Super-convergence in maximum norm of the gradient for the Shortley-Weller method

Lisl Weynans

Super-convergence in maximum norm of the gradient for the Shortley-Weller method.

L. Weynans[1]

[1] Team Memphis, INRIA Bordeaux-Sud-Ouest & CNRS UMR 5251,
Université de Bordeaux, France

ISSN 0249-6399 ISRN INRIA/RR--8757--FR+ENG

RESEARCH REPORT
N° 8757
Janvier 2017
Project-Teams Memphis

Super-convergence in maximum norm of the gradient for the Shortley-Weller method.

L. Weynans[1*]

[1]Team Memphis, INRIA Bordeaux-Sud-Ouest & CNRS UMR 5251, Université de Bordeaux, France

Project-Teams Memphis

Research Report n° 8757 — Janvier 2017 — 14 pages

Abstract: We prove in this paper the second-order super-convergence in L^∞-norm of the gradient for the Shortley-Weller method. Indeed, with this method the discrete gradient is known to converge with second-order accuracy even if the truncation error near the boundary is only first-order, and the solution itself only converges with second-order. We present a proof in the finite-difference spirit, inspired by the paper of Ciarlet [1] and taking advantage of a discrete maximum principle to obtain estimates on the coefficients of the inverse matrix. This reasoning leads us to prove third-order convergence for the numerical solution near the boundary of the domain, and then second-order convergence for the discrete gradient in the whole domain. The advantage of this finite-difference approach is that it can provide locally pointwise convergence results depending on the local truncation error and the location on the computational domain, as well as convergence results in L^∞-norm.

Key-words: Finite-difference, Poisson equation, super-convergence, discrete Green's function, Shortley-Weller method

* Corresponding author: lisl.weynans@inria.fr

RESEARCH CENTRE BORDEAUX – SUD-OUEST

351, Cours de la Libération
Bâtiment A 29
33405 Talence Cedex

Super-convergence du gradient en norme infinie pour la méthode de Shortley-Weller.

Résumé : Nous présentons dans ce rapport une preuve de la super-convergence à l'ordre deux du gradient, en norme L^∞ pour la méthode de Shortley-Weller. En effet, avec cette méthode le gradient discret converge à l'ordre deux même si l'erreur de troncature près du bord du domaine est d'ordre un seulement, et que la solution elle-même ne converge aussi qu'à l'ordre deux. La preuve est réalisée avec une technique de différences finies, inspirée par l'article de Ciarlet [1], et utilisant un principe du maximum discret pour obtenir des estimations des coefficients de la matrice inverse. Ce raisonnement nous permet de prouver que la solution numérique converge à l'ordre trois près du bord du domaine, puis que le gradient discret converge à l'ordre deux dans tout le domaine. Cette approche par différences finies permet d'obtenir des résultats de convergence locaux, en fonction des différentes valeurs de l'erreur de troncature et de la position du point considéré sur le domaine de calcul. Elle permet aussi d'obtenir des résultats en norme L^∞.

Mots-clés : Différences finies, super-convergence, fonction de Green discrète, méthode de Shortley-Weller

Contents

1 Introduction

The Shortley-Weller method is a classical finite-difference method to solve the Poisson equation with Dirichlet boundary conditions in irregular domains. It is known to converge with second-order accuracy, although the truncation error of the numerical scheme is only first-order near the boundary. Furthermore, it has been numerically observed that the gradient of the numerical solution also converges with second-order accuracy. Recently, Yoon and Min raised in [7] the issue that mathematical justifications of this super-convergence phenomenon were lacking. Then they provided in [8] a proof of this super-convergence in a discrete L^2-norm.

Here we present a proof of the super-convergence of the gradient in a discrete L^∞-norm, with a finite-difference technique.

To our knowledge, all proofs of the super-convergence of the gradient, using finite-differences as in [8], or a finite-element formalism as in [4] and [3], have been established for discrete L^2-norm as we will discuss in §6.

We propose a variant of the method introduced by Ciarlet in [1]. This method is based on the use of the discrete maximum principle, for monotone matrices, leading us to obtain bounds on the coefficients of the inverse matrix. We first provide some notations, recall the Shortley-Weller method and present our results in §2. Then we present the technique of Ciarlet [1] adapted to our case in §3. We prove with this technique that the numerical solution converges with second-order accuracy in the whole domain, and with third-order accuracy near the boundary in §4. This proof is similar to the one presented in [5]. This intermediate result leads us to formulate in §5 a discrete Poisson equation for the discrete gradient, with Dirichlet boundary conditions that are second-order accurate, and finally to prove the second-order convergence of the gradient. We compare our approach to the literature in §6.

2 Notations and statement of results

In the following, we consider a domain Ω belonging to $\mathbb{R}^2$ or $\mathbb{R}^3$, with a boundary Γ.

The Shortley-Weller method is aimed to solve the Poisson equation in the domain Ω with Dirichlet conditions on the boundary:

$$\left\{ \begin{array}{r} -\triangle u = f \text{ on } \Omega, \\ u = g \text{ on } \Gamma. \end{array} \right. \tag{1}$$

For our analysis, we need that a unique solution u exists and is smooth enough for our truncation error analyses to be valid. We also need the solution of problem (1) with $f = 1$ and $g = 0$ to be C^1 near the boundary, because this property provides us useful estimates of the discrete Green functions.

Consequently, for the sake of simplicity, we assume in the whole paper that the source term f, the boundary Γ, and the boundary condition g are such that these two properties are satisfied. For instance, u belongs to $C^{3,1}(\Omega)$ (the set of functions whose 3th-order derivatives are Lipschitz continuous) or to $C^4(\Omega)$.

In this context, the boundary Γ may be only piecewise smooth and have corners. But it is known that if the boundary has corners with angle greater than a limit value, then the solution may lose its regularity near these corners. In this case, our analysis is not valid anymore. This behaviour is illustrated in the appendix. The case of convergence when singularities occur near the interface has been handled in [3] with a finite-element approach, obtaining a $O(h^{1.5})$ convergence in a discrete H^1- norm.

The problem (1) is discretized on a uniform cartesian grid, see Fig. 1. For the sake of clarity, the figures will represent the discretization points in two dimensions only, but the formulation

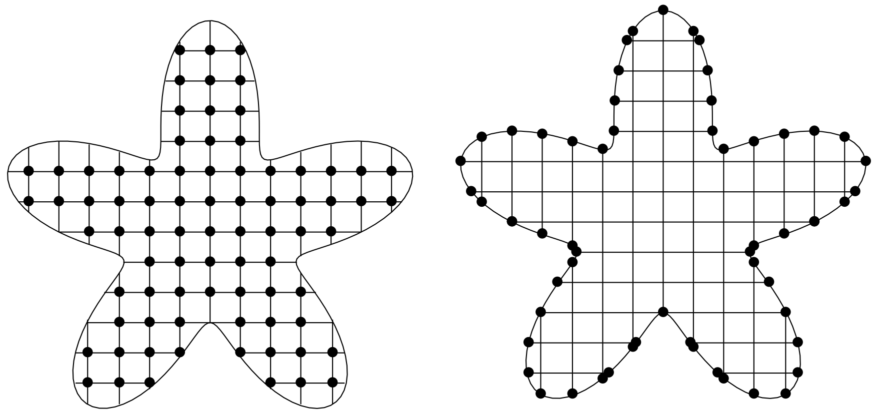

Figure 1: Left: interior nodes, belonging to Ω_h, right: boundary nodes, belonging to Γ_h.

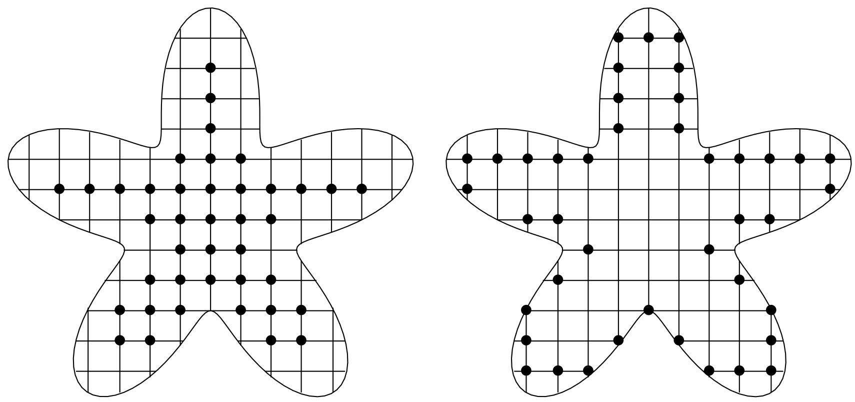

Figure 2: Left: nodes regular node, i.e. belonging to Ω_h^{**}, right: irregular nodes, i.e. belonging to Ω_h^{*}.

of the problem and the proofs of convergence will be performed in three dimensions. The grid spacing is denoted h, and the coordinates of the points on the grid are defined by $(x_i, y_j, z_k) = (i\,h, j\,h, k\,h)$. The points on the cartesian grid are named either with letters such as P or Q, or with letters and indices such as $M_{i,j,k} = (x_i, y_j, z_k)$ if one wants to have informations about the location of the point.

The set of grid points located inside the domain Ω is denoted Ω_h. These points are called interior nodes. The set of points located at the intersection of the axes of the grid and the boundary Γ is denoted Γ_h. These points are called boundary nodes and are used for imposing the boundary conditions in the numerical scheme, see Fig. 1 for an illustration. We say that a grid node is regular if none of its direct neighbors is on the boundary Γ_h, and that it is irregular if at least one of its neighbors belongs to Γ_h. The set of regular grid nodes is denoted Ω_h^{**}, and the set of irregular grid nodes is denoted Ω_h^{*}. See Fig. 2 for an illustration.

The Shortley-Weller scheme for solving the Poisson equation with Dirichlet boundary conditions is based on a dimension by dimension approach. In the following, for the sake of clarity we use the same notations as in the paper of Min [7].

Let the six neighboring nodes of a grid node P inside the domain be named as $P_i,\ 1 \le i \le 6$

and the distances to the neighbors as $h_i,\ 1 \le i \le 6$. If P is a regular node that $h_i = h$ for all $1 \le i \le 6$. If P is an irregular node then at least one of the h_i is different from h.

The discretization of the Laplace operator with the Shortley-Weller method reads:

$$\begin{aligned}-\triangle_h u_h(P) =& \left(\frac{2}{h_1h_2}+\frac{2}{h_3h_4}+\frac{2}{h_5h_6}\right)u_h(P) - \frac{2}{h_1(h_1+h_2)}u_h(P_1) - \frac{2}{h_2(h_1+h_2)}u_h(P_2)\\ &- \frac{2}{h_3(h_3+h_4)}u_h(P_3) - \frac{2}{h_4(h_3+h_4)}u_h(P_4) - \frac{2}{h_5(h_5+h_6)}u_h(P_5) - \frac{2}{h_6(h_5+h_6)}u_h(P_6).\end{aligned}$$

The matrix associated with this linear system has all its diagonal terms strictly positive, all extra-diagonal terms negative and is irreducibly diagonally dominant. Consequently it is a monotone matrix. Therefore, all coefficients of the inverse matrix are positive. This property will allow us to apply a discrete maximum principle useful to bound the coefficients of the inverse matrix.

We first present a the convergence proof of the solution itself, leading notably to third order estimates for the convergence on points belonging to Ω_h^*. This third-order convergence will be useful in the convergence proof of the discrete gradient of the solution, because it will provide second-order boundary conditions for an Laplace operator applied to the components of the gradient. This proof is similar to the one presented in [5] and in [8]. We present it in spite of its redundancy with the latter references in a purpose of clarity, because some elements of this proof will be useful for the convergence of the discrete gradient.

We denumeration by u_h the numerical solution of problem (1) with the Shortley-Weller method. The local error on a node P is defined by $e_h(P) = u(P) - u_h(P)$.

Theorem 1. *For the Shortley-Weller method, the local error $e_h(P)$ satisfies*

$$\begin{aligned}&|e_h(P)| \le O(h^2) \quad \forall P \in \Omega_h,\\ &|e_h(P)| \le O(h^3) \quad \textit{for all points } P \textit{ such that their distance } \phi(P) \textit{ to the boundary is } O(h).\end{aligned}$$

Concerning the convergence of the gradient, in practice, we will only study the convergence of the discrete version of $\partial_x u$, because the $x-$, $y-$ and $z-$directions have symmetric behaviours. Let us define the discrete version of $\partial_x u$. We consider two adjacent points belonging to Ω_h: $M_{i,j,k} = (x_i, y_j, z_k)$ and $M_{i+1,j,k} = (x_{i+1}, y_j, z_k)$. We denote by $M_{i+1/2,j,k}$ the middle of the segment $[M_{i,j,k}, M_{i+1,j,k}]$. For the sake of clarity we also denote by $M_{i+1/2,j,k}$ the middle of the segment $[M_{i,j,k}, Q]$, if Q, the nearest point on the right side of $M_{i,j,k}$ belongs to Γ_h. Similarly, we also denote by $M_{i+1/2,j,k}$ the middle of the segment $[Q, M_{i+1,j,k}]$, if Q the nearest point on the left side of $M_{i+1,j,k}$ belongs to Γ_h.

We define the discrete x-derivative $D_x u_h(M_{i+1/2,j,k})$ at the point $M_{i+1/2,j,k}$ as

$$D_x u_h(M_{i+1/2,j,k}) = \frac{u_h(M) - u_h(N)}{x_M - x_N},$$

where M and N are the points belonging to $\Omega_h \cup \Gamma_h$ such that $M_{i+1/2,j,k}$ is defined as the middle of $[MN]$.

We consider the set S_h of all points $M_{i+1/2,j,k}$ located inside the domain Ω. We define $\tilde{\Omega}_h$ the subset containing the points of S_h where we can define the discrete Laplace operator $(-\triangle_h)$ with all points of the seven-point stencil belonging to S_h. All other points of S_h constitute the subset $\tilde{\Gamma}_h$. The points of this subset satisfy by construction the property (see an illustration on Fig. 2)

$$\phi(P) \le 2\,h \quad \forall P \in \tilde{\Gamma}_h. \tag{2}$$

Now, we divide the subset $\tilde{\Omega}_h$ into two subsets of points: we say that a grid node in $\tilde{\Omega}_h$ is regular if none of its direct neighbors is on the boundary $\tilde{\Gamma}_h$, and that it is near the boundary if

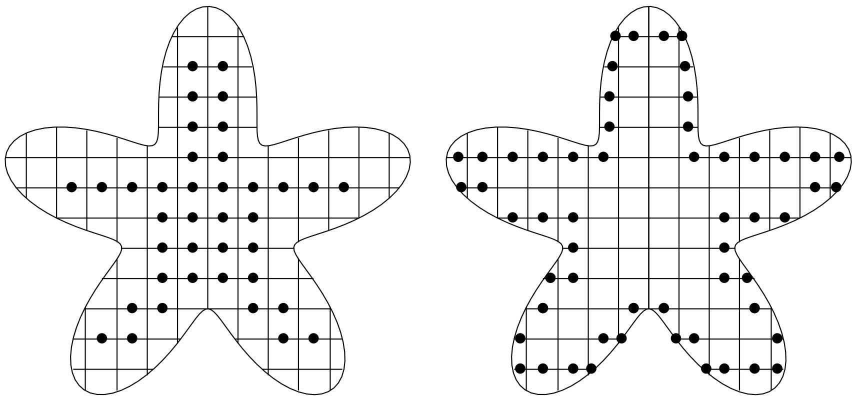

Figure 3: Left: nodes belonging to $\tilde{\Omega}_h$, right: nodes belonging to $\tilde{\Gamma}_h$

at least one if its neighbors belongs to $\tilde{\Gamma}_h$. The set of regular grid nodes is denoted by $\tilde{\Omega}_h^{**}$, and the set of grid nodes near the boundary is denoted by $\tilde{\Omega}_h^{*}$.

Theorem 2. *For the Shortley-Weller method, the local error on the discrete x-derivative is second-order accurate*

$$|\partial_x u(M_{i+1/2,j,k}) - D_x u_h(M_{i+1/2,j,k})| \leq O(h^2) \quad \forall M_{i+1/2,j,k} \in S_h.$$

3 Discrete maximum principle to prove convergence

Here we recall the principle of the method presented in [1] to prove high-order convergence for finite-differences operators with the help of the discrete maximum principle. We do not use exactly the same type of discretization matrix as in [1], due to the different way to account for boundary conditions, so here we present the reasoning in the case of our discretization matrix.

3.1 Discrete Green's function

For each $Q \in \Omega_h$, define the discrete Green's function $G_h(P,Q), P \in \Omega_h$ as the solution of the discrete problem:

$$\left\{ \begin{array}{rl} -\triangle_h G_h(:,Q)(P) = \left\{ \begin{array}{ll} 0, & P \neq Q \\ 1, & P = Q \end{array} \right. & P \in \Omega_h, \\ G_h(P,Q) = 0, \quad P \in \Gamma_h. & \end{array} \right. \tag{3}$$

In fact, each array $(G_h(P,Q)_{P\in\Omega_h}$ represents a column of the inverse matrix of the discrete operator $(-\triangle_h)$. For the sake of brevity we denote by $G_h(:,Q)$ the column corresponding to the grid node Q. The matrix of $(-\triangle_h)$ being monotone, as we noticed in §2, it means that all values of $G_h(P,Q)$ are positive.

With this definition of $G_h(P,Q)$ we can write the solution of the numerical problem as a sum of the source terms multiplied by the local values of the discrete Green function:

$$u_h(P) = \sum_{Q\in\Omega_h} G_h(P,Q)\,(-\triangle_h u_h)(Q), \quad \forall P \in \Omega_h.$$

With these notations, the Dirichlet boundary conditions are taken into account in the source terms.

3.2 Estimating the coefficients of the discrete Green's function

The following is an adaptation of the proof presented in [1].

Lemma 1. *Let S be a subset of grid nodes (thus corresponding also to a subset of the indices of the matrix), and W an array and $\alpha > 0$ such that:*

$$\begin{cases} W(P) \geq 0 \quad \forall P \in \Omega_h, \\ (-\triangle_h W)(P) \geq 0 \quad \forall P \in \Omega_h, \\ (-\triangle_h W)(P) \geq \alpha^{-i} \text{ for all } P \in S. \end{cases}$$

Then

$$\sum_{Q \in S} G_h(P,Q) \leq \alpha^i W(P).$$

Proof. Using the definition of the discrete Green function, we can write

$$(-\triangle_h \sum_{Q \in S} G_h(:,Q))(P) = \begin{cases} 1 & \text{if } P \in S, \\ 0 & \text{if } P \notin S. \end{cases}$$

Therefore,

$$-\triangle_h (W - \alpha^{-i} \sum_{Q \in S} G_h(:,Q))(P) \geq 0 \quad \forall P \in \Omega_h.$$

As all coefficients of the inverse of $-\triangle_h$ are positive, it leads to

$$W(P) - \alpha^{-i} \sum_{Q \in S} G_h(P,Q) \geq 0 \quad \forall P \in \Omega_h,$$

and finally we obtain an estimate of the coefficients of $\sum_{Q \in S} G_h(:,Q)$ in terms of the coefficients of W:

$$\sum_{Q \in S} G_h(P,Q) \leq \alpha^i W(P).$$

4 Convergence study of the solution

In this section we look for adequate subsets S and functions W in order to prove second-order convergence in L^∞-norm in the whole numerical domain, and third-order convergence for the grid nodes whose distance to the boundary is $O(h)$. This third-order convergence will be used to obtain second-order boundary conditions for a similar problem involving the discrete gradient of the numerical solution in §5.

Proof. We denote by $\tau(P)$ the truncation error of the Shortley-Weller method on a point P belonging to Ω_h. With a classical Taylor expansion one can show that $\tau(P) = O(h^2)$ if P belongs to Ω_h^{**}, and $\tau(P) = O(h)$ only a priori if P belongs to Ω_h^*. The local error satisfies the same linear system as the numerical solution $u_h(P)$, but with the truncation error as a source term:

$$-\triangle_h e_h(P) = \tau(P) \quad \forall P \in \Omega_h.$$

We want to obtain some bounds on $\sum_{Q \in \Omega_h^{**}} G_h(P,Q)$ and $\sum_{Q \in \Omega_h^*} G_h(P,Q)$ with the method described in §1. We handle these two subsets of Ω_h separately because they do not have the same truncation error.

We consider a point $M = (x_M, y_M, z_M)$ inside Ω. We define the discrete function:

$$W(Q) = \frac{C - (x_Q - x_M)^2 - (y_Q - y_M)^2 - (z_Q - z_M)^2}{4},$$

with (x_Q, y_Q, z_Q) the coordinates of the point Q, and C such that $W(Q) \geq 0$ for all $Q \in \Omega_h$. For instance we take $C = 2\,(diam(\Omega))^2$. On every grid node P belonging to Ω_h,

$$(-\triangle_h W)(P) = 1.$$

Therefore, we can directly write, using the lemma 1 the fact that the matrix is monotone:

$$\sum_{Q\in\Omega_h} G_h(P,Q) \quad \leq \quad W(P) \leq \frac{(diam(\Omega))^2}{2}, \quad \forall P \in \Omega_h. \tag{4}$$

To prove the second-order convergence of the solution, it remains to prove an appropriate estimate for $Q \in \Omega_h^*$. We define the discrete function

$$\tilde{W}(Q) = \begin{cases} 0 & \text{if } Q \in \Omega_h^*, \\ 1 & \text{otherwise.} \end{cases}$$

This function satisfies

$$\begin{cases} -\triangle_h(\tilde{W})(Q) \geq \frac{1}{h^2} & \text{if } Q \in \Omega_h^*, \\ -\triangle_h(\tilde{W})(Q) = 0 & \text{otherwise.} \end{cases}$$

we can directly write, using the fact that the matrix is monotone and the lemma 1

$$\sum_{Q\in\Omega_h^*} G_h(P,Q) \quad \leq \quad h^2\tilde{W}(P) \leq h^2. \tag{5}$$

Finally, combining (4) and (5), we obtain a second-order estimate of the local error on every point $P \in \Omega_h$

$$\begin{aligned} |u(P) - u_h(P)| &= |\sum_{Q\in\Omega_h} G_h(P,Q)\tau(Q)|, \\ &\leq \sum_{Q\in\Omega_h^{**}} G_h(P,Q)O(h^2) + \sum_{Q\in\Omega_h^*} G_h(P,Q)O(h), \\ &\leq \frac{diam(\Omega)}{4} O(h^2) + h^2 O(h) = O(h^2). \end{aligned}$$

To prove the third order accuracy on nodes in Ω_h^*, we consider grid points P such that $\phi(P) = O(h)$, with ϕ the positive distance to the boundary, and we want to prove that for such points,

$$\sum_{Q\in\Omega_h} G_h(P,Q) \quad \leq \quad O(h)$$

We consider the elliptic problem

$$\begin{cases} -\triangle u = 1 \text{ on } \Omega, \\ \quad u = 0 \text{ on } \Gamma. \end{cases} \tag{6}$$

We denote v the solution of this problem. We have assumed in the introduction that v was C^1 near the boundary. Therefore, it satisfies $v(P) = O(h)$ for points P such that $\phi(P) = O(h)$.

We apply the numerical operator $-\Delta_h$ to this function v. As this operator is consistent on every grid node, for h small enough we can write

$$-\triangle_h v(P) \geq \frac{1}{2}, \quad \forall P \in \Omega_h.$$

Therefore, using the result of lemma 1, and because the matrix is monotone,

$$\sum_{Q \in \Omega_h} G_h(P,Q) \leq 2v(P) \leq O(h), \quad \forall P \in \Omega_h, \tag{7}$$

Then, restricting ourselves to the columns belonging to Ω_h^{**},

$$\sum_{Q \in \Omega_h^{**}} G_h(P,Q) \leq O(h), \quad \forall P \text{ such that } \phi(P) = O(h). \tag{8}$$

Finally, combining (5) and (8) we can write for all P such that $\phi(P) = O(h)$

$$\begin{aligned}
|u(P) - u_h(P)| &= |\sum_{Q \in \Omega_h} G_h(P,Q)\tau(Q)|, \\
&\leq |\sum_{Q \in \Omega_h^{**}} G_h(P,Q)\, O(h^2)| + |\sum_{Q \in \Omega_h^{*}} G_h(P,Q)\, O(h)|, \\
&\leq O(h)O(h^2) + h^2 O(h) = O(h^3),
\end{aligned}$$

which means that the numerical solution is third-order accurate on grid points located at a distance $O(h)$ from the boundary.

5 Second-order convergence for the discrete gradient

Proof. We assume that we know the values of u_h, as it is the numerical solution of the linear system (1). We have proven in §4 that u_h converges with third-order accuracy for points P such that $\phi(P) \leq 2\,h$. Therefore, if $M_{i+1/2,j,k}$ belongs to $\tilde{\Gamma}_h$, then the numerical solution on $M_{i+1/2,j,k}$ is a third-order approximation of the exact solution. As a consequence, if $M_{i+1/2,j,k}$ belongs to $\tilde{\Gamma}_h$, then $D_x u(M_{i+1/2,j,k})$ is a second-order approximation of the x-derivative on point $M_{i+1/2,j,k}$.

Now we notice that we can build on points belonging to $\tilde{\Omega}_h$ a discretization $-\tilde{\triangle}_h$ of the Laplace operator with the Shortley-Weller method for the discrete x-derivative, with the discrete x-derivative of the function f as a source term, and the values of $D_x u$ on $\tilde{\Gamma}_h$ used as boundary conditions:

$$\begin{cases} (-\tilde{\triangle}_h v)(P) = D_x f(P) & \forall P \text{ on } \tilde{\Omega}_h, \\ v(P) = D_x u(P) & \forall P \text{ on } \tilde{\Gamma}_h. \end{cases}$$

This discretization has the truncation errors:

- $O(h^2)$ for the nodes belonging to $\tilde{\Omega}_h^{**}$,
- $O(1)$ for the nodes belonging to $\tilde{\Omega}_h^{*}$.

If the exact gradient was known on $\tilde{\Gamma}_h$, then the truncation error would be also $O(1)$ due to the fact that the Shortley-Weller operator does not commute with the discrete gradient D_x. The boundary conditions are defined with second-order accuracy, and thus lead to an additionnal $O(1)$ term in the truncation error.

Now, similar estimates as in §4 can be obtained for the coefficients of the inverse matrix, because the matrix of the current linear system, being also a discretization with the Shortley-Weller method, has the same structure as the matrix of the linear system for u_h. If we denote by $\tilde{G}_h(P,Q)$ the discrete Green's function corresponding to $-\tilde{\triangle}_h$, then it satisfies

$$\sum_{Q\in\tilde{\Omega}_h} \tilde{G}_h(P,Q) \leq \frac{(diam(\Omega))^2}{2}, \quad \forall P \in \tilde{\Omega}_h, \tag{9}$$

$$\sum_{Q\in\tilde{\Omega}_h^*} \tilde{G}_h(P,Q) \leq h^2, \quad \forall P \in \tilde{\Omega}_h. \tag{10}$$

Therefore, the expression on the local error on point P belonging to $\tilde{\Omega}_h$ reads

$$\begin{aligned} |\partial_x u(M_{i+1/2,j,k}) - D_x u_h(M_{i+1/2,j,k})| &= |\sum_{Q\in\tilde{\Omega}_h} \tilde{G}_h(P,Q)\tilde{\tau}(Q)|, \\ &\leq |\sum_{Q\in\tilde{\Omega}_h^{**}} \tilde{G}_h(P,Q)\tilde{\tau}(Q)| + |\sum_{Q\in\tilde{\Omega}_h^*} \tilde{G}_h(P,Q)\tilde{\tau}(Q)|, \\ &\leq |\sum_{Q\in\tilde{\Omega}_h^{**}} \tilde{G}_h(P,Q)O(h^2)| + |\sum_{Q\in\tilde{\Omega}_h^*} \tilde{G}_h(P,Q)O(1)|, \\ &\leq O(1)O(h^2) + O(h^2)O(1) = O(h^2). \end{aligned}$$

6 Discussion

This work was originally motivated by the remark in the recent paper of Yoon and Min [6] about the lack of mathematical analysis about the super-convergence of the Shortley-Weller method. This paper was followed by [8] where the autors provided a proof of this super-convergence in a discrete L^2-norm, using a discrete divergence theorem.

To our knowledge, few other works in the literature have studied the super-convergence of the gradient for elliptic finite-difference schemes, among them [2], [4] and [3].

In [2] Ferreira and Grigorieff deal with more general elliptic operators, with variable coefficients and mixed derivatives, and prove second-order convergence in H^1 norm.The proof uses negative norms and is based on the fact that the finite difference scheme is a certain non-standard finite element scheme on triangular grids combined with a special form of quadrature.

In [4] Li et al. study the super-convergence of solution derivatives for the Shortley-Weller method for Poisson's equation, considering also this method as a special kind of finite element method. They obtained second-order convergence in H^1 norm for rectangular domains, and an order 1.5 for polygonal domains. The work in [3] adresses the case of unbounded derivatives near the boundary Γ, on polygonal domains.

Our approach differs from the latter because we do not use a finite-element approach. Instead we propose a proof based on a finite-difference analysis, which is a variant of the method of Ciarlet [1]: we use a discrete maximum principle to obtain estimates on the coefficients of the inverse matrix, but in our case the bound on the coefficients can also vary with the rows of the inverse matrix. This variant is useful to obtain a specific bound for points located near the boundary and obtain the third-order convergence of the solution at these points. Then the same methodology is applied to the discrete gradient, obtaining second-order accuracy.

The approach developed in this paper has the advantage to be simple to carry out, and to be able to provide locally pointwise estimates, instead of the usual convergence results in the L^2 or H^1 norms.

7 Conclusion

We have proven that the discrete gradient obtained by the Shortley-Weller method for the Poisson equation converges with second-order accuracy. This is a super-convergence property because the numerical solution itself also converges with second order accuracy. This property is proven with a variant of Ciarlet's technique to obtain high-order convergence estimates for monotone finite-differences matrices. With carefuly chosen test functions we are able to bound the coefficients of the discrete Green function associated with the matrix of the Shortley-Weller method. One key ingredient is consider the discrete gradient as the solution of another Poisson equation with Dirichlet boundary conditions that have a second-order accuracy. A further development would be to extend this work to the case of more general elliptic operators.

A Numerical illustration: corners and regularity of the solution

We illustrate here the possible loss of regularity of the solution in the case of corners, that was evocated in §2. Depending on the angles of this corners, the solution can indeed be less than C^1 near the boundary, even if the source term and the boundary conditions are very smooth.

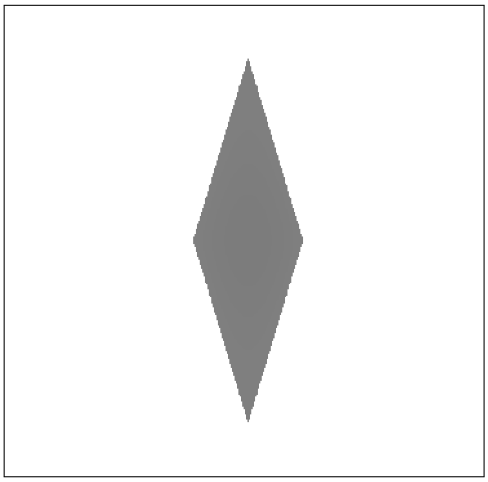

Figure 4: Exemple of domains with corners

On figure 4, we consider two domains that are only piecewise smooth: the diamond-shaped one, denoted Ω_1, and its complementary, denoted Ω_2. The angles of the first one do not exceed the value π, while some of the second one do actually.

We solve numerically the following problem

$$\begin{cases} -\triangle u = 1 \text{ on } \Omega, \\ u = 0 \text{ on } \Gamma. \end{cases} \tag{11}$$

in these two domains: $\Omega = \Omega_1$ and $\Omega = \Omega_2$, with the Shortley-Weller method. The numerical solution is in fact the sum of all the discrete Green functions associated to grid points in the numerical domain. In this paper, to obtain the estimate 7 on the discrete Green function, we make the assumption that the solution u of problem 11 is at least C^1 near the boundary, so that it satisfies $u(x) = O(h)$ for points located at a distance $O(h)$ of the domain boundary.

We compute in both cases, the L^∞-norm of the numerical solution on grid points in Ω_h^* (that is, irregular grid points). The tables 1 et 2 present these results. We observe that for domain Ω_1, the numerical solution on Ω_h^* converges to zero at order one. For domain Ω_2, the convergence order is strictly smaller than one, which means that the discrete Green functions do not satisfy the property that we need for our convergence estimates.

N	L^∞ norm	Convergence order
100	4.80E-003	-
200	2.45E-003	0.970
400	1.26E-003	0.965
600	8.43E-004	0.971
800	6.34E-004	0.97 3
1000	5.077E-004	0.976

Table 1: Convergence to zero for irregular grid points for problem (11) in domain Ω_1

N	L^∞ norm	Convergence order
100	1.462E-002	-
200	8.104E-003	0.856
400	4.503E-003	0.853
600	3.195E-003	0.851
800	2.506E-003	0.850
1000	2.075E-003	0.850

Table 2: Convergence to zero for irregular grid points for problem (11) in domain Ω_2

References

[1] P.G. Ciarlet. Discrete maximum principle for finite differences operators. *Aequat. Math.*, 4:338–352, 1970.

[2] J.A. Ferreira and R.D. Grigorieff. On the supraconvergence of elliptic finite difference schemes. *Applied Num. Math.*, 28:275–292, 1998.

[3] Z.-C. Li, H.-Y. Hu, S.T. Wang, and Q. Fang. Superconvergence of solution derivatives for the shortley-weller difference approximation to poisson's equation with singularities on polygonal domains. *Applied Num. Math.*, 58:689–704, 2008.

[4] Z.-C. Li, T. Yamamoto, and Q. Fang. Superconvergence of solution derivatives for the shortley-weller difference approximation of poisson's equation. part i: smoothness problems. *J. Comp. Applied Math.*, 151:307–333, 2003.

[5] M. Matsunaga and T. Yamamoto. Superconvergence of the shortley-weller approximation for dirichlet problems. *J. Comp. Applied Math.*, 116:263–273, 2000.

[6] G. Yoon and C. Min. A review of the supra-convergences of shortley-weller method for poisson equation. *J. KSIAM*, 18(1):51–60, 2014.

[7] G. Yoon and C. Min. Analyses on the finite difference method by gibou et al. for the poisson equation. *J. Comput. Phys.*, 280:184–194, 2015.

[8] G. Yoon and C. Min. Convergence analysis of the standard central finite difference method for poisson equation. *J Sci Comput*, 67:602–617, 2016.

RESEARCH CENTRE
BORDEAUX – SUD-OUEST

351, Cours de la Libération
Bâtiment A 29
33405 Talence Cedex

Publisher
Inria
Domaine de Voluceau - Rocquencourt
BP 105 - 78153 Le Chesnay Cedex
inria.fr

ISSN 0249-6399