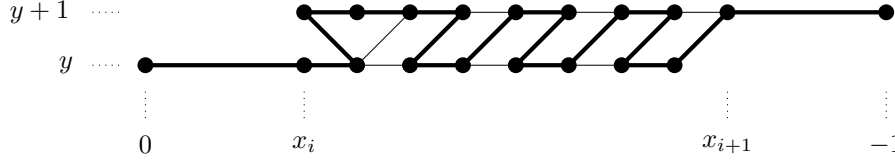


Fig. 13: An (x_{i+1}, y) - $(x_i, y + 1)$ -path-collection for a block B_i of length 8.

Lemma 12 *If for some $i \in \mathbb{Z}/d_2\mathbb{Z}$ and for some $j \geq 1$, the blocks $B_i, B_{i+1}, \dots, B_{i+j}$ of G^D are such that $-1 \notin \{i, i + 1, \dots, i + j\}$, B_i and B_{i+j} are of odd length and $B_{i+1}, \dots, B_{i+j-1}$ are of even length at least 4, then G^D has an (x_{i+j+1}, y) - $(x_i, y + 2)$ -path-collection for all y .*

Proof: By Lemma 3, the blocks B_i and B_{i+j} are of length at least 3.

Let

$$\begin{aligned} P_i & : (x_i + 1, y), (x_i + 2, y + 1), (x_i + 3, y + 1), (x_i + 2, y), \\ & (x_i + 3, y), (x_i + 4, y + 1), (x_i + 5, y + 1), (x_i + 4, y), \\ & \dots, (x_{i+1} - 2, y), (x_{i+1} - 1, y + 1), (x_{i+1}, y + 1), (x_{i+1} - 1, y). \end{aligned}$$

For $1 \leq q \leq j - 1$, let

$$\begin{aligned} P_{i+q} & : (x_{i+q}, y), (x_{i+q} + 1, y), \\ & (x_{i+q} + 2, y), (x_{i+q} + 3, y + 1), (x_{i+q} + 4, y + 1), (x_{i+q} + 3, y), \\ & (x_{i+q} + 4, y), (x_{i+q} + 5, y + 1), (x_{i+q} + 6, y + 1), (x_{i+q} + 5, y), \\ & \dots, (x_{i+q+1} - 2, y), (x_{i+q+1} - 1, y + 1), (x_{i+q+1}, y + 1), (x_{i+q+1} - 1, y). \end{aligned}$$

Furthermore, let

$$\begin{aligned} P_{i+j} & : (x_{i+j} + 2, y), (x_{i+j} + 3, y + 1), (x_{i+j} + 4, y + 1), (x_{i+j} + 3, y), \\ & (x_{i+j} + 4, y), (x_{i+j} + 5, y + 1), (x_{i+j} + 6, y + 1), (x_{i+j} + 5, y), \\ & \dots, (x_{i+j+1} - 3, y), (x_{i+j+1} - 2, y + 1), (x_{i+j+1} - 1, y + 1), (x_{i+j+1} - 2, y). \end{aligned}$$

For $1 \leq q \leq j$, let

$$\begin{aligned} Q_{i+q} & : (x_{i+q-1} + 4, y + 2), (x_{i+q-1} + 5, y + 2), \dots, (x_{i+q} + 2, y + 2), \\ & (x_{i+q} + 1, y + 1), (x_{i+q} + 2, y + 1), (x_{i+q} + 3, y + 2). \end{aligned}$$

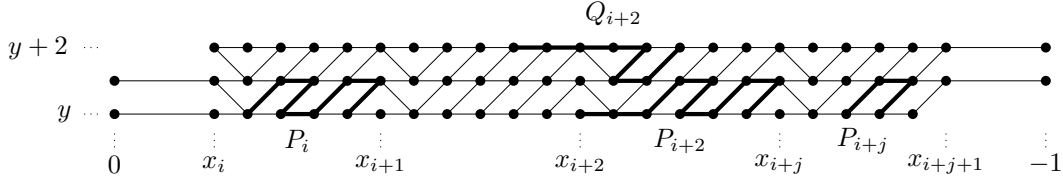


Fig. 14: P_i, P_{i+2}, P_{i+j} , and Q_{i+2} for $j = 3$.

Now, R_y and R_{y+1} , where

$$\begin{aligned}
 R_y & : (0, y), (1, y), \dots, (x_i, y), \\
 & P_i, \\
 & P_{i+1}, P_{i+2}, \dots, P_{i+j-1}, \\
 & (x_{i+j}, y), (x_{i+j} + 1, y), P_{i+j}, (x_{i+j+1} - 1, y), \\
 & (x_{i+j+1}, y + 1), (x_{i+j+1} + 1, y + 1), \dots, (-1, y + 1)
 \end{aligned}$$

and

$$\begin{aligned}
 R_{y+1} & : (0, y + 1), (1, y + 1), \dots, (x_i + 1, y + 1), \\
 & (x_i, y + 2), (x_i + 1, y + 2), (x_i + 2, y + 2), (x_i + 3, y + 2), \\
 & Q_{i+1}, Q_{i+2}, \dots, Q_{i+j}, \\
 & (x_{i+j+1} - 1, y + 2), (x_{i+j+1}, y + 2), \dots, (-1, y + 2)
 \end{aligned}$$

define an $(x_{i+j+1}, y)-(x_i, y + 2)$ -path-collection of G^D . See Figures 14 and 15 for an illustration. \square

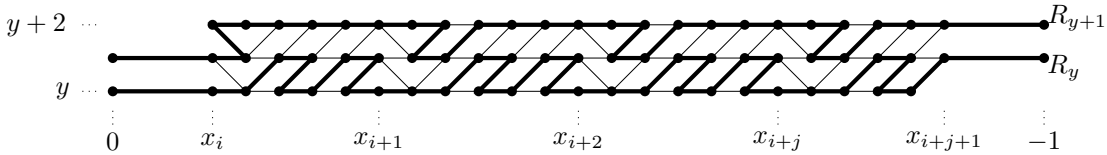


Fig. 15: An $(x_{i+j+1}, y)-(x_i, y + 2)$ -path-collection for $j = 3$.

Lemma 13 *If for some $i \in \mathbb{Z}/d_2\mathbb{Z}$ and for some $j \geq 1$, the blocks $B_i, B_{i+1}, \dots, B_{i+j}$ of G^D are such that $-1 \notin \{i, i + 1, \dots, i + j\}$, B_i and B_{i+j} are of length 3 and $B_{i+1}, \dots, B_{i+j-1}$ are of length 2, then G^D has an $(x_{i+j+1}, y)-(x_i, y + j + 1)$ -path-collection for all y .*

Proof: Note that $x_{i+j+1} = x_i + 2j + 4$. For $0 \leq q \leq j - 1$, let

$$R_{i+q} : (0, y + q), (1, y + q), \dots, (x_i + 1, y + q),$$

$$\begin{aligned}
 & (x_i + 2, y + q + 1), (x_i + 3, y + q + 1), \dots, (x_i + 2q + 3, y + q + 1), \\
 & (x_i + 2q + 2, y + q), (x_i + 2q + 3, y + q), \dots, (x_i + 2j + 3, y + q), \\
 & (x_i + 2j + 4, y + q + 1), (x_i + 2j + 5, y + q + 1), \dots, (-1, y + q + 1).
 \end{aligned}$$

and let

$$\begin{aligned}
 R_{i+j} & : (0, y + j), (1, y + j), \dots, (x_i + 1, y + j), \\
 & (x_i, y + j + 1), (x_i + 1, y + j + 1), \dots, (x_i + 2j + 3, y + j + 1), \\
 & (x_i + 2j + 2, y + j), (x_i + 2j + 3, y + j), \\
 & (x_i + 2j + 4, y + j + 1), (x_i + 2j + 5, y + j + 1), \dots, (-1, y + j + 1).
 \end{aligned}$$

Now, $R_i, R_{i+1}, \dots, R_{i+j}$ define an $(x_{i+j+1}, y) - (x_i, y + j + 1)$ -path-collection of G^D . See Figure 16 for an illustration. \square

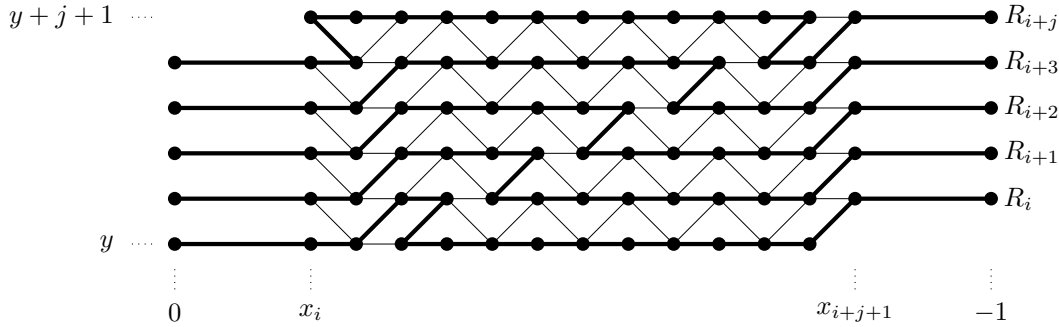


Fig. 16: An $(x_{i+j+1}, y) - (x_i, y + j + 1)$ -path-collection for $j = 4$.

Lemma 14 *If for some $i \in \mathbb{Z}/d_2\mathbb{Z}$ and for some $j \geq 0$, the sequence $\mathcal{S} = B_i, B_{i+1}, \dots, B_{i+j}$ of blocks of G^D is such that $-1 \notin \{i, i + 1, \dots, i + j\}$ and the number of blocks of odd length among $B_i, B_{i+1}, \dots, B_{i+j}$ is even, then G^D has an $(x_{i+j+1}, y) - (x_i, y + \Delta y)$ -path-collection for some Δy and for all y .*

Proof: By definition, the union of suitable path-collections is a path-collection: If for some x, x', x'', y, y', y'' , G^D has an $(x, y) - (x', y')$ -path-collection and an $(x', y') - (x'', y'')$ -path-collection, then G^D has an $(x, y) - (x'', y'')$ -path-collection. We can partition \mathcal{S} into subsequences, where each subsequence is either a block of even length or a sequence $B_k, B_{k+1}, \dots, B_{k+l}$ of blocks with $k \in \mathbb{Z}/d_2\mathbb{Z}$ and $l \geq 1$, such that blocks B_k and B_{k+l} have odd length and blocks $B_{k+1}, \dots, B_{k+l-1}$ have even length. For a subsequence that consists of one even block B_k with $k \in \mathbb{Z}/d_2\mathbb{Z}$, Lemma 11 implies that there exists a $(x_{k+1}, y) - (x_k, y + 1)$ path collection for every y . If $\frac{d_1}{d_2} < 2$, then Lemma 3 implies that the lengths of the blocks are 2 and 3. For a subsequence that consists of at least two blocks $B_k, B_{k+1}, \dots, B_{k+l}$ with $k \in \mathbb{Z}/d_2\mathbb{Z}$ and $l \geq 1$, Lemma 13 implies that there exists an $(x_{k+l+1}, y) - (x_k, y + l + 1)$ -path-collection for every y . If $\frac{d_1}{d_2} \geq 2$, then Lemma 3 implies that the lengths of the blocks are at least 3. For a subsequence that consists

of at least two blocks $B_k, B_{k+1}, \dots, B_{k+l}$ with $k \in \mathbb{Z}/d_2\mathbb{Z}$ and $l \geq 1$, Lemma 12 implies that there exists an (x_{k+l+1}, y) - $(x_k, y+2)$ -path-collection for every y . Hence, a suitable union of path-collections forms an (x_{i+j+1}, y) - $(x_i, y+\Delta y)$ -path-collection for a suitable Δy and all y . \square

Lemma 15 *If for some $-i \in \mathbb{Z}/d_2\mathbb{Z}$, the blocks $B_{-i}, B_{-i+1}, \dots, B_{-1}$ of G^D are such that B_{-i} is of odd length and B_{-i+1}, \dots, B_{-1} are of even length at least 4, then for all y , G^D has a path with endvertices $(-1, y+1)$ and $(-1, y+2)$ that consists of all vertices of rows y and $y+1$ and the vertices $(x_{-i}, y+2), (x_{-i}+1, y+2), \dots, (-1, y+2)$.*

Proof: For $1 \leq q \leq i-1$, let

$$Q_{-q} : (x_{-q+1} - 3, y), (x_{-q+1} - 4, y), \dots, (x_{-q}, y), \\ (x_{-q} - 1, y), (x_{-q}, y+1), (x_{-q} - 1, y+1), (x_{-q} - 2, y)$$

and let

$$Q_{-i} : (x_{-i+1} - 3, y), (x_{-i+1} - 2, y+1), (x_{-i+1} - 3, y+1), (x_{-i+1} - 4, y), \\ (x_{-i+1} - 5, y), (x_{-i+1} - 4, y+1), (x_{-i+1} - 5, y+1), (x_{-i+1} - 6, y), \\ \dots, (x_{-i} + 2, y), (x_{-i} + 3, y+1), (x_{-i} + 2, y+1), (x_{-i} + 1, y).$$

Furthermore, let for $1 \leq q \leq i-1$

$$P_{-q} : (x_{-q}, y+2), (x_{-q} + 1, y+2), \\ (x_{-q} + 2, y+2), (x_{-q} + 1, y+1), (x_{-q} + 2, y+1), (x_{-q} + 3, y+2), \\ (x_{-q} + 4, y+2), (x_{-q} + 3, y+1), (x_{-q} + 4, y+1), (x_{-q} + 5, y+2), \\ \dots, (x_{-q+1} - 2, y+2), (x_{-q+1} - 3, y+1), (x_{-q+1} - 2, y+1), (x_{-q+1} - 1, y+2).$$

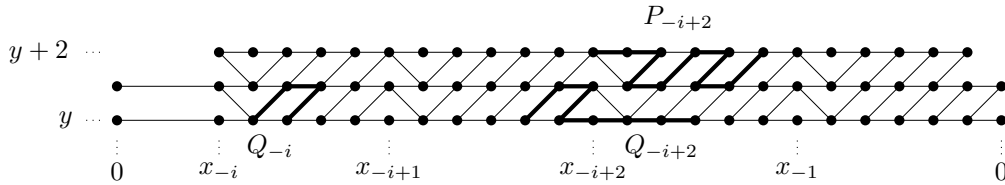


Fig. 17: Q_{-i} , Q_{-i+2} , and P_{-i+2} for $i = 4$.

Now, the sequence

$$(-1, y+1), (-2, y), \\ Q_{-1}, Q_{-2}, \dots, Q_{-i}, \\ (x_{-i}, y), (x_{-i} - 1, y), \dots, (-1, y), \\ (0, y+1), (1, y+1), \dots, (x_{-i} + 1, y+1), \\ (x_{-i}, y+2), (x_{-i} + 1, y+2), \dots, (x_{-i+1} - 1, y+2), \\ P_{-i+1}, P_{-i+2}, \dots, P_{-1}$$

defines a path that satisfies the conditions of the lemma. See Figures 17 and 18 for an illustration. \square

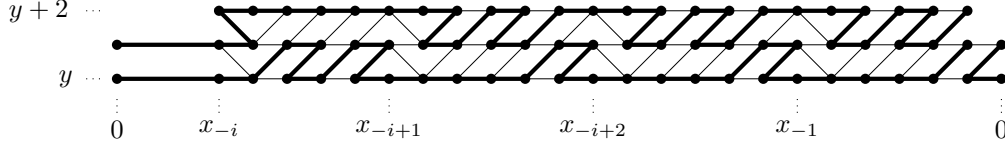


Fig. 18: A path for $i = 4$.

A cycle C in G^D is called *special*, if $V(C) = [\min(V(C)), \max(V(C))]$.

Lemma 16 For every $D = \{d_1, d_2\} \subseteq \mathbb{N}$ with $d_1 > d_2$, $d_1 d_2$ even, and $\gcd(d_1, d_2) = 1$, there is some $n \in \mathbb{N}$ with $n \equiv 0 \pmod{d_1 + d_2}$ such that G^D has a special cycle C of order $n + 1$ with $V(C) = [0, n]$.

Proof: Clearly, vertex n is in column 0. Since $d_1 d_2$ is even and $\gcd(d_1, d_2) = 1$, we obtain that $d_1 + d_2$ is odd and hence the number of blocks of odd length is odd, i.e. at least 1. Let $i \in \mathbb{Z}/d_2\mathbb{Z}$, such that block B_i is of odd length and the blocks B_{i+1}, \dots, B_{-1} are of even length. Clearly, by Lemma 3, the length of the blocks B_{i+1}, \dots, B_{-1} are at least 4. By Lemma 15, G^D has a path Q with endvertices $(-1, 1)$ and $(-1, 2)$ that consists of all vertices of rows 0 and 1 and the vertices $(x_i, 2), (x_i + 1, 2), \dots, (-1, 2)$. Since the number of blocks of B_0, \dots, B_{i-1} of odd length is even, by Lemma 14, G^D has an $(x_i, 2)$ - $(0, y')$ -path-collection \mathcal{R} for some y' . Note, that if G^D has only one block of odd length, then $\mathcal{R} = \emptyset$. In this case we define $y' = 2$. Let

$$P = \left(Q \cup \bigcup_{R \in \mathcal{R}} R \right) + \bigcup_{y=1}^{y'-2} \{(-1, y), (0, y+1)\}.$$

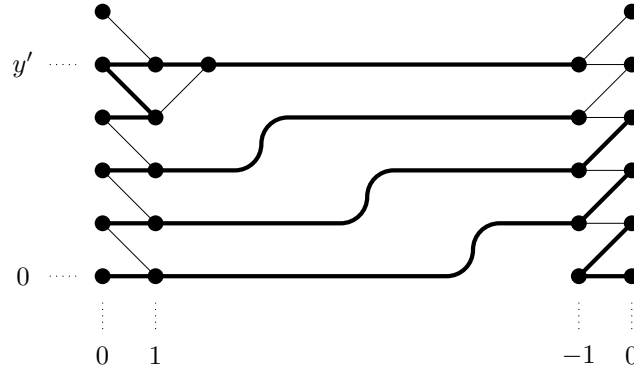


Fig. 19: The path P .

By construction, P is a path with endvertices $(-1, y' - 1)$ and $(-1, y')$ that consists of all vertices of rows $0, 1, \dots, y'$. The vertex $(0, y')$ has the neighbors $(1, y' - 1)$ and $(1, y')$ in P . Since the vertex $(1, y')$ is an upper vertex, $(1, y')$ has the neighbors $(0, y')$ and $(2, y')$ in P and $\{(1, y' - 1), (2, y')\} \in E(G^D)$. Now,

$$\begin{aligned} C = P & \\ & + \{ \{(1, y' - 1), (2, y')\}, \{(-1, y' - 1), (0, y')\}, \{(-1, y'), (0, y' + 1)\}, \{(0, y' + 1), (1, y')\} \} \\ & - \{ \{(1, y' - 1), (0, y')\}, \{(1, y'), (2, y')\} \} \end{aligned}$$

is a special cycle of G^D of order $n + 1$ with $n = (y' + 1)(d_1 + d_2)$ and $V(C) = [0, n]$. See Figures 19 and 20 for an illustration. \square

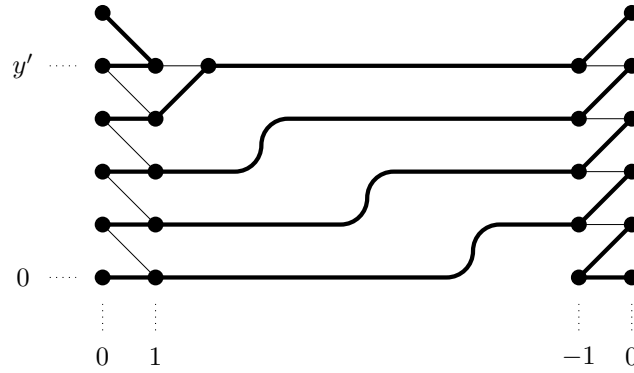


Fig. 20: The cycle C in the proof of Lemma 16.

Lemma 17 For every $D = \{d_1, d_2\} \subseteq \mathbb{N}$ with $d_1 > d_2$, $d_1 d_2$ odd, and $\gcd(d_1, d_2) = 1$, there is some $n \in \mathbb{N}$ with $n \equiv 0 \pmod{d_1 + d_2}$ such that G^D has a special cycle C of order $n + 2$ with $V(C) = [0, n + 1]$.

Proof: Clearly, vertex n is in column 0. First we assume that $d_2 = 1$. In that case, G^D has only one block and the vertex $n + 1$ is in column -1 . Let $P = \emptyset$ for $d_1 = 3$, otherwise let

$$\begin{aligned} P : & (1, 0), (2, 1), (3, 1), (2, 0), \\ & (3, 0), (4, 1), (5, 1), (4, 0), \\ & \dots, (-5, 0), (-4, 1), (-3, 1), (-4, 0). \end{aligned}$$

The sequence

$$C : (0, 0), P, (-3, 0), (-2, 0), (-1, 1), (-2, 1), (-1, 2), (0, 2), (1, 1), (0, 1), (-1, 0), (0, 0)$$

defines a special cycle of G^D of order $2(d_1 + d_2) + 2$ with $V(C) = [0, 2(d_1 + d_2) + 1]$. See Figure 21 for an illustration.

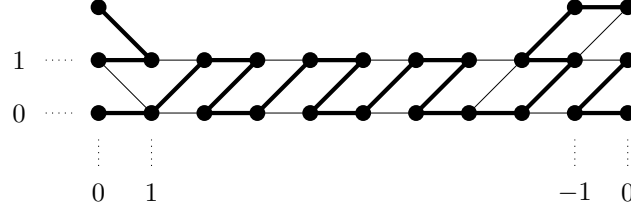


Fig. 21: The special cycle C for $d_1 = 9$ and $d_2 = 1$.

Now we assume that $d_2 > 1$. Hence, by Lemma 3, G^D has more than one block. This implies that vertex $n + 1$ is lower. Let $k \in \mathbb{Z}/d_2\mathbb{Z}$, such that vertex $n + 1$ belongs to block B_k . Since $d_1 + d_2$ is even, exactly those vertices are even integers that are in a column with an even index. Since vertex $n + 1$ is lower and an odd integer, the number of blocks among $B_k, B_{k+1}, \dots, B_{-1}$ of odd length is odd, i.e. at least one. Let $i \in \mathbb{Z}/d_2\mathbb{Z}$ be such that block B_i is of odd length and the blocks $B_{i+1}, B_{i+2}, \dots, B_{-1}$ are of even length. Clearly, by Lemma 3, the length of the blocks $B_{i+1}, B_{i+2}, \dots, B_{-1}$ are at least 4. By Lemma 15, G^D has a path Q_1 with endvertices $(-1, 1)$ and $(-1, 2)$ that consists of all vertices of rows 0 and 1 and the vertices $(x_i, 2), (x_{i+1}, 2), \dots, (-1, 2)$. Since the number of blocks of $B_k, B_{k+1}, \dots, B_{i-1}$ of odd length is even, by Lemma 14, G^D has an $(x_i, 2)$ - (x_k, y') -path-collection \mathcal{R}_1 for some y' . Note, that if $i = k$, then $\mathcal{R}_1 = \emptyset$. In this case we define $y' = 2$. By the same arguments, G^D has a path Q_2 with endvertices $(-1, y' + 2)$ and $(-1, y' + 3)$ that consists of all vertices of rows $y' + 1$ and $y' + 2$ and the vertices $(x_i, y' + 3), (x_{i+1}, y' + 3), \dots, (-1, y' + 3)$ and G^D has an $(x_i, y' + 3)$ - $(x_k, 2y' + 1)$ -path-collection \mathcal{R}_2 .

By definition, for every $y' + 1 \leq y \leq 2y'$, the edges $\{(0, y), (1, y)\}$ and $\{(x_k - 1, y), (x_k, y)\}$ belong to Q_2 or a path in \mathcal{R}_2 . Furthermore, the path

$$P_0 : (x_k + 1, 2y'), (x_k, 2y' + 1), (x_k + 1, 2y' + 1), (x_k + 2, 2y' + 1)$$

is a subpath of a path in $\{Q_2\} \cup \mathcal{R}_2$. Let

$$P_1 : (0, y'), (1, y'), \dots, (x_k - 1, y')$$

and let

$$P_2 : (x_k, 2y' + 1), (x_k + 1, 2y' + 1), (x_k, 2y' + 2), \\ (x_k - 1, 2y' + 1), (x_k - 2, 2y' + 1), \dots, (1, 2y' + 1).$$

Now,

$$C = (Q_1 \cup Q_2 \cup \mathcal{R}_1 \cup \mathcal{R}_2) \\ - \left(E(P_0) \cup \bigcup_{y=y'+1}^{2y'} \{ \{(0, y), (1, y)\} \} \cup \bigcup_{y=y'+1}^{2y'} \{ \{(x_k - 1, y), (x_k, y)\} \} \right) \\ + E(P_1) \cup E(P_2) \\ + \bigcup_{y=1}^{y'} \{ \{(-1, y), (0, y + 1)\} \}$$

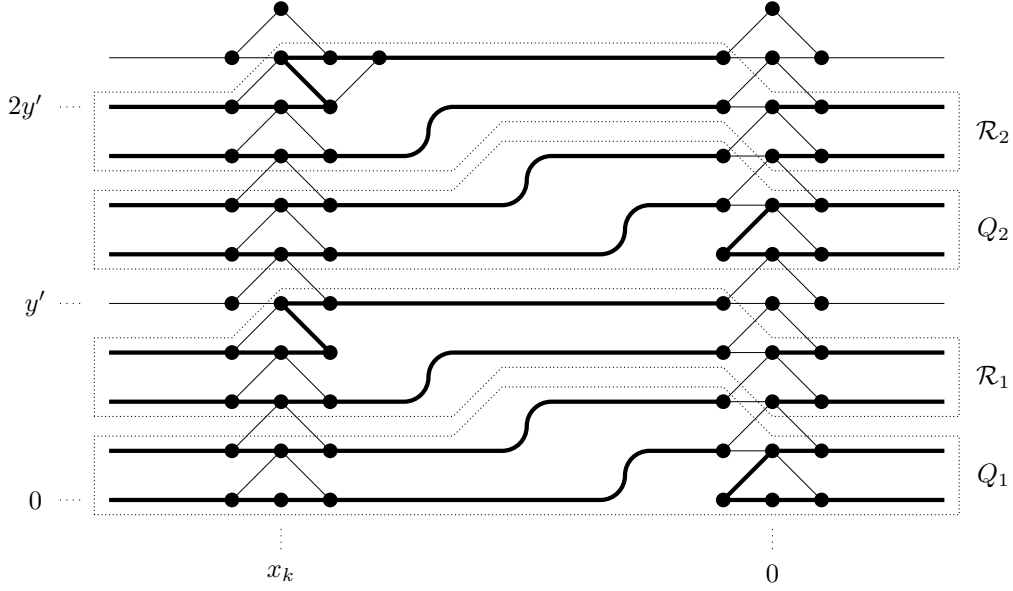


Fig. 22: $Q_1 \cup Q_2 \cup R_1 \cup R_2$.

$$\begin{aligned}
& + \bigcup_{y=y'}^{2y'} \{ \{ (x_k - 1, y), (x_k, y + 1) \} \} \\
& + \bigcup_{y=y'+2}^{2y'+1} \{ \{ (-1, y), (0, y + 1) \} \} \\
& + \bigcup_{y=y'+1}^{2y'+1} \{ \{ (0, y + 1), (1, y) \} \} \\
& + \{ \{ (x_k + 1, 2y'), (x_k + 2, 2y' + 1) \} \}
\end{aligned}$$

defines a special cycle of G^D of order $n + 2$ with $n = (2y' + 2)(d_1 + d_2) + 2$ and $V(C) = [0, n + 1]$. See Figures 22 and 23 for an illustration. \square

Let C be a special cycle of G^D and let $n' = \max(V(C))$. If for all $a, b \in V(C)$ with $n' - d_1 + 1 \leq a < b \leq n'$, $\{a, b\} \neq \{n' - 2d_2, n' - d_2\}$, and $|a - b| \in D$, we have $\{a, b\} \in E(C)$, then we call C good.

We are now in a position to prove the main results of this section.

Proof of Theorem 9: If $D = \{1, 3\}$, then the result follows by induction on n . $C : 0, 1, 2, 3, 0$ is a Hamiltonian cycle of G_4^D . Let C_n be a Hamiltonian cycle of G_n^D . Since the vertex $n - 1$ has degree 2 in

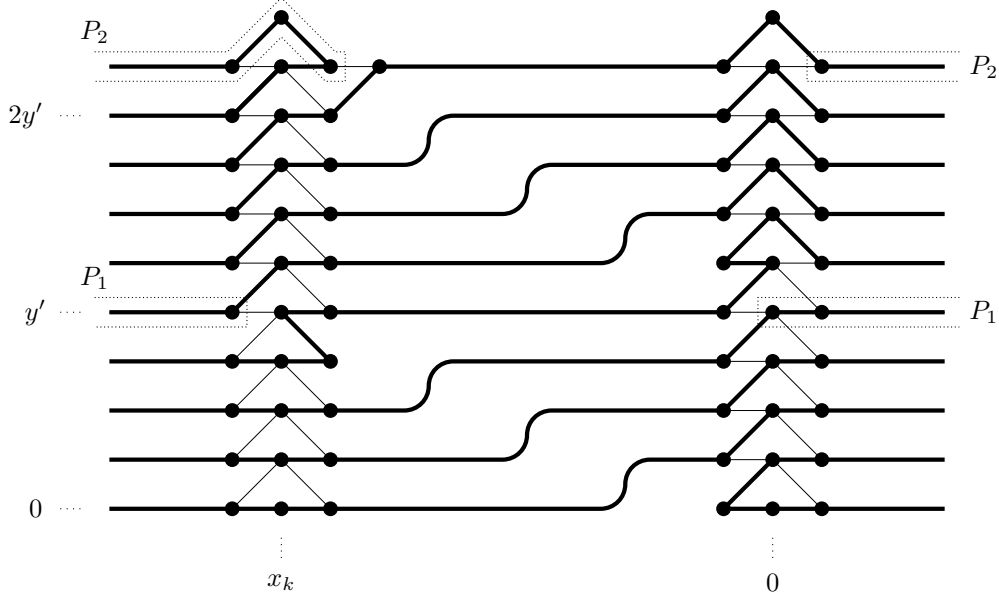


Fig. 23: The cycle C in the proof of Lemma 17.

G_n^D , $\{n-2, n-1\} \in E(C_n)$. Hence,

$$C_{n+2} = C_n + \{\{n-2, n+1\}, \{n+1, n\}, \{n, n-1\}\} - \{\{n-2, n-1\}\}$$

is a Hamiltonian cycle of G_{n+2}^D .

Hence we can assume that $D \neq \{1, 3\}$. Note that we can shift special cycles: If $C : v_0, \dots, v_l, v_0$ is a special cycle in G^D , then also $C + h : v_0 + h, \dots, v_l + h, v_0 + h$ is a special cycle in G^D . Furthermore, we can merge special cycles: If C_1 and C_2 are special cycles with $\min(V(C_2)) = \max(V(C_1)) + 1$, $\{a, b\} \in E(C_1)$, $\{c, d\} \in E(C_2)$, and $\{a, c\}, \{b, d\} \in E(G^D)$, then

$$(C_1 \cup C_2) + \{\{a, c\}, \{b, d\}\} - \{\{a, b\}, \{c, d\}\}$$

is a special cycle with vertex set $[\min(V(C_1)), \max(V(C_2))]$. If for $i \leq a < b \leq j$, $\{a, b\}$ is an edge of G^D and at least one of a, b has degree 2 in $G^D[[i, j]]$, then the edge $\{a, b\}$ belongs to every special cycles C of G^D with $V(C) = [i, j]$.

Claim 1 *If C_1 and C_2 are good special cycles of G^D with $\min(V(C_2)) = \max(V(C_1)) + 1$ and $D \neq \{1, 3\}$, then there is a good special cycle C with $V(C) = [\min(V(C_1)), \max(V(C_2))]$.*

Proof of Claim 1: Let $n' = \max(V(C_1))$.

Case 1 $d_1 \neq 2d_2 + 1$.

Since $d_1 \neq 2d_2 + 1$ and C_1 is good, $e_1 = \{n' - d_1 + 1, n' - d_1 + d_2 + 1\} \in E(C_1)$. Clearly, $e_2 = \{n' + 1, n' + d_2 + 1\} \in E(C_2)$. Hence,

$$C = (C_1 \cup C_2) + \{\{n' - d_1 + 1, n' + 1\}, \{n' - d_1 + d_2 + 1, n' + d_2 + 1\}\} - \{e_1, e_2\}$$

is a good special cycle with $V(C) = [\min(V(C_1)), \max(V(C_2))]$. This concludes the first case.

Case 2 $d_1 = 2d_2 + 1$.

Since $D \neq \{1, 3\}$, we have $d_2 > 1$. Since $d_1 = 2d_2 + 1$, and C_1 is good, $e_1 = \{n' - d_1 + 2, n' - d_1 + d_2 + 2\} \in E(C_1)$. Since $d_2 > 1$, $e_2 = \{n' + 2, n' + d_2 + 2\} \in E(C_2)$. Hence,

$$C = (C_1 \cup C_2) + \{\{n' - d_1 + 2, n' + 2\}, \{n' - d_1 + d_2 + 2, n' + d_2 + 2\}\} - \{e_1, e_2\}$$

is a good special cycle with $V(C) = [\min(V(C_1)), \max(V(C_2))]$. This concludes the second case and the proof of Claim 1. \square

Claim 2 G^D has a good special cycle of order $2 \pmod{d_1 + d_2}$.

Proof of Claim 2: By Lemma 17, G^D has a special cycle of order $2 \pmod{d_1 + d_2}$. Let C_1 be a special cycle of G^D of order $2 \pmod{d_1 + d_2}$ and let $n' = \max(V(C_1))$. It follows from [25, 21] that G^D has a special cycle of order $d_1 + d_2$. Note that every vertex in $\{j, j + 1, \dots, j + d_1 + d_2 - 1\}$ has degree 2 in $G^D[[j, j + d_1 + d_2 - 1]]$, for $j \in \mathbb{Z}$ and hence a special cycle of order $d_1 + d_2$ is good. Let C_2 be a special cycle of G^D of order $d_1 + d_2$ with $\min(V(C_2)) = n' + 1$. Since vertex n' has degree 2 in $G^D[V(C_1)]$, $\{n' - d_2, n'\} \in E(C_1)$ and since vertex $n' + d_1$ has degree 2 in $G^D[V(C_2)]$, $\{n' + d_1 - d_2, n' + d_1\} \in E(C_2)$. Hence,

$$(C_1 \cup C_2) + \{\{n' - d_2, n' + d_1 - d_2\}, \{n', n' + d_1\}\} - \{\{n' - d_2, n'\}, \{n' + d_1 - d_2, n' + d_1\}\}$$

is a good special cycle of G^D . This concludes the proof of Claim 2. \square

Let p_1 with $p_1 \equiv 2 \pmod{d_1 + d_2}$, such that G^D has a good special cycle of order p_1 . By Claim 2, such a p_1 exists. As said before, G^D has a good special cycle of order $p_2 = d_1 + d_2$. Since $\gcd(p_1, p_2) = 2$, it follows from the extended Euclidean algorithm that every sufficiently large even integer is a positive integral linear combination of p_1 and p_2 . Therefore and by Claim 1, the desired result follows by shifting and merging copies of good special cycles of order p_1 and p_2 . \square

Proof of Theorem 10:: The proof is analogous to the proof of Theorem 9. Instead of using Lemma 17 we use Lemma 16. Proceeding as in the proof of Theorem 9 we obtain p_1 with $p_1 \equiv 1 \pmod{d_1 + d_2}$ and hence $\gcd(p_1, p_2) = 1$. This clearly allows to establish the theorem for all sufficiently large n and not just for sufficiently large even n . \square

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