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Uniform convergence of conditional distributions for absorbed one-dimensional diffusions

Nicolas Champagnat[1,2,3], Denis Villemonais[1,2,3]

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Abstract

This article studies the quasi-stationary behaviour of absorbed one-dimensional diffusions. We obtain a necessary and sufficient condition for the exponential convergence to a unique quasi-stationary distribution in total variation, uniformly with respect to the initial distribution. Our approach is based on probabilistic and coupling methods, contrary to the classical approach based on spectral theory results. We provide several conditions ensuring this criterion, which apply to most practical cases. As a by-product, we prove that most strict local martingale diffusions are strict in a stronger sense: their expectation at any given positive time is actually uniformly bounded with respect to the initial position. We provide several examples and extensions, including the sticky Brownian motion and some one-dimensional processes with jumps. We also give exponential ergodicity results on the Q-process.

Keywords: one-dimensional diffusions; absorbed process; quasi-stationary distribution; Q-process; uniform exponential mixing property; strict local martingales; one dimensional processes with jumps.

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1 Introduction

This article studies the quasi-stationary behaviour of general one-dimensional diffusion processes in an interval E of $\mathbb{R}$, absorbed at its finite bound-

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aries. When the process is absorbed, it is sent to some cemetary point ∂. This covers the case of solutions to one-dimensional stochastic differential equations (SDE), but also of diffusions with singular speed measures, as the sticky Brownian motion.

We recall that a *quasi-stationary distribution* for a continuous-time Markov process $(X_t, t \geq 0)$ on the state space $E \cup \{\partial\}$, is a probability measure α on E such that

$$\mathbb{P}_\alpha(X_t \in \cdot \mid t < \tau_\partial) = \alpha(\cdot), \quad \forall t \geq 0,$$

where $\mathbb{P}_\alpha$ denotes the distribution of the process X given that X_0 has distribution α, and

$$\tau_\partial := \inf\{t \geq 0 : X_t = \partial\}.$$

We refer to [14, 20, 18] for general introductions to the topic.

Our goal is to give conditions ensuring the existence of a quasi-stationary distribution α on E such that, for all probability measures μ on E and all $t \geq 0$,

$$\|\mathbb{P}_\mu(X_t \in \cdot \mid t < \tau_\partial) - \alpha\|_{TV} \leq Ce^{-\gamma t}, \tag{1.1}$$

where $\|\cdot\|_{TV}$ is the total variation norm and C and γ are positive constants. In particular, α is the unique quasi-stationary distribution. We also study the long time behavior of absorption probabilities and the exponential ergodicity in total variation of the Q-process associated to X, defined as the diffusion X conditioned never to hit ∂.

The usual tools to prove convergence as in (1.1) involve coupling arguments (Doeblin's condition, Dobrushin coupling constant, see e.g. [15]). Typically, contraction in total variation norm for the non-conditioned semigroup can be obtained using standard coupling techniques for one-dimensional diffusions on $\mathbb{R}$ such that $-\infty$ and $+\infty$ are entrance boundaries. However, a diffusion conditioned not to hit 0 before a given time $t > 0$ is a time-inhomogeneous diffusion process with a singular, non-explicit drift for which these methods fail. For instance, the solution $(X_s)_{s\geq 0}$ to the SDE

$$dX_s = b(X_s)ds + \sigma(X_s)dB_s$$

with smooth coefficients and conditioned not to hit 0 up to a time $t > 0$ has the law of the solution $(X_s^{(t)})_{s\in[0,t]}$ to the SDE

$$dX_s^{(t)} = b(X_s^{(t)})ds + \sigma(X_s^{(t)})dB_s + \sigma(X_s^{(t)})\ [\nabla \ln \mathbb{P}_\cdot(t - s < \tau_\partial)]\,(X_s^{(t)})ds.$$

Since $\mathbb{P}_x(t-s<\tau_\partial)$ vanishes when x converges to 0, the drift term in the above SDE is singular and cannot in general be quantified.

Hence, convergence of conditioned diffusion processes have been obtained up to now using (sometimes involved) spectral theoretic arguments, which most often require regularity of the coefficients (see for instance [2, 12, 9, 6]). Up to our knowledge, only one reference studies general diffusions [16] but none of the above references provide uniform convergence with respect to the initial distribution. This is of great importance in applications, since in general the initial distribution might greatly influence the time needed to observe stabilisation of the conditional distribution of the process. This difficulty already arises for processes without conditioning and is even more crucial for conditioned processes since the practical interest of a quasi-stationary distribution requires to compare the typical time of extinction with the typical time of stabilization of conditional distributions (see the discussion in [14, Ex. 2]).

More generally, the approaches of the above references all rely on spectral results specific to the case of one-dimensional diffusions (e.g. Sturm-Liouville theory in [12, 9]) and cannot extend to other classes of processes. On the contrary, we rely in this work on recent probabilistic criteria for convergence of conditioned processes obtained in [4, Theorem 2.1], which are more flexible and can easily extend for example to one-dimensional processes with jumps.

Our main result is a *necessary and sufficient* criterion for the uniform exponential convergence of conditional distributions in total variation. In the case of a general diffusion on $[0,+\infty)$ absorbed at $\partial=0$, this condition is given by

(B) The diffusion X comes down from infinity (i.e. $+\infty$ is an entrance boundary) and there exists $t, A>0$ such that

$$\mathbb{P}_x(\tau_\partial>t)\leq As(x), \quad \forall x>0,$$

where s is the scale function of the diffusion X.

In the sequel, without loss of generality, we will focus on the case of a diffusion on natural scale and modify (B) accordingly. We will also focus on the case $E=(0,+\infty)$, but our results easily extend to any other interval.

We also provide an explicit formula relating the speed measure of a diffusion on natural scale and its quasi-stationary distribution. As a by-product, we show that, given a probability measure α on $(0,\infty)$ satisfying suitable conditions and a positive constant λ_0, there exists a unique diffusion on natural scale, with explicit speed measure, admitting α as unique quasi-stationary distribution with associated absorbing rate λ_0. We also use this

explicit formula to compute the quasi-stationary distribution of the sticky Brownian motion on $(-1,1)$, absorbed at $\{-1,1\}$.

An important part of our study is devoted to check that Condition (B) is satisfied in very general situations. First, we show how it can be easily checked with elementary probabilistic tools, in a large range of practical situations. Second, more involved arguments actually allow to prove (B) for all absorbed diffusion processes coming down from infinity, provided the speed measure satisfies a very weak condition on its oscillation near 0.

We also prove that Condition (B) is equivalent to the fact that some local martingale $(Z_t, t \geq 0)$ on $[0,\infty)$ constructed from X is a *strict local martingale* in the following strong sense: there exists $t > 0$ such that

$$\sup_{x>0} \mathbb{E}_x(Z_t) < \infty. \tag{1.2}$$

We prove that the property (1.2) is true for all $t > 0$ and for nearly all strict local martingale diffusions on $[0,\infty)$ (with controlled oscillation of the speed measure near $+\infty$). This new result has its own interest in the theory of strict local martingales [19]. For instance, it is well-known that the solution on $(0,+\infty)$ to the SDE

$$dZ_t = Z_t^\alpha dB_t$$

is a strict local martingale if and only if $\alpha > 1$, which means that $\mathbb{E}_x(Z_t) < x$ for all $x, t > 0$. Our result implies that, for all $\alpha > 1$ and all $t > 0$, (1.2) holds true, and hence that $\mathbb{E}_x(Z_t) < x$ for all $x > 0$ is actually equivalent to $\sup_{x>0} \mathbb{E}_x(Z_t) < \infty$. This is also true for example for the SDE

$$dZ_t = Z_t \ln(Z_t)^\beta dB_t,$$

which is a strict local martingale iff $\beta > 1/2$.

The paper is organized as follows. In Section 2, we precisely define the absorbed diffusion processes and recall their construction as time-changed Brownian motions. Section 3 contains the statements of all our main results: the exponential convergence (1.1), the asymptotic behavior of absorption probabilities and the exponential ergodicity of the Q-process are stated in Subsection 3.1; Subsections 3.2 and 3.3 give explicit sufficient conditions to check (B) and the link with strict local martingales; in Subsection 3.4, we provide a sufficient condition on a probability measure α to be the quasi-stationary distribution of a one-dimensional diffusion on natural scale; finally, Subsection 3.5 is devoted to the precise comparison of our results with the existing literature and to the study of examples, including a case

of one-dimensional processes with jumps. Section 4 contains several results on strict local martingales, including the criterion ensuring (1.2). Finally, we give in Section 5 the proofs of the main results of Section 3.

2 Absorbed diffusion processes

Our goal is to construct general diffusion processes $(X_t, t \geq 0)$ on $[0, +\infty)$, absorbed at $\partial = 0$. The typical situation corresponds to stochastic population dynamics of continuous densities with possible extinction. These processes are standard objects, but for sake of completeness, we recall their construction from a standard Brownian motion.

The distribution of X given $X_0 = x \in [0, \infty)$ will be denoted $\mathbb{P}_x$, and the semigroup of the process is given by $P_t f(x) = \mathbb{E}_x[f(X_t)]$ for all bounded measurable $f : [0, \infty) \to \mathbb{R}$ and all $x \geq 0$.

A stochastic process $(X_t, t \geq 0)$ on $[0, +\infty)$ is called a diffusion if it has a.s. continuous paths in $[0, \infty)$, satisfies the strong Markov property and is *regular*. By regular, we mean that for all $x \in (0, \infty)$ and $y \in [0, \infty)$, $\mathbb{P}_x(T_y < \infty) > 0$, where T_y is the first hitting time of y by the process X. Given such a process, there exists a continuous and strictly increasing function s on $[0, \infty)$, called the *scale function*, such that $(s(X_{t \wedge T_0}), t \geq 0)$ is a local martingale [5]. The stochastic process $(s(X_t), t \geq 0)$ is itself a diffusion process with identity scale function. Since we shall assume that 0 is regular or exit and that $T_0 < \infty$ a.s., we necessarily have $s(0) > -\infty$ and $s(\infty) = \infty$, and we can assume $s(0) = 0$. Hence, replacing $(X_t, t \geq 0)$ by $(s(X_t), t \geq 0)$, we can assume without loss of generality that $s(x) = x$. Such a process is said to be on *natural scale* and satisfies for all $0 < a < b < \infty$,

$$\mathbb{P}_x(T_a < T_b) = \frac{b - x}{b - a}.$$

To such a process X, one can associate a unique locally finite positive measure $m(dx)$ on $(0, \infty)$, called the *speed measure* of X, which gives positive mass to any open subset of $(0, +\infty)$ and such that $X_t = B_{\sigma_t}$ for all $t \geq 0$ for some standard Brownian motion B, where

$$\sigma_t = \inf\{s > 0 : A_s > t\}, \text{ with } A_s = \int_0^\infty L_s^x \, m(dx). \tag{2.1}$$

and L^x is the local time of B at level x. Conversely, any such time change of a Brownian motion defines a regular diffusion on $[0, \infty)$ [8, Thm. 23.9].

We will use in the sequel the assumption that

$$\int_0^\infty y\, m(dy) < \infty. \tag{2.2}$$

Since the measure m is locally finite on $(0,\infty)$, this assumption reduces to local integrability of $ym(dy)$ near 0 and $+\infty$. These two conditions are respectively equivalent to $\mathbb{P}_x(\tau_\partial < \infty) = 1$ for all $x \in (0,\infty)$ [8, Thm. 23.12], where $\tau_\partial := T_0$ is the first hitting time of $\partial = 0$ by the process X, and to the existence of $t > 0$ and $y > 0$ such that

$$\inf_{x>y} \mathbb{P}_x(T_y < t) > 0.$$

This means that ∞ is an *entrance boundary* for X [8, Thm. 23.12], or equivalently that the process X *comes down from infinity* [2].

We recall that 0 is absorbing for the process X iff $\int_0^1 m(dx) = \infty$ [8, Thm. 23.12], which is not necessarily the case under the assumption that $\int_0^1 y\, m(dy) < \infty$. Therefore, we modify the definition of X so that ∂ becomes an absorbing point as follows:

$$X_t = \begin{cases} B_{\sigma_t} & \text{if } 0 \le t < \tau_\partial \\ \partial & \text{if } t \ge \tau_\partial. \end{cases} \tag{2.3}$$

Note that this could be done equivalently by replacing m by $m + \infty\delta_0$.

With this definition, X is a local martingale. Note that A_s is strictly increasing since m gives positive mass to any open interval, and hence σ_t is continuous and

$$\sigma_{\tau_\partial} = T_0^B, \quad \text{or, equivalently,} \quad \tau_\partial = A_{T_0^B}, \tag{2.4}$$

where T_x^B is the first hitting time of $x \in \mathbb{R}$ by the process $(B_t, t \ge 0)$. In particular, $X_t = B_{\sigma_t \wedge T_0^B}$, $\forall t \ge 0$.

Let us recall that, when m is absolutely continuous with respect to Lebesgue's measure on $[0,\infty)$ with density $1/\sigma^2(x)$, the diffusion X is the unique (weak) solution of the stochastic differential equation

$$dX_t = \sigma(X_t) dW_t,$$

where $(W_t, t \ge 0)$ is a standard Brownian motion.

Further properties of these processes will be given Section 4.

3 Quasi-stationary distribution for one dimensional diffusions

The key assumption for the results of this section is the following one.

(B) The measure m satisfies

$$\int_1^\infty y\, m(dy) < \infty$$

and there exist two constants $t_1, A > 0$ such that

$$\mathbb{P}_x(\tau_\partial > t_1) \leq Ax, \quad \forall x > 0. \tag{3.1}$$

Let us recall that the assumption $\int_0^\infty y\, m(dy) < \infty$ means that the diffusion process X comes down from infinity and hit 0 a.s. after a finite time.

3.1 Exponential convergence to quasi-stationary distribution

In the following theorem, we establish that Assumption (B) is equivalent to the exponential convergence to a unique quasi-stationary distribution. This result is proved in Section 5 and relies on [4, Thm. 2.1].

Theorem 3.1. *Assume that X is a one-dimensional diffusion on natural scale on $[0, \infty)$ with speed measure $m(dx)$ such that $\tau_\partial < \infty$ a.s., i.e. $\int_0^1 y\, m(dy) < \infty$. Then we have equivalence between*

(i) *Assumption* (B).

(ii) *There exist a probability measure α on $(0, \infty)$ and two constants $C, \gamma > 0$ such that, for all initial distribution μ on $(0, \infty)$,*

$$\|\mathbb{P}_\mu(X_t \in \cdot \mid t < \tau_\partial) - \alpha(\cdot)\|_{TV} \leq Ce^{-\gamma t}, \ \forall t \geq 0. \tag{3.2}$$

In this case, α is the unique quasi-stationary distribution for the process and there exists $\lambda_0 > 0$ such that $\mathbb{P}_\alpha(t < \tau_\partial) = e^{-\lambda_0 t}$. Moreover, α is absolutely continuous with respect to m and

$$\frac{d\alpha}{dm}(x) = 2\lambda_0 \int_0^\infty (x \wedge y)\, \alpha(dy). \tag{3.3}$$

In addition,

$$\int_0^\infty y\, \alpha(dy) < \infty.$$

Remark 1. We deduce from (3.3) that he density of α with respect to m is positive, bounded, (strictly) increasing on $(0, +\infty)$, and differentiable at each point of $(0, +\infty)$.

The next result is a consequence of the equivalence of Theorem 3.1 and also relies on the results established in [4]

Proposition 3.2. *Assume that X is a one-dimensional diffusion on natural scale with speed measure $m(dx)$ such that $\tau_\partial < \infty$ a.s. and such that (B) is satisfied. We define η on $[0, +\infty)$ as the normalized right hand side of* (3.3)*:*

$$\eta(x) = \frac{\displaystyle\int_0^\infty (x \wedge y)\, \alpha(dy)}{\displaystyle\int_0^\infty \int_0^\infty (y \wedge z)\, \alpha(dy)\, \alpha(dz)}. \tag{3.4}$$

Then

$$\eta(x) = \lim_{t\to\infty} \frac{\mathbb{P}_x(t < \tau_\partial)}{\mathbb{P}_\alpha(t < \tau_\partial)} = \lim_{t\to+\infty} e^{\lambda_0 t} \mathbb{P}_x(t < \tau_\partial), \tag{3.5}$$

where the convergence holds for the uniform norm on $[0, +\infty)$. Moreover, the function η is non-decreasing, bounded, $\alpha(\eta) = 1$, $\eta(x) \le Cx$ for all $x \ge 0$ for some constant $C > 0$, and η belongs to the domain of the infinitesimal generator L of X on the Banach space of bounded measurable functions on $[0, \infty)$ equipped with the L^∞ norm and

$$L\eta = -\lambda_0 \eta. \tag{3.6}$$

Note that it follows from (3.3) and (3.4) that $d\alpha$ is proportional to $\eta\, dm$.

Let us recall another important consequence, established in [4], of the equivalence of Theorem 3.1. It deals with the so-called Q-processes, defined as the diffusion process X conditioned never to hit 0.

Theorem 3.3. *Assume that X is a one-dimensional diffusion on natural scale on $[0, \infty)$ stopped at 0 with speed measure $m(dx)$ such that $\int_0^1 y\, m(dy) < \infty$ and satisfying Assumption (B). Then we have the following properties.*

(i) Existence of the Q-process. *There exists a family $(\mathbb{Q}_x)_{x>0}$ of probability measures on Ω defined by*

$$\lim_{t\to+\infty} \mathbb{P}_x(A \mid t < \tau_\partial) = \mathbb{Q}_x(A)$$

for all $\mathcal{F}_s$-measurable set A, and the process $(\Omega, (\mathcal{F}_t)_{t\ge0}, (X_t)_{t\ge0}, (\mathbb{Q}_x)_{x>0})$ is a diffusion process on $(0, \infty)$ (which does not hit 0).

(ii) Transition kernel. *The transition kernel of X under $(\mathbb{Q}_x)_{x>0}$ is given by*

$$\tilde{p}(x;t,dy) = e^{\lambda_0 t}\frac{\eta(y)}{\eta(x)}p(x;t,dy).$$

In other words, for all bounded measurable φ on $(0,\infty)$ and all $t \geq 0$,

$$\widetilde{P}_t\varphi(x) = \frac{e^{\lambda_0 t}}{\eta(x)}P_t(\eta\varphi)(x) \tag{3.7}$$

where $(\widetilde{P}_t)_{t\geq 0}$ is the semi-group of X under $\mathbb{Q}$.

(iii) Exponential ergodicity. *The probability measure β on $(0,\infty)$ defined by*

$$\beta(dx) = \eta(x)\alpha(dx).$$

is the unique invariant distribution of X under $\mathbb{Q}$. Moreover, for any initial distributions μ_1, μ_2 on $(0,\infty)$,

$$\|\mathbb{Q}_{\mu_1}(X_t \in \cdot) - \mathbb{Q}_{\mu_2}(X_t \in \cdot)\|_{TV} \leq (1 - c_1c_2)^{\lfloor t/t_0 \rfloor}\|\mu_1 - \mu_2\|_{TV},$$

where $\mathbb{Q}_\mu = \int_{(0,\infty)} \mathbb{Q}_x\, \mu(dx)$.

3.2 Simple criteria ensuring Condition (B)

We start with some basic properties related to Condition (B). The first one shows that the converse inequality in (3.1) is always true. The second one is an (apparently weaker) equivalent condition.

Proposition 3.4. *Let X be a diffusion process on natural scale on $(0,+\infty)$.*

(i) *For all $t > 0$, there exists a constant $a > 0$ such that*

$$\mathbb{P}_x(t < \tau_\partial) \geq ax, \quad \forall x \in (0,1).$$

(ii) *Condition (B) is equivalent to*

(B') *X comes down from infinity and there exists $x_0, t_1, A' > 0$ such that*

$$\mathbb{P}_x(t_1 < \tau_\partial \wedge T_{x_0}) \leq A'x, \quad \forall x \in (0,x_0).$$

Proof. For Point (i), we observe that, by the strong Markov property at time T_1,

$$\mathbb{P}_x(t < \tau_\partial) \geq \mathbb{E}_x[1_{T_1<\tau_\partial}\mathbb{P}_1(t < \tau_\partial)] = \mathbb{P}_x(T_1 < \tau_\partial)\mathbb{P}_1(t < \tau_\partial) = x\mathbb{P}_1(t < \tau_\partial).$$

Point (ii) is an easy consequence of the fact that $\mathbb{P}_x(\tau_\partial > T_{x_0}) = x/x_0$, since X is on natural scale. □

The next result gives a simple condition implying (B). Its proof is particularly simple. We give in the next section a slightly more general criterion which makes use of much more involved results on the density of diffusion processes (related to spectral theory).

Theorem 3.5. *With the previous notation, assume that m satisfies* (2.2) *and for all $x \in (0,1)$,*

$$I(x) := \int_{(0,x)} y\, m(dy) \leq C x^{\rho} \tag{3.8}$$

for some constants $C > 0$ and $\rho > 0$. Then, for all $t > 0$, there exists $A_t < \infty$ such that

$$\mathbb{P}_x(t < \tau_\partial) \leq A_t\, x,\ \forall x > 0.$$

Remark 2. In particular, (3.8) is satisfied if m is absolutely continuous w.r.t. Lebesgue's measure Λ on $(0,\infty)$ and if

$$\frac{dm}{d\Lambda}(x) \leq \frac{C}{x^{\alpha}}$$

in the neighborhood of 0, where $\alpha < 2$. This corresponds to a diffusion process solution to the SDE

$$dX_t = \sigma(X_t) dB_t$$

with $\sigma(x) \geq c x^{\alpha/2}$ in the neighborhood of 0.

Let us also notice that the previous result ensures that Assumption (B) is satisfied as soon as 0 is a regular point for the diffusion, i.e. $\int_{(0,1)} m(dx) < \infty$.

Proof. The proof is based on the following lemma.

Lemma 3.6. *For all $k \geq 1$ and $x \in \mathbb{R}_+$, the function $M_k(x) = \mathbb{E}_x\left(\tau_\partial^k\right)$ is bounded and satisfies*

$$M_k(x) = 2k \int_0^\infty x \wedge y\, M_{k-1}(y)\, m(dy)\ < \infty. \tag{3.9}$$

First, notice that we can assume without loss of generality that $\rho \notin \mathbb{Q}$. We prove by induction that there exists a constant $C_k > 0$ such that for all $x \in (0,1)$,

$$M_k(x) \leq C_k x^{(k\rho)\wedge 1}.$$

This is trivial for $k=0$. Let us assume it is true for $k \geq 0$ and let us denote by α_k the constant $(k\rho) \wedge 1$. For all $x \in (0,1)$, by (3.9),

$$\begin{aligned}
M_{k+1}(x) &\leq C\left(\int_{(0,x)} yM_k(y)m(dy) + x\int_{[x,1)} M_k(y)m(dy) + x\int_{[1,\infty)} M_k(y)m(dy)\right) \\
&\leq C\left(\int_{(0,x)} y^{1+\alpha_k}\, m(dy) + x\int_{[x,1)} y^{\alpha_k}\, m(dy) + x\int_{[1,\infty)} m(dy)\right) \\
&\leq C\left[x^{\alpha_k} I(x) + x\left(I(1) + (1-\alpha_k)\int_x^1 y^{\alpha_k-2} I(y)\, dy\right) + x\int_{[1,\infty)} m(dy)\right] \\
&\leq Cx^{\alpha_k+\rho} + Cx + Cx\int_x^1 y^{\alpha_k+\rho-2}\, dy,
\end{aligned}$$

where the constant $C>0$ depends on k and may change from line to line, and the third inequality follows from integration by parts. Since $\rho \notin \mathbb{Q}$, we have

$$\int_x^1 y^{\alpha_k+\rho-2}\, dy \leq \begin{cases} 1/(\alpha_k+\rho-1) & \text{if } \alpha_k+\rho-1>0, \\ x^{\alpha_k+\rho-1}/(1-\alpha_k-\rho) & \text{if } \alpha_k+\rho-1<0. \end{cases}$$

This concludes the induction. □

Proof of Lemma 3.6. Let us define the functions $M_k(x)$ recursively from the formula (3.9) and $M_0(x)=1$. An immediate induction procedure and (2.2) proves that, for all $k \geq 0$, the function M_k is bounded.

The Green function of one-dimensional diffusions on natural scale [8, Lemma 23.10] implies that, for any $0<a<b<\infty$,

$$\mathbb{E}_x\left(\int_0^{T_{a,b}} M_{k-1}(X_t)\, dt\right) = 2\int_{(a,b)} \frac{(x\wedge y-a)(b-x\vee y)}{b-a} M_{k-1}(y)\, m(dy).$$

By monotone convergence and since M_{k-1} is bounded, we obtain by letting $a \to 0$ and $b \to \infty$ that

$$M_k(x) = k\mathbb{E}_x\left(\int_0^{\tau_\partial} M_{k-1}(X_t)\, dt\right).$$

Let us now prove by induction that $M_k(x) = \mathbb{E}(\tau_\partial^k)$ for all $k \geq 0$. The property is true for $k=0$. Assume that $M_{k-1}(x) = \mathbb{E}_x(\tau_\partial^{k-1})$, then

$$M_k(x) = k\mathbb{E}_x\left(\int_0^{\tau_\partial} \mathbb{E}_{X_t}(\tau_\partial^{k-1})\, dt\right) = k\mathbb{E}_x\left(\int_0^{\tau_\partial} (\tau_\partial - t)^{k-1}\, dt\right) = \mathbb{E}_x(\tau_\partial^k),$$

where we used the Markov property. □

3.3 Link between strict local martingales and Condition (B)

The following result gives a necessary and sufficient condition to check Condition (B). It will be obtained as a consequence of Corollary 4.4 in Section 4.2.

Theorem 3.7. *For any constants $A > 0$ and $t > 0$,*

$$\mathbb{P}_x(t < \tau_\partial) \leq Ax, \ \forall x \geq 0 \iff \sup_{z>0} \mathbb{E}(Z_t \mid Z_0 = z) \leq A,$$

where $(Z_t, t \geq 0)$ is a diffusion on $(0, +\infty)$ on natural scale with speed measure

$$\widetilde{m}(dz) = \frac{1}{z^2}(f * m)(dz), \tag{3.10}$$

where $f * m$ is the image measure of m by the application $f(x) = 1/x$.

Remark 3. In particular, if Condition (B) is satisfied, then $(Z_t)_{t\geq 0}$ is a strict local martingale, *i.e.* a local martingale which is not a martingale.

The notion of strict local martingales is of great importance in the theory of financial bubbles. We refer the interested reader to Protter [19] and references therein. Note that under the assumption that $\mathbb{P}_x^X(t < \tau_\partial) \leq Ax$ for all $x > 0$, we obtain a strong form of strict martingale property for Z, since we prove that its expectation at time t is uniformly bounded.

When restated in terms of the speed measure $\widetilde{m}$ (see Theorem 4.1), the next result gives general conditions ensuring that Z is a strict local martingale in the strong previous sense. We refer to Section 4.1 for precise definitions, statements.

The next result gives a sufficient condition for (B), weaker than the one of Theorem 3.5, but using more involved and specific arguments. It will be obtained as a corollary of Theorem 4.2 in Section 4.2.

Theorem 3.8. *Assume that m satisfies* (2.2) *and*

$$\int_0^1 \frac{1}{x} \sup_{y \leq x} \left(\frac{1}{y} \int_{(0,y)} z^2 \, m(dz) \right) dx < \infty. \tag{3.11}$$

Then, for all $t > 0$, there exists $A_t < \infty$ such that

$$\mathbb{P}_x(t < \tau_\partial) \leq A_t \, x, \ \forall x > 0.$$

Remark 4. Note that, as noticed in [13], in the case where m has a density with respect to Lebesgue's measure Λ on $(0, \infty)$, the condition (3.11) covers

almost all the practical situations where (2.2) is satisfied. For example, it is true if

$$\frac{dm}{d\Lambda}(x) \leq \frac{C}{x^2 \log \frac{1}{x} \cdots \log_{k-1} \frac{1}{x} \left(\log_k \frac{1}{x}\right)^{1+\epsilon}},$$

for all x in a neighbourhood of 0 and for some $\epsilon > 0$ and $k \geq 1$, where $\log_k(x) = \log_{k-1}(\log x)$ (see [13, Ex. 2]).

The last theorem is based on abstract but powerful general analytical results on the density of diffusion processes, which are hard to extend to more general settings. Our condition (B) is simple enough to be check under stronger assumptions using elementary probability tools, as we did in Theorem 3.5, and to apply to many other settings (processes with jumps or piecewise deterministic paths...). We give in Subsection 3.5.4 a simple example of a one-dimensional process with jumps where exponential convergence to the quasi-stationary distribution can be proved by an easy extension of our method.

3.4 Consequences of the expression of α as a function of m

In Theorem 3.1, we proved that, under Assmuption (B), α is absolutely continuous with respect to m and, more precisely, that

$$\frac{d\alpha}{dm}(x) = 2\lambda_0 \int_0^\infty (x \wedge y)\alpha(dy), \ \forall x \in (0, +\infty). \tag{3.12}$$

Our goal here is to provide a converse property, i.e. to give a necessary and sufficient condition on a given positive measure α, such that it is the unique quasi-stationary distribution of a diffusion process on natural scale on $[0, +\infty)$ with exponential convergence of the conditional laws.

We will say (with a slight abuse of notation) that a positive measure m on $(0, \infty)$ satisfies (B) if it is locally finite, gives positive mass on each open subset of $(0, \infty)$, satisfies $\int_0^1 y\, m(dy) < \infty$ and the diffusion process on $[0, \infty)$ on natural scale with speed measure m and stopped at 0 satisfies Condition (B).

Theorem 3.9. *Fix $\lambda_0 > 0$ and a probability measure α on $(0, +\infty)$ such that the measure $(\frac{1}{x} \vee 1)\alpha(dx)$ satisfies (B). Then, there exists a unique (in law) diffusion process $(X_t)_{t\geq 0}$ on $[0, +\infty)$ on natural scale almost surely absorbed at 0 satisfying*

$$\|\mathbb{P}_x(X_t \in \cdot \mid t < \tau_\partial) - \alpha\|_{TV} \leq Ce^{-\gamma t}, \quad \forall x \in (0, +\infty),$$

for some positive constants C, γ and such that

$$\mathbb{P}_\alpha(t < \tau_\partial) = e^{-\lambda_0 t}, \quad \forall t \geq 0.$$

In addition, the speed measure of the process X is given by

$$m(dx) = \frac{\alpha(dx)}{2\lambda_0 \int_0^\infty (x \wedge y)\alpha(dy)}, \quad \forall x \in (0, \infty).$$

Conversely, if the speed measure satisfies (B), then the corresponding quasi-stationary distribution α is such that $(\frac{1}{x} \vee 1)\alpha(dx)$ satisfies (B).

Because of the relation (3.12) and the equivalence between (B) and uniform exponential convergence to a quasi-stationary distribution, this result follows from the next lemma.

Lemma 3.10. *Let α be a probability measure on $(0, +\infty)$. Define the measure m on $(0, +\infty)$ as*

$$m(dx) = \frac{\alpha(dx)}{\int_0^\infty (x \wedge y)\alpha(dy)}, \ \forall x \in (0, \infty). \tag{3.13}$$

Then the measure $(\frac{1}{x} \vee 1)\alpha(dx)$ satisfies (B) if and only if the measure m satisfies (B).

Proof. Note that, if m_1, m_2 are two measures on $(0, +\infty)$ such that $m_1 \leq m_2$ and such that m_2 satisfies (B), then m_1 also satisfies (B). This is a direct consequence of the construction of diffusion processes given in Section 2 and of (2.1).

Therefore, it is enough to check that there exist two constant $c_1, c_2 > 0$ such that

$$c_1(\frac{1}{x} \vee 1) \leq \frac{dm}{d\alpha}(x) \leq c_2(\frac{1}{x} \vee 1),$$

or equivalently that there exist two constants $c_1', c_2' > 0$ such that

$$c_1'(x \wedge 1) \leq \int_0^\infty (x \wedge y)\alpha(dy) \leq c_2'(x \wedge 1). \tag{3.14}$$

To prove this, we will make use of the property

$$\int_0^\infty (1 \vee y)\, \alpha(dy) < \infty. \tag{3.15}$$

It is clearly true if the measure $(\frac{1}{x} \vee 1)\alpha(dx)$ satisfies (B). Conversely, if m satisfies (B), Theorem 3.1 implies that α is the unique quasi-stationary distribution associated to m with $\lambda_0 = 1/2$ and that (3.15) holds true.

We have

$$\int_0^\infty (x \wedge y)\alpha(dy) \leq \left(x \int_0^\infty \alpha(dy)\right) \wedge \left(\int_0^\infty y\, \alpha(dy)\right).$$

Hence (3.15) implies the existence of $c_2' < \infty$ in (3.14).

In addition,

$$\int_0^\infty (x \wedge y)\alpha(dy) \geq \begin{cases} x \int_1^\infty \alpha(dy), & \text{if } x \leq 1, \\ \int_1^\infty \alpha(dy), & \text{if } x > 1. \end{cases}$$

This ends the proof of the lemma. □

3.5 Examples

3.5.1 On general diffusions

Let us first recall that our results also cover the case of general diffusion processes $(Y_t, t \geq 0)$ on $[0, \infty)$ absorbed at 0, which hit 0 in a.s. finite time. Under these assumptions, there exists a scale function $s : [0, \infty) \to [0, \infty)$ of the process Y such that $s(0) = 0$ and $s(\infty) = \infty$, and our results apply to the process $X_t = s(Y_t)$ on natural scale. Expressed in terms of the process Y, the necessary and sufficient Condition (B) becomes: the process Y comes down from infinity, hits 0 a.s. in finite time and there exist $t_1, A > 0$ such that

$$\mathbb{P}(t_1 < \tau_\partial \mid Y_0 = y) \leq As(y).$$

Let us also mention that, as will appear clearly in the proof, our methods can be easily extended to diffusion processes on a bounded interval, where one of the boundary point is an entrance boundary and the other is exit or regular, and also to cases where both boundary points are either exit or regular.

To illustrate the generality of the processes that are covered by our criteria, we give two simple examples where the speed measure m is singular with respect to Lebesgue's measure Λ. The case of speed measures absolutely continuous w.r.t. Lebesgue's measure (i.e. of SDEs) will be discussed in the next subsection. Our results also cover the case of a speed measure with discontinuous density with respect to Lebesgue's measure. Such diffusion are naturally obtained as rescaled solution of SDEs driven by their local times, including skew Brownian motions (see for instance [11]).

Example 1 We recall that a diffusion process on $\mathbb{R}$ with speed measure $\Lambda + \delta_0$ is called a *sticky Brownian motion* [7, 1]. We consider a diffusion process on $[0, \infty)$ which comes down from infinity ($\int_1^\infty y\, m(dy) < \infty$) and which is a sticky Brownian motion on $[0, 1]$, "sticked" at the points $a_1, a_2, \ldots$, where $(a_i)_{i\geq 1}$ is decreasing, converges to 0 and $a_1 < 1$, i.e.

$$m_{|(0,1)} = \Lambda_{|(0,1)} + \sum_{i\geq 1} \delta_{a_i}.$$

Assuming that there exist constants $C, \rho > 0$ such that for all $j \geq 1$,

$$\sum_{i\geq j} a_i \leq C a_j^\rho, \tag{3.16}$$

then for all $x \in (0, 1)$, defining $i_0 := \inf\{j \geq 1 : a_j < x\}$,

$$\int_{(0,x)} y\, m(dy) = \frac{x^2}{2} + \sum_{i\geq i_0} a_i \leq \frac{x^2}{2} + C a_{i_0}^\rho \leq \frac{x^2}{2} + C x^\rho,$$

and we can apply Theorem 3.5.

For example, the choice $a_i = i^{-\frac{1}{1-\rho}}$, for all $i \geq 1$, satisfies (3.16).

Example 2 Let X be a sticky Brownian motion stopped at -1 and 1. This means that X is a diffusion on natural scale with speed measure $m(dx) = \Lambda(dx) + \delta_0(dx)$ on $(-1, 1)$, absorbed at -1 and 1. It is straightforward to adapt our results to processes on $[-1, 1]$. For the process of this example, the corresponding Assumption (B) is clearly fulfilled since both boundaries -1 and 1 are regular for X (see Remark 2). Then the unique quasi-stationary distribution α of X is

$$\alpha(dx) = \frac{\gamma^*}{2} \sin\left(\gamma^*(1 + x) \wedge (1 - x)\right) m(dx),$$

where γ^* is the unique solution in $(0, \pi]$ of cotan $\gamma = \gamma/2$.

Indeed, it satisfies the following adaptation of formula (3.3)

$$\frac{d\alpha}{dm}(x) = \lambda_0 \int_{-1}^{1} (x \wedge y + 1)(1 - x \vee y)\, \alpha(dy), \ \forall x \in (-1, 1). \tag{3.17}$$

In particular, the measure α is absolutely continuous with respect to Lebesgue's measure Λ on $(-1, 1) \setminus \{0\}$ and satisfies

$$\left(\frac{d\alpha}{d\Lambda}\right)''(x) = -2\lambda_0 \frac{d\alpha}{d\Lambda}(x), \ \forall x \neq 0.$$

Using the symmetry of the problem and the 0 boundary conditions, we deduce that there exist constants $a, b \in \mathbb{R}$ such that

$$\alpha(dx) = a\delta_0(dx) + b\sin\left(\gamma(1+x) \wedge (1-x)\right)\Lambda(dx),$$

where $\gamma = \sqrt{2\lambda_0}$. Note that this implies that $\gamma \in (0, \pi]$. In addition, equality (3.17) at $x = 0$ entails

$$a = \lambda_0 \left(2b \int_0^1 (1-y)\sin(\gamma(1-y))dy + a\right)$$

and hence

$$(1 - \gamma^2/2)a = b\left(\sin\gamma - \gamma\cos\gamma\right). \tag{3.18}$$

In addition, $d\alpha/dm(x)$ is continuous at 0, hence

$$a = b\sin\gamma. \tag{3.19}$$

Finally, α is a probability measure, so that

$$a + \frac{2b}{\gamma}(1 - \cos\gamma) = 1. \tag{3.20}$$

Now, dividing (3.18) by (3.19), we obtain $\gamma = \gamma^*$. Equality (3.20) becomes

$$1 = a - b\sin\gamma + \frac{2b}{\gamma}.$$

By (3.19), we deduce that $b = \gamma/2$ and finally that $a = \frac{\gamma\sin\gamma}{2}$.

3.5.2 On processes solutions of stochastic differential equations

In the case where the speed measure m is absolutely continuous w.r.t. the Lebesgue measure on $(0, \infty)$, our diffusion processes on natural scale are solutions to SDEs of the form

$$dX_t = \sigma(X_t)dB_t, \tag{3.21}$$

where σ is a measurable function from $(0, \infty)$ to itself such that the speed measure $m(dx) = \frac{1}{\sigma^2(x)}\,dx$ is locally finite on $(0, \infty)$. Following the scale function trick of Section 3.5.1, our results actually covers all SDEs of the form

$$dY_t = \sigma(Y_t)dB_t + b(Y_t)dt$$

such that $b/\sigma^2 \in L^1_{\text{loc}}((0,\infty))$ (see Chapter 23 of [8]).

A major part of the literature [2, 12, 9] on the subject considers a less general class of diffusion processes $(Y_t, t \geq 0)$ on $[0,\infty)$ which are solution of a SDE of the form

$$dY_t = dB_t - q(Y_t)dt. \tag{3.22}$$

This form is more suited to Sturm-Liouville theory, which is the basis of all the previously cited works, but is not used her. Let us recall that the scale function of the process $(Y_t, t \geq 0)$ solution to (3.22) is given by

$$s(x) = \int_0^x \exp\left(2\int_1^y q(z)\,dz\right)\,dy,$$

and that $X_t = s(Y_t)$ is solution to the SDE

$$dX_t = \exp\left(2\int_1^{s^{-1}(X_t)} q(z)\,dz\right)\,dB_t. \tag{3.23}$$

We see in particular that the diffusion (3.23) obtained from (3.22) is far from being as general as (3.21). In particular, the diffusion coefficient in (3.23) is C^1 since q is assumed at least continuous in all the references previously cited.

3.5.3 Comparison of our results with the literature

As said in the introduction, our result is the first one to prove uniform speed of convergence to the quasi-stationary distribution, independently of the initial distribution. This is of great importance because, in applications, the initial distribution is often only known approximately and, without uniform speed of convergence, one cannot be sure that the quasi-stationary distribution can be observed empirically (see [14]). We also emphasize that our main result gives a necessary and sufficient condition for these properties.

In fact, the only reference proving that non-compactly supported initial distributions belong to the domain of attraction of the quasi-stationary distribution is [2, Thm. 7.3]. However, this result is proved for SDEs of the form (3.22) with $q \in C^1((0,\infty))$, and additional assumptions are required. We can make the link with Theorem 3.5, considering the case where $q(x) = \frac{\alpha}{2(2-\alpha)}\frac{1}{x}$ for all $x \in (0,1)$ (we leave aside the assumptions on q in the neighborhood of $+\infty$). This corresponds to a diffusion coefficient $\sigma(x) = x^{\alpha/2}$ in (3.23), as in Remark 2. As explained in [2, Rk. 4.6], the case covered by their result corresponds to $\alpha < 3/2$. Our Theorem 3.5 only requires $\alpha < 2$.

The questions of existence and uniqueness of a quasi-stationary distribution are much better understood. Still, our results improve the existing ones in several aspects. The article [16] is the only one which considers general diffusions. It is based on results from spectral theory [13], which are exactly the same we used to prove Theorem 3.8, so their result assumes the same conditions on the speed measure in the neighborhood of 0. However, they also need to assume similar conditions as in Theorem 4.1 in the neighborhood of $+\infty$, which are not needed here.

All the other cited references [2, 12, 9] consider SDEs of the form (3.22), with $q \in C^0((0,\infty))$ (hence the speed measure m with C^1 density) in [9], or $q \in C^1((0,\infty))$ (hence the speed measure m with C^2 density) in [2, 12]. When we restrict our results to these speed measures, the only existence and uniqueness result that does not enter in our setting is the one of [12], where it is only assumed that $\int_0^1 y\, m(dy) < \infty$ without requiring bounds like those in Remark 4. The question whether Assumption (B) is satisfied in these limit cases remains open.

We also obtain uniform convergence results to the eigenvalue η (Prop. 3.2) and uniform exponential ergodicity results on the Q-process (Theorem 3.3, see below) that are not studied in the previously cited works, except in [2] for existence and simple ergodicity of the Q-process. In addition, our probabilistic approach is particularly adapted to the case of absorbed diffusions with killing [3] and can be easily extended to cases where spectral methods do not apply easily, e.g. for processes with jumps, as the following example shows.

3.5.4 A simple example with jumps

The following simple example shows how our method can be easily extended to processes with jumps.

We consider a diffusion process $(X_t, t \geq 0)$ on $[0,\infty)$ with speed measure m satisfying Assumption (B). Let us denote by $\mathcal{L}$ the infinitesimal generator of X. We consider the Markov process $(\widetilde{X}_t, t \geq 0)$ with infinitesimal generator

$$\widetilde{\mathcal{L}} f(x) = \mathcal{L} f(x) + (f(x+1) - f(x)) 1_{x \geq 1},$$

for all f in the domain of $\mathcal{L}$. In other words, we consider a càdlàg process following a diffusion process with speed measure m between jump times, which occur at the jump times of an independent Poisson process $(N_t, t \geq 0)$ of rate 1, with jump size $+1$ if the process is above 1, and 0 otherwise. We denote by $\widetilde{\tau}_\partial$ its first hitting time of 0.

The proof of the following proposition is postponed to Subsection 5.4.

Proposition 3.11. *Under the previous assumptions, there exist a unique probability measure α on $(0,\infty)$ and two constants $C,\gamma>0$ such that, for all initial distribution μ on $(0,\infty)$,*

$$\left\|\mathbb{P}_\mu(\widetilde{X}_t\in\cdot\mid t<\widetilde{\tau}_\partial)-\alpha(\cdot)\right\|_{TV}\leq Ce^{-\gamma t},\ \forall t\geq 0. \tag{3.24}$$

In particular, the probability measure α is the unique quasi-stationary distribution of the process.

4 Strict local martingales

We give in this section a general condition ensuring that a strict local martingale diffusion satisfies the strong strict local martingale property (1.2). We then make the link with condition (B) and prove Theorems 3.7 and 3.8.

4.1 Strict local martingales in a strong sense

The next result gives a sufficient criterion for a diffusion process Z on $[0,\infty)$ on natural scale to satisfy $\sup_{z>0}\mathbb{E}_z(Z_t)<\infty$.

Theorem 4.1. *Let $\widetilde{m}$ be a locally finite measure on $(0,\infty)$ giving positive mass to any open subset of $(0,\infty)$, and let Z be a diffusion process on $[0,\infty)$ stopped at 0 on natural scale with speed measure $\widetilde{m}$. Then, if*

$$\int_1^\infty y\,\widetilde{m}(dy)<\infty \tag{4.1}$$

and

$$\int_1^\infty \frac{1}{x}\sup_{y\geq x}\left(y\int_y^\infty \widetilde{m}(dz)\right)dx<\infty. \tag{4.2}$$

Then, for all $t>0$, there exists $A_t<\infty$ such that

$$\mathbb{E}_z(Z_t)\leq A_t,\ \forall z>0.$$

Recall that (4.1) means that $+\infty$ is an entrance boundary for Z, which is equivalent to the fact that Z is a strict local martingale [10]. Similarly as in Remark 4, Condition (4.2) is true in most situations where (4.1) is true. It could fail only when $\tilde{m}$ has strong oscillations close to $+\infty$.

Proof. Since $(Z_t, t \geq 0)$ is a positive local martingale, hence a supermartingale, we have for all $z \geq 1$ and $t > 0$,

$$\begin{aligned}\mathbb{E}_z(Z_t) &= \mathbb{E}_z(Z_t 1_{t<T_1^Z}) + \mathbb{E}_z(Z_t 1_{T_1^Z<t}) \\ &\leq \mathbb{E}_z(Z_{t\wedge T_1^Z} 1_{t<T_1^Z}) + \mathbb{E}_z[1_{T_1^Z<t}\mathbb{E}_1(Z_{t-T_1^Z})] \\ &\leq \mathbb{E}_z(Z_{t\wedge T_1^Z}) + 1,\end{aligned}$$

where T_1^Z is the first hitting time of 1 by Z. Hence, we only need to prove that $\sup_{z\geq 1} \mathbb{E}_z(Z_t^{T_1}) < \infty$, where Z^{T_1} is the diffusion process Z stopped at time T_1^Z.

Since Z^{T_1} is a diffusion process on $[1, \infty)$, stopped at 1, on natural scale and with speed measure $\nu(\cdot) = \widetilde{m}(\cdot \cap (1, \infty))$ satisfying

$$\nu([1, 2]) < \infty,$$

we can apply the result of [13, Thm. 4.1, Ex. 2]. This result ensures that, under conditions (4.1) and (4.2), the probability density function of $Z_t^{T_1}$ with respect to ν, denoted by $p(x, y, t)$, is well defined for all $t > 0$ and there exists a constant $A'_t > 0$ such that

$$\sup_{1\leq x,y<+\infty} p(x, y, t) \leq A'_t, \quad \forall t > 0.$$

As a consequence, for all $t > 0$,

$$\mathbb{E}_z(Z_t^{T_1}) = \int_1^\infty y\, p(z, y, t)\, d\nu(y) \leq A'_t \int_1^\infty y\, d\widetilde{m}(y). \qquad \square$$

4.2 Links between strict local martingales and quasi-stationary distributions

In this section, we make the link between absorbed diffusion processes and strict local martingales. We also prove Theorems 3.7 and 3.8.

The next result does not require that m satisfies (2.2).

Theorem 4.2. *Let X be a diffusion process on $[0, +\infty)$ on natural scale stopped at 0 with speed measure m on $(0, +\infty)$, and let $\mathbb{P}_x$ denote its law when $X_0 = x$, defined on the canonical space of continuous functions from $\mathbb{R}_+$ to itself, equipped with its canonical filtration. For all $x > 0$, we define the following h-transform of $\mathbb{P}_x$:*

$$d\widetilde{\mathbb{Q}}^x|_{\mathcal{F}_\tau} = \frac{X_\tau}{x} d\mathbb{P}^x|_{\mathcal{F}_\tau},$$

for all stopping time τ such that $(X_{t\wedge\tau}, t \geq 0)$ is a uniformly integrable martingale. Then, under $(\widetilde{\mathbb{Q}}^x)_{x>0}$, the process $Z_t := 1/X_t$ is a diffusion process stopped at 0 on natural scale with speed measure

$$\widetilde{m}(dz) = \frac{1}{z^2}(f * m)(dz), \tag{4.3}$$

where $f * m$ is the image measure of m by the application $f(x) = 1/x$.

In particular, if the measure $m(dx)$ is absolutely continuous w.r.t. Lebesgue's measure Λ on $(0, +\infty)$, then $\widetilde{m}(dz)$ is absolutely continuous w.r.t. Λ and

$$\frac{d\widetilde{m}}{d\Lambda}(z) = \frac{1}{z^4}\frac{dm}{d\Lambda}\left(\frac{1}{z}\right), \quad \forall z > 0. \tag{4.4}$$

Combining this with Theorem 4.1 implies Theorem 3.8. In particular, (3.11) is obtained from (4.2) by a change of variable.

Remark 5. Note that the family of probability measures $(\mathbb{Q}_x)_{x>0}$ of Theorem 3.3 is different from the family $(\widetilde{\mathbb{Q}}_x)_{x>0}$. Under both measures, $\mathbb{Q}_x(\tau_\partial = \infty) = \widetilde{\mathbb{Q}}_x(\tau_\partial = \infty) = 1$ (for $\widetilde{\mathbb{Q}}_x$, this is a consequence of [17, Cor. 2.6]), but the first one is obtained by conditioning X on late extinction, whereas the second one is obtained by conditioning X on hitting large values before hitting 0. More precisely, for all $a > x > 0$, $\widetilde{\mathbb{Q}}_x\big|_{\mathcal{F}_{T_a}} = \mathbb{P}_x(\cdot \mid T_a < T_0)\big|_{\mathcal{F}_{T_a}}$ [17, Thm. 2.2]. Among the noticeable differences, the process X is ergodic under $\mathbb{Q}_x$ by Theorem 3.3, whereas $\lim_{t\to\infty} X_t = \infty$ a.s. under $\widetilde{\mathbb{Q}}_x$ [17, Cor. 2.6].

Proof of Theorem 4.2. The fact that $\widetilde{\mathbb{Q}}_x$, $x > 0$, are probability measures and that the process $Z = 1/X$ is a diffusion on $[0, \infty)$ stopped at 0 on natural scale under $(\widetilde{\mathbb{Q}}_x)_{x>0}$ are proved in [17] (see Lemma 2.5 for the last point). Hence, we only have to compute its speed measure, i.e. the unique locally finite measure $\widetilde{m}$ on $(0, \infty)$, giving positive mass to any open subset of $(0, \infty)$, such that for all $0 < a < z < b < +\infty$ and for all bounded measurable function $g : (0, \infty) \to \mathbb{R}$,

$$\mathbb{E}^{\widetilde{\mathbb{Q}}_{1/z}}\left[\int_0^{T^Z_{a,b}} g(Z_t)\,dt\right] = 2\int_a^b \frac{(z \wedge y - a)(b - z \vee y)}{b - a} g(y)\,\widetilde{m}(dy), \tag{4.5}$$

where $T^Z_{a,b} := \inf\{t \geq 0 : Z_t \in \{a, b\}\}$. Let us check that the measure $\widetilde{m}$

defined in (4.3) satisfies this relation: the change of variable formula entails

$$
\begin{aligned}
2\int_{[a,b]} \frac{(z\wedge y-a)(b-z\vee y)}{b-a}\frac{g(y)}{y^2}\,(f*m)(dy)& \\
&= 2\int_{[1/b,1/a]} \frac{(z\wedge \frac{1}{x}-a)(b-z\vee \frac{1}{x})}{b-a} x^2 g(1/x)\,m(dx) \\
&= 2z\int_{1/b}^{1/a} \frac{(\frac{1}{a}-\frac{1}{z}\vee x)(x\wedge \frac{1}{z}-\frac{1}{b})}{\frac{1}{a}-\frac{1}{b}} x g(1/x)\,m(dx) \\
&= z\mathbb{E}_{1/z}\left[\int_0^{T_{1/b,1/a}} X_t g(1/X_t)\,dt\right],
\end{aligned}
$$

where the last equality comes from the fact that X is a diffusion on natural scale with speed measure m under $(\mathbb{P}_x)_{x\geq 0}$. Since $(X_{t\wedge T_{1/b,1/a}}, t\geq 0)$ is a bounded martingale under $\mathbb{P}_{1/z}$, we obtain by definition of $(\widetilde{\mathbb{Q}}_x)_{x>0}$

$$
\begin{aligned}
\mathbb{E}^{\widetilde{\mathbb{Q}}_{1/z}}\left[\int_0^{T^Z_{a,b}} g(Z_t)\,dt\right] &= \mathbb{E}^{\widetilde{\mathbb{Q}}_{1/z}}\left[\int_0^{T_{1/b,1/a}} g(1/X_t)\,dt\right] \\
&= z\mathbb{E}_{1/z}\left[X_{T_{1/b,1/a}}\int_0^{T_{1/b,1/a}} g(1/X_t)\,dt\right].
\end{aligned}
$$

Now, Itô's formula implies that, a.s. for all $t\geq 0$,

$$
X_t\int_0^t g(1/X_s)\,ds = \int_0^t\left(\int_0^s g(1/X_u)\,du\right)dX_s + \int_0^t X_s g(1/X_s)\,ds.
$$

Since $(X_{t\wedge T_{1/b,1/a}}, t\geq 0)$ is a bounded martingale under $\mathbb{P}_{1/z}$, for all $n\geq 1$,

$$
\mathbb{E}_{1/z}\left[X_{n\wedge T_{1/b,1/a}}\int_0^{n\wedge T_{1/b,1/a}} g(1/X_s)\,ds\right] = \mathbb{E}_{1/z}\left[\int_0^{n\wedge T_{1/b,1/a}} X_s g(1/X_s)\,ds\right].
$$

Since $\mathbb{E}_{1/z}[T_{1/b,1/a}]<\infty$ (this is a general property of diffusions), Lebesgue's theorem implies (4.5), which ends the proof of Theorem 4.2. □

The next result can be deduced from Theorem 4.2 by a simple change of variable.

Corollary 4.3. *With the previous notation,*

$$
\int_0^1 y\,m(dy)<\infty \quad\Longleftrightarrow\quad \int_1^\infty y\,\widetilde{m}(dy)<\infty
$$

and

$$
\int_1^\infty y\,m(dy)<\infty \quad\Longleftrightarrow\quad \int_0^1 y\,\widetilde{m}(dy)<\infty.
$$

The next corollary gives an important relationship between the absorption probability of X under $\mathbb{P}_x$ and the expectation of Z under $\widetilde{\mathbb{Q}}_x$. It also proves Theorem 3.7.

Corollary 4.4. *With the previous notation, for all $t > 0$ and $x > 0$, we have*

$$\frac{\mathbb{P}_x^X(t < \tau_\partial)}{x} = \mathbb{E}_{1/x}^Z(Z_t),$$

where $\mathbb{P}_x^X$ and $\mathbb{P}_z^Z$ are the respective distributions of the diffusion processes X such that $X_0 = x$ and Z such that $Z_0 = z$, both on natural scale and with respective speed measures m and $\widetilde{m}$.

In particular,

$$(Z_t)_{t\geq 0} \text{ is a martingale} \iff \int_0^1 y\, m(dy) = \infty.$$

Moreover, for any constant $A > 0$,

$$\sup_{z>0} \mathbb{E}_z^Z(Z_t) \leq A \iff \mathbb{P}_x^X(t < \tau_\partial) \leq Ax, \ \forall x \geq 0.$$

If one of these conditions happens for some $A > 0$, then $\int_0^1 ym(dy) < \infty$ and $(Z_t)_{t\geq 0}$ is a strict local martingale.

Proof of Corollary 4.4. We have, for all $x > 0$ and all $t > 0$,

$$\mathbb{E}_{1/x}^Z(Z_t) = \mathbb{E}^{\widetilde{\mathbb{Q}}_x}\left(\frac{1}{X_t}\right) = \frac{1}{x}\mathbb{P}_x(X_t > 0). \qquad \square$$

5 Proof of the results of Section 3

The main part of Theorem 3.1, Proposition 3.2 and Theorem 3.3 directly follow from the results on general Markov processes of [4]. More precisely, the following condition (A) is equivalent to Condition (ii) of Theorem 3.1 ([4, Thm. 2.1]), and implies properties (3.5) and (3.6) of Proposition 3.2 ([4, Prop. 2.3]) and the whole Theorem 3.3 ([4, Thm. 3.1]).

Assumption (A) There exists a probability measure ν on $(0, \infty)$ such that

(A1) there exists $t_0, c_1 > 0$ such that for all $x > 0$,

$$\mathbb{P}_x(X_{t_0} \in \cdot \mid t_0 < \tau_\partial) \geq c_1 \nu(\cdot);$$

(A2) there exists $c_2 > 0$ such that for all $x > 0$ and $t \geq 0$,

$$\mathbb{P}_\nu(t < \tau_\partial) \geq c_2 \mathbb{P}_x(t < \tau_\partial).$$

Hence, we need first to prove that (B) is equivalent to (A) (in Subsection 5.1), second, to prove (3.4) and that $\eta(x) \leq Cx$ (in Subsection 5.2), and third to prove (3.3) and that $\int y\, \alpha(dy) < \infty$ (in Subsection 5.3). We next give in Subsection 5.4 the proof of Proposition 3.11 on processes with jumps.

5.1 Proof of the equivalence between (A) and (B)

Note that [4, Thm. 2.1] also assumes that

$$\mathbb{P}_x(t < \tau_\partial) > 0, \quad \forall x > 0, \quad \forall t > 0, \tag{5.1}$$

so we need first to check that this is true for our diffusion processes. This follows easily from the strong Markov and the regularity properties of X (it is for example enough to use that $\mathbb{P}_x(T_{x/2} < +\infty) > 0$ and $\mathbb{P}_{x/2}(T_x < +\infty) > 0$).

We actually prove below that (A1) and (B) are equivalent and that these properties imply (A2). In other words, for one-dimensional diffusion processes, (3.2) is actually equivalent to (A1) alone.

We first prove that (A1) implies (B). If (A1) holds true, there exist $0 < a < b$ such that

$$\inf_{x>0} \mathbb{P}_x(X_{t_1} \in [a, b] \mid t_1 < \tau_\partial) = \underline{c} > 0. \tag{5.2}$$

Now, for all $x > b$,

$$\frac{\mathbb{P}_x(T_b < t_1)}{\mathbb{P}_b(t_1 < \tau_\partial)} \geq \frac{\mathbb{P}_x(X_{t_1} \in [a, b])}{\mathbb{P}_b(t_1 < \tau_\partial)} \geq \mathbb{P}_x(X_{t_1} \in [a, b] \mid t_1 < \tau_\partial).$$

Since we proved above that $\mathbb{P}_b(t_1 < \tau_\partial) > 0$, we deduce that $\inf_{x>b} \mathbb{P}_x(T_b < t_1) > 0$, i.e. that ∞ is an entrance boundary for X. Equation (5.2) also implies that, for all $x < a$,

$$\begin{aligned}\mathbb{P}_x(t_1 < \tau_\partial) &\leq \frac{\mathbb{P}_x(X_{t_1} \in [a, b])}{\underline{c}} \\ &\leq \frac{\mathbb{E}_x(X_{t_1 \wedge T_a})}{a\underline{c}} = \frac{x}{a\underline{c}}.\end{aligned}$$

Hence (i) is proved.

The difficult part of the proof is the implication (B)⇒(A).

Step 1: the conditioned process escapes a neighborhood of 0 in finite time. The goal of this step is to prove that there exists $\varepsilon, c > 0$ such that

$$\mathbb{P}_x(X_{t_1} \geq \varepsilon \mid t_1 < \tau_\partial) \geq c, \quad \forall x > 0. \tag{5.3}$$

To prove this, we first observe that, since X is a local martingale, for all $x \in (0,1)$,

$$x = \mathbb{E}_x(X_{t_1 \wedge T_1}) = \mathbb{P}_x(t_1 < \tau_\partial)\mathbb{E}_x(X_{t_1 \wedge T_1} \mid t_1 < \tau_\partial) + \mathbb{P}_x(T_1 < \tau_\partial \leq t_1).$$

By the Markov property,

$$\begin{aligned}\mathbb{P}_x(T_1 < \tau_\partial \leq t_1) &\leq \mathbb{E}_x\left[1_{T_1 < \tau_\partial \wedge t_1}\mathbb{P}_1(\tau_\partial \leq t_1)\right] \\ &\leq \mathbb{P}_x(T_1 < \tau_\partial)\mathbb{P}_1(\tau_\partial \leq t_1) \\ &= x\mathbb{P}_1(\tau_\partial \leq t_1).\end{aligned}$$

Hence (3.1) entails

$$\mathbb{E}_x(1 - X_{t_1 \wedge T_1} \mid t_1 < \tau_\partial) \leq 1 - \frac{1}{A'},$$

where $A' := A/\mathbb{P}_1(t_1 < \tau_\partial)$. Note that, necessarily, $A' > 1$. Markov's inequality then implies that, for all $x \in (0,1)$,

$$\mathbb{P}_x(X_{t_1 \wedge T_1} \leq \frac{1}{2A'-1} \mid t_1 < \tau_\partial) \leq \frac{1 - 1/A'}{1 - 1/(2A'-1)} = 1 - \frac{1}{2A'}. \tag{5.4}$$

Since $\mathbb{P}_{1/(2A'-1)}(t_1 < \tau_\partial) > 0$, there exists $\varepsilon \in (0, 1/(2A'-1))$ such that

$$\mathbb{P}_{1/(2A'-1)}(t_1 < T_\varepsilon) > 0. \tag{5.5}$$

Hence, for all $x \in (0,1)$,

$$\begin{aligned}\mathbb{P}_x(X_{t_1} \geq \varepsilon) &\geq \mathbb{P}_x(T_{1/(2A'-1)} < t_1)\mathbb{P}_{1/(2A'-1)}(T_\varepsilon > t_1) \\ &\geq \mathbb{P}_x(X_{t_1 \wedge T_1} \geq 1/(2A'-1))\mathbb{P}_{1/(2A'-1)}(T_\varepsilon > t_1) \\ &\geq \frac{\mathbb{P}_x(t_1 < \tau_\partial)}{2A'}\mathbb{P}_{1/(2A'-1)}(T_\varepsilon > t_1)\end{aligned}$$

by (5.4). This ends the proof of (5.3) for $x < 1$. For $x \geq 1 > 1/(2A'-1) > \varepsilon$, the continuity and the strong Markov property for X entail

$$\mathbb{P}_x(X_{t_1} > \varepsilon \mid t_1 < \tau_\partial) \geq \mathbb{P}_x(X_{t_1} > \varepsilon) \geq \mathbb{P}_x(T_\varepsilon > t_1) \geq \mathbb{P}_{1/(2A'-1)}(T_\varepsilon > t_1) > 0.$$

Hence (5.3) is proved.

Step 2: Construction of coupling measures for the unconditioned process. Our goal is to prove that there exist two constants $t_2, c_1 > 0$ such that, for all $x \geq \varepsilon$,

$$\mathbb{P}_x(X_{t_2} \in \cdot) \geq c_1 \nu, \tag{5.6}$$

where

$$\nu = \mathbb{P}_\varepsilon(X_{t_2} \in \cdot \mid t_2 < \tau_\partial).$$

This kind of relations can be obtained with classical coupling arguments, which we detail here for completeness. Fix $x \geq \varepsilon$ and construct two independent diffusions X^ε and X^x with speed measure $m(dx)$, and initial value ε and x respectively. Let $\theta = \inf\{t \geq 0 : X_t^\varepsilon = X_t^x\}$. By the strong Markov property, the process

$$Y_t^x = \begin{cases} X_t^x & \text{if } t \leq \theta, \\ X_t^\varepsilon & \text{if } t > \theta \end{cases}$$

has the same law as X^x. Since $\theta \leq \tau_\partial^x := \inf\{t \geq 0 : X_t^x = 0\}$, for all $t > 0$, $\mathbb{P}(\theta < t) \geq \mathbb{P}(\tau_\partial^x < t)$. Since ∞ is an entrance boundary and 0 is accessible for X, there exists $t_2 > 0$ such that

$$\inf_{y>0} \mathbb{P}_y(\tau_\partial < t_2) = c_1' > 0.$$

Hence

$$\mathbb{P}_x(X_{t_2} \in \cdot) = \mathbb{P}(Y_{t_2}^x \in \cdot) \geq \mathbb{P}(X_{t_2}^\varepsilon \in \cdot,\ \tau_\partial^x < t_2) \geq c_1' \mathbb{P}_\varepsilon(X_{t_2} \in \cdot).$$

Therefore, (5.6) is proved with $c_1 = c_1' \mathbb{P}_\varepsilon(t_2 < \tau_\partial)$.

Step 3: Proof of (A1).
Using successively the Markov property, Step 2 and Step 1, we have for all $x > 0$

$$\begin{aligned}
\mathbb{P}_x(X_{t_1+t_2} \in \cdot \mid t_1 + t_2 < \tau_\partial) &\geq \mathbb{P}_x(X_{t_1+t_2} \in \cdot \mid t_1 < \tau_\partial) \\
&\geq \int_\varepsilon^\infty \mathbb{P}_y(X_{t_2} \in \cdot) \mathbb{P}_x(X_{t_1} \in dy \mid t_1 < \tau_\partial) \\
&\geq c_1 \int_\varepsilon^\infty \nu(\cdot) \mathbb{P}_x(X_{t_1} \in dy \mid t_1 < \tau_\partial) \\
&= c_1 \nu(\cdot) \mathbb{P}_x(X_{t_1} \geq \varepsilon \mid t_1 < \tau_\partial) \geq c_1 c \nu(\cdot).
\end{aligned}$$

This entails (A1) with $t_0 = t_1 + t_2$. Hence we have proved the equivalence between (A1) and (B).

Step 4: Proof of (A2).
The general idea of the proof is close to the case of birth and death processes in [4].

For all $0 < a < b < \infty$, we have

$$\mathbb{E}_x(T_a \wedge T_b) = 2\int_a^b \frac{(x \wedge y - a)(b - x \vee y)}{b - a}\, m(dy), \quad \forall a < x < b.$$

Hence,

$$\mathbb{E}_x(T_a \wedge T_b) \leq 2\int_a^b (x \wedge y - a)\, m(dy) \leq 2\int_a^\infty y\, m(dy).$$

Since the process is non-explosive, the left hand side converges to $\mathbb{E}_x(T_a)$ when $b \to \infty$. But $\int_0^\infty y\, m(dy) < \infty$ by assumption, so that, for all $\varepsilon > 0$, there exists $a_\varepsilon > 0$ such that

$$\sup_{x \geq a_\varepsilon} \mathbb{E}_x(T_{a_\varepsilon}) \leq \varepsilon.$$

Therefore, $\sup_{x \geq a_\varepsilon} \mathbb{P}_x(T_{a_\varepsilon} \geq 1) \leq \varepsilon$ and, applying recursively the Markov property, $\sup_{x \geq a_\varepsilon} \mathbb{P}_x(T_{a_\varepsilon} \geq k) \leq \varepsilon^k$. Then, for all $\lambda > 0$, there exists $y_\lambda \geq 1$ such that

$$\sup_{x \geq y_\lambda} \mathbb{E}_x(e^{\lambda T_{y_\lambda}}) < +\infty. \tag{5.7}$$

The proof of the following lemma is postponed to the end of this section.

Lemma 5.1. *There exists $a > 0$ such that $\nu([a, +\infty[) > 0$ and, for all $k \in \mathbb{N}$,*

$$\mathbb{P}_a(X_{kt_0} \geq a) \geq e^{-\rho k t_0},$$

with $\rho > 0$.

Take a as in the previous lemma. From (5.7), one can choose $b > a$ large enough so that

$$\sup_{x \geq b} \mathbb{E}_x\left(e^{\rho T_b}\right) < \infty. \tag{5.8}$$

We are also going to make use of two inequalities. Since

$$\mathbb{P}_a(t < \tau_\partial) \geq \mathbb{P}_a(T_b < s_0)\mathbb{P}_b(t < \tau_\partial)$$

for all $s_0 \geq 0$, we obtain the first inequality: $\forall t \geq 0$,

$$\sup_{x\in[a,b]} \mathbb{P}_x(t < \tau_\partial) = \mathbb{P}_b(t < \tau_\partial) \leq C\mathbb{P}_a(t < \tau_\partial) = C \inf_{x\in[a,b]} \mathbb{P}_x(t < \tau_\partial), \quad (5.9)$$

where C is a positive constant. We recall that the regularity assumption ensures that $\mathbb{P}_a(T_b < s_0) > 0$ for s_0 large enough. The second inequality is an immediate consequence of the Markov property: $\forall s < t$,

$$\begin{aligned}\mathbb{P}_a(X_{\lceil s/t_0\rceil t_0} \geq a)\mathbb{P}_a(t - s < \tau_\partial) &= \mathbb{P}_a(X_{\lceil s/t_0\rceil t_0} \geq a) \inf_{x\in[a,\infty)} \mathbb{P}_x(t - s < \tau_\partial) \\ &\leq \mathbb{P}_a(t < \tau_\partial). \end{aligned} \quad (5.10)$$

In the following computation, we use successively (5.8), (5.9) and (5.10). For all $x \geq b$, with a constant $C > 0$ that may change from line to line,

$$\begin{aligned}\mathbb{P}_x(t < \tau_\partial) &\leq \mathbb{P}_x(t < T_b) + \int_0^t \mathbb{P}_b(t - s < \tau_\partial)\, \mathbb{P}_x(T_b \in ds) \\ &\leq Ce^{-\rho t} + C \int_0^t \mathbb{P}_a(t - s < \tau_\partial)\, \mathbb{P}_x(T_b \in ds) \\ &\leq Ce^{-\rho(\lceil t/t_0\rceil - 1)t_0} + C\mathbb{P}_a(t < \tau_\partial) \int_0^t \frac{1}{\mathbb{P}_a(X_{\lceil s/t_0\rceil t_0} \geq a)}\, \mathbb{P}_x(T_b \in ds) \\ &\leq C\mathbb{P}_a(t < \tau_\partial) + C\mathbb{P}_a(t < \tau_\partial) \int_0^t e^{\rho(s+t_0)}\, \mathbb{P}_x(T_b \in ds),\end{aligned}$$

where we used twice Lemma 5.1 in the last line. We deduce form (5.8) that, for all $t \geq 0$,

$$\sup_{x\geq b} \mathbb{P}_x(t < \tau_\partial) \leq C\mathbb{P}_a(t < \tau_\partial).$$

Since $\nu([a, +\infty[) > 0$, this ends the proof of (A2).

Proof of Lemma 5.1. Fix $a > 0$ such that $\nu([a, +\infty[) > 0$. Step 3 of the previous proof and (5.1) entail

$$\mathbb{P}_a(X_{t_0} \geq a) \geq c_1\nu([a, +\infty[)\mathbb{P}_a(t_0 < \tau_\partial) \stackrel{\text{def}}{=} e^{-\rho t_0} > 0.$$

Now, using Markov property,

$$\mathbb{P}_a(X_{kt_0} \geq a) \geq \left(\inf_{x\geq a} \mathbb{P}_x(X_{t_0} \geq a)\right)^k.$$

Since $\inf_{x\geq a} \mathbb{P}_x(X_{t_0} \geq a) = \mathbb{P}_a(X_{t_0} \geq a)$ by coupling arguments, the proof is completed. □

5.2 Proof of Proposition 3.2

Since (A) is equivalent to (B), the convergence in (3.5) for the uniform norm and (3.6) are direct consequences of Proposition [4, Prop 2.3]. This proposition also entails that η is bounded, positive on $(0, +\infty)$ and vanishes on 0, and that $\alpha(f) = 1$. The fact that η is non-decreasing comes from (3.5) and from the fact that $\mathbb{P}_x(t < \tau_\partial)$ is non-decreasing in $x \geq 0$ by standard comparison arguments. The fact that $\eta(x) \leq Cx$ follows from assumption (B) since

$$P_{t_1}\eta(x) \leq \|\eta\|_\infty \mathbb{P}_x(t_1 \leq \tau_\partial) \leq \|\eta\|_\infty Ax, \quad \forall x \geq 0.$$

It only remains to prove (3.4). For all measurable $f \geq 0$ and all $0 \leq a < x < b \leq \infty$,

$$\mathbb{E}_x\left(\int_0^{T_a \wedge T_b} f(X_t)\,dt\right) = 2\int_a^b \frac{(x \wedge y - a)(b - x \vee y)}{b - a}\, f(y)\, m(dy). \quad (5.11)$$

For $a = 0$ and $b = \infty$, we obtain

$$\mathbb{E}_x\left(\int_0^{\tau_\partial} f(X_t)\,dt\right) = 2\int_0^\infty (x \wedge y)\, f(y)\, m(dy). \quad (5.12)$$

For $f = \eta$, we deduce that

$$\int_0^\infty \mathbb{E}_x\left(\eta(X_t)1_{t<\tau_\partial}\right)\, dt = 2\int_0^\infty (x \wedge y)\, \eta(y)\, m(dy).$$

Since $\eta(0) = 0$ and $L\eta = -\lambda_0\eta$, we have $\mathbb{E}_x\left(\eta(X_t)1_{t<\tau_\partial}\right) = P_t\eta(x) = e^{-\lambda_0 t}\eta(x)$. Then

$$\frac{\eta(x)}{\lambda_0} = 2\int_0^\infty (x \wedge y)\, \eta(y)\, m(dy).$$

This entails (3.4) provided we prove that $\alpha(dx)$ is proportional to $\eta(x)m(dx)$ (observe that the normalizing constant is determined by the condition $\alpha(\eta) = 1$). This will be done in the next Subsection.

5.3 Proof of (3.3) and that $\int y\,\alpha(dy) < \infty$

Integrating (5.12) with respect to $\alpha(dx)$, we obtain

$$\int_0^\infty \mathbb{E}_\alpha\left(f(X_t)1_{t<\tau_\partial}\right)\, dt = 2\int_0^\infty \int_0^\infty (x \wedge y)\, f(y)\, m(dy)\, \alpha(dx).$$

Since $\mathbb{E}_\alpha\left(f(X_t)1_{t<\tau_\partial}\right) = \alpha(f)\, e^{-\lambda_0 t}$, we deduce that

$$\frac{\alpha(f)}{\lambda_0} = 2\int_0^\infty \int_0^\infty (x\wedge y)\, f(y)\, m(dy)\, \alpha(dx).$$

This entails (3.3). We now prove that $\alpha(dx)$ is proportional to $\eta(x)m(dx)$.

The following reversibility result for diffusions on natural scale is more or less classical but we need a version with precise bounds on the test functions in the case of a diffusion coming down from infinity and hitting 0 a.s. in finite time. The proof is given at the end of the subsection for sake of completeness.

Lemma 5.2. *Let X be a diffusion on $[0,\infty)$ in natural scale with speed measure m satisfying* (2.2)*. Then it is reversible with respect to m in the sense that, for all bounded non-negative measurable functions f on $(0,+\infty)$ and all nonnegative measurable function g on $(0,+\infty)$ such that $g(x)\le Cx$ for some $C>0$,*

$$\int_0^\infty f(x)P_t g(x)m(dx) = \int_0^\infty g(x)P_t f(x)m(dx), \quad \forall t\ge 0, \tag{5.13}$$

where both sides are finite.

Applying this lemma for $g=\eta$, we obtain that, for all bounded measurable non-negative f,

$$\int_0^\infty f(x)\eta(x)m(dx) = e^{\lambda_0 t}\int_0^\infty \eta(x)P_t f(x)m(dx)$$

Now, it follows from (3.2) and (3.5) that

$$e^{\lambda_0 t}P_t f(x) = e^{\lambda_0 t}\mathbb{P}_x(t<\tau_\partial)\mathbb{E}_x(f(X_t)\mid t<\tau_\partial) \to \eta(x)\alpha(f)$$

when $t\to+\infty$, where the convergence is uniform in x. Since $\int \eta\, dm<\infty$, Lebesgue's theorem entails that

$$\alpha(f)\int_0^\infty \eta^2(x)m(dx) = \int_0^\infty f(x)\eta(x)m(dx)$$

for all bounded measurable f. Hence $\alpha\propto \eta\, dm$. Note also that $f=\eta$ gives $\int_0^\infty \eta\, dm = \int_0^\infty \eta^2\, dm$.

It only remains to prove that $\int y\,\alpha(dy)<\infty$. Since η is bounded and

$$\eta(x)\propto \int_0^\infty (x\wedge y)\,\alpha(dy),$$

this follows from the monotone convergence theorem.

Proof of Lemma 5.2. We start by proving that both sides of (5.13) are finite. This is obvious for the r.h.s. because of (2.2). For the l.h.s., since the positive local martingale $(X_t, t \geq 0)$ is a supermartingale, we have $P_t g(x) = \mathbb{E}_x[g(X_t)1_{t<\tau_\partial}] \leq C\mathbb{E}_x[X_t] \leq Cx$, which allows to conclude.

By Lebesgue's theorem, it is enough to prove (5.13) for a f, g in a dense subset of the set of continuous functions on $(0, +\infty)$ with compact support. Note that a function g with compact support in $(0, +\infty)$ satisfies $g(x) \leq Cx$ for some $C > 0$. For all $s \in [0, t]$, we define

$$\psi(s) = \int_0^\infty P_s f(x)\, P_{t-s} g(x)\, m(dx).$$

We use the characterization of the infinitesimal generator of diffusion processes of [5, Thm. 2.81]: let $\mathcal{D}$ be the set of functions f bounded continuous on $[0, \infty)$, such that the right derivative f^+ of f exists, is finite, continuous from the right and of bounded variation on all compact intervals of $(0, \infty)$, and such that $df^+ = h\, dm$, where df^+ denotes the measure on $(0, \infty)$ such that

$$f^+(y) - f^+(x) = df^+(x, y]$$

and h is some bounded continuous function on $[0, \infty)$ with $h(0) = 0$. Then, for all $f \in \mathcal{D}$,

$$Lf(x) := \lim_{t \to 0} \frac{P_t f(x) - f(x)}{t} = \frac{df^+}{dm}(x), \quad \forall x \geq 0,$$

where the convergence holds for the uniform norm on $[0, \infty)$.

So let $f, g \in \mathcal{D}$ have compact support. In particular, there exists $C > 0$ such that $P_s f(x) + P_s g(x) \leq Cx$ for all $x, s \geq 0$. Since $f, g \in \mathcal{D}$, $P_s f$ and $P_s g$ also belong to $\mathcal{D}$ and Lebesgue's theorem entails that, for all $s \in [0, t]$

$$\begin{aligned}\psi'(s) &= \int_0^\infty LP_s f(x)\, P_{t-s} g(x)\, m(dx) - \int_0^\infty P_s f(x)\, LP_{t-s} g(x)\, m(dx) \\ &= \int_0^\infty P_{t-s} g(x)\, d(LP_s f)^+(x) - \int_0^\infty P_s f(x)\, d(LP_{t-s} g)^+(x).\end{aligned}$$

The right-hand side is equal to zero by the integration by parts formula for Stieljes integrals.

Since $\mathcal{D}$ is dense in the set of continuous functions on $(0, \infty)$ with compact support, the proof is completed. □

5.4 Proof of Proposition 3.11

The proof is similar to the one of Theorem 3.1 (see Subsection 5.1) and we only detail the steps that need to be modified. Our aim is to check that conditions (A1) and (A2) hold.

Step 1. (A1) is satisfied.

Since m satisfies the conditions of Theorem 3.1, we deduce that (A1) holds for X. As a consequence, there exist two constants $c_1 > 0$ and $t_0 > 0$ and a probability measure ν on $(0, +\infty)$, such that, for all $A \subset (0, \infty)$ measurable,

$$\mathbb{P}_x(X_{t_0} \in A) \geq c_1 \nu(A) \mathbb{P}_x(t_0 < \tau_\partial), \quad \forall x \in (0, +\infty).$$

By construction of $\widetilde{X}$, we have

$$\mathbb{P}_x(\widetilde{X}_{t_0} \in A) \geq e^{-t_0} \mathbb{P}_x(X_{t_0} \in A).$$

Fix $x \in (0, 1)$. Using the fact that $X_t = \widetilde{X}_t$ for all $t \leq T_1$ under $\mathbb{P}_x$, we deduce that

$$\begin{aligned}
\mathbb{P}_x(t_0 < \widetilde{\tau}_\partial) &\leq \mathbb{P}_x(\{t_0 < \tau_\partial\} \cup \{\widetilde{X}_{t_0} \neq X_{t_0}\}) \\
&\leq \mathbb{P}_x(t_0 < \tau_\partial) + \mathbb{P}_x(T_1 \leq \tau_\partial) \\
&\leq Ax + x \\
&\leq \frac{A+1}{a} \mathbb{P}_x(t_0 < \tau_\partial),
\end{aligned}$$

where $a > 0$ is the positive constant from Proposition 3.4 (i). As a consequence, for all $A \subset (0, \infty)$ measurable,

$$\mathbb{P}_x(\widetilde{X}_{t_0} \in A) \geq \frac{c_1 a e^{-t_0}}{A+1} \nu(A) \, \mathbb{P}_x(t_0 < \widetilde{\tau}_\partial), \tag{5.14}$$

which concludes the proof of (A1) for $\widetilde{X}$.

Step 2. (A2) is satisfied.

By construction of the process $\widetilde{X}$ from the process X and an independent Poisson process, it is clear that, for all $x \leq y$, all $t > 0$ and $a \in (0, x)$, we have

$$\mathbb{P}_x(\widetilde{T}_a \leq t) \geq \mathbb{P}_y(\widetilde{T}_a \leq t),$$

where $\widetilde{T}_a = \inf\{t \geq 0,\ \widetilde{X}_t = a\}$, and that $\mathbb{P}_x(\widetilde{X}_t \geq a) \geq \mathbb{P}_x(X_t \geq a)$. Hence we only need to prove that $\widetilde{X}$ satisfies (5.7), the rest of the proof being the same as in Step 4 of Subsection 5.1.

Fix $\varepsilon > 0$ and set $t_\varepsilon = -\log(1-\varepsilon/2) > 0$. For all $a > 0$, we have, by independence of the Poisson process $(N_t)_{t\geq 0}$,

$$\inf_{x\in(a,+\infty)} \mathbb{P}_x(\widetilde{T}_a \leq t_\varepsilon) = \lim_{x\to\infty} \mathbb{P}_x(\widetilde{T}_a \leq t_\varepsilon) \geq \left(1-\frac{\varepsilon}{2}\right) \lim_{x\to\infty} \mathbb{P}_x(T_a \leq t_\varepsilon).$$

Since X comes down from infinity, there exists $a_\varepsilon > 0$ such that

$$\lim_{x\to+\infty} \mathbb{P}_x(T_{a_\varepsilon} \leq t_\varepsilon) \geq 1 - \frac{\varepsilon}{2}.$$

Hence, for all $\varepsilon > 0$ small enough, there exists $a_\varepsilon > 0$ such that, for all $x \geq a_\varepsilon$,

$$\mathbb{P}_x(1 < \widetilde{T}_{a_\varepsilon}) \leq \mathbb{P}_x(t_\varepsilon < \widetilde{T}_{a_\varepsilon}) \leq 1 - \left(1-\frac{\varepsilon}{2}\right)^2 \leq \varepsilon.$$

This entails (5.7) and (A2) as in Step 4 of the proof of Theorem 3.1.

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