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# ON THE CONSTRUCTION OF GENERAL CUBATURE FORMULA BY FLAT EXTENSIONS

M. ABRIL BUCERO, C. BAJAJ, AND B. MOURRAIN

ABSTRACT. We describe a new method to compute general cubature formulae. The problem is initially transformed into the computation of truncated Hankel operators with flat extensions. We then analyse the algebraic properties associated to flat extensions and show how to recover the cubature points and weights from the truncated Hankel operator. We next present an algorithm to test the flat extension property and to additionally compute the decomposition. To generate cubature formulae with a minimal number of points, we propose a new relaxation hierarchy of convex optimization problems minimizing the nuclear norm of the Hankel operators. For a suitably high order of convex relaxation, the minimizer of the optimization problem corresponds to a cubature formula. Furthermore cubature formulae with a minimal number of points are associated to faces of the convex sets. We illustrate our method on some examples, and for each we obtain a new minimal cubature formula.

## 1. CUBATURE FORMULA

1.1. **Statement of the problem.** Consider the integral for a continuous function  $f$ ,

$$I[f] = \int_{\Omega} w(\mathbf{x})f(\mathbf{x})d\mathbf{x}$$

where  $\Omega \subset \mathbb{R}^n$  and  $w$  is a positive function on  $\Omega$ .

We are looking for a cubature formula which has the form

$$(1) \quad \langle \sigma | f \rangle = \sum_{j=1}^r w_j f(\zeta_j)$$

where the points  $\zeta_j \in \mathbb{C}^n$  and the weights  $w_j \in \mathbb{R}$  are independent of the function  $f$ . They are chosen so that

$$\langle \sigma | f \rangle = I[f], \forall f \in V,$$

where  $V$  is a finite dimensional vector space of functions. Usually, the vector space  $V$  is the vector space of polynomials of degree  $\leq d$ , because

a well-behaved function  $f$  can be approximated by a polynomial, so that  $Q[f]$  approximates the integral  $I[f]$ .

Given a cubature formula (1) for  $I$ , its algebraic degree is the largest degree  $d$  for which  $I[f] = \langle \sigma | f \rangle$  for all  $f$  of degree  $\leq d$ .

**1.2. Related works.** Prior approaches to the solution of cubature problem can be grouped into roughly two classes. One, where the goal is to estimate the fewest weighted, aka cubature points possible for satisfying a prescribed cubature rule of fixed degree [9, 25, 27, 30, 31, 34]. The other class, focusses on the determination and construction of cubature rules which would yield the fewest cubature points possible [7, 35, 39, 40, 41, 42, 45, 46]. In [35], for example, Radon introduced a fundamental technique for constructing minimal cubature rules where the cubature points are common zeros of multivariate orthogonal polynomials. This fundamental technique has since been extended by many, including for e.g. [34, 42, 46] where notably, the paper [46] use multivariate ideal theory, while [34] uses operator dilation theory. In this paper, we propose another approach to the second class of cubature solutions, namely, constructing a suitable finite dimensional Hankel matrix and extracting the cubature points using sub operators of the Hankel matrix [18]. This approach is related to [21, 22, 24] and which in turn are based on the methods of multivariate truncated moment matrices, their positivity and extension properties [11, 12, 13].

Applications of such algorithms determining cubature rules and cubature points over general domains occur in isogeometric modeling and finite element analysis using generalized Barycentric finite elements [17, 1, 36, 37]. Additional applications abound in numerical integration for low dimensional (6-100 dimensions) convolution integrals that appear naturally in computational molecular biology [3, 2], as well in truly high dimensional (tens of thousands of dimensions) integrals that occur in finance [33, 8].

**1.3. Reformulation.** Let  $R = \mathbb{R}[\mathbf{x}]$  be the ring of polynomials in the variables  $\mathbf{x} = (x_1, \dots, x_n)$  with coefficients in  $\mathbb{R}$ . Let  $R_d$  be the set of polynomials of degree  $\leq d$ . The set of linear forms on  $R$ , that is, the set of linear maps from  $R$  to  $\mathbb{R}$  is denoted by  $R^*$ . The value of a linear form  $\Lambda \in R^*$  on a polynomial  $p \in R$  is denoted by  $\langle \Lambda | p \rangle$ . The set  $R^*$  can be identified with the ring of formal power series in new variables  $\mathbf{z} = (z_1, \dots, z_n)$ :

$$\begin{aligned} R^* &\rightarrow \mathbb{R}[[\mathbf{z}]] \\ \Lambda &\mapsto \Lambda(\mathbf{z}) = \sum_{\alpha \in \mathbb{N}^n} \langle \Lambda | \mathbf{x}^\alpha \rangle \mathbf{z}^\alpha. \end{aligned}$$

The coefficients  $\langle \Lambda | \mathbf{x}^\alpha \rangle$  of these series are called the *moments* of  $\Lambda$ . The evaluation at a point  $\zeta \in \mathbb{R}^n$  is an element of  $R$ , denoted by  $\mathbf{e}_\zeta$ , and defined by  $\mathbf{e}_\zeta : f \in R \mapsto f(\zeta) \in \mathbb{R}$ . For any  $p \in R$  and any  $\Lambda \in R^*$ , let  $p \star \Lambda : q \in R \mapsto \Lambda(pq)$ .

**Cubature problem:** Let  $V \subset R$  be the vector space of polynomials and consider the linear form  $\bar{I} \in V^*$  defined by

$$\begin{aligned} \bar{I} : V &\rightarrow \mathbb{R} \\ \mathbf{v} &\mapsto I[\mathbf{v}] \end{aligned}$$

Computing a cubature formula for  $I$  on  $V$  then consists in finding a linear form

$$\sigma = \sum_{i=1}^r w_i \mathbf{e}_{\zeta_i} : f \mapsto \sum_{j=1}^r w_j f(\zeta_j).$$

which coincides on  $V$  with  $\bar{I}$ . In other words, given the linear form  $\bar{I}$  on  $R_d$ , we wish to find a linear form  $\sigma = \sum_{i=1}^r w_i \mathbf{e}_{\zeta_i}$  which extends  $\bar{I}$ .

## 2. CUBATURE FORMULAE AND HANKEL OPERATORS

To find such a linear form  $\sigma \in R^*$ , we exploit the properties of its associated bilinear form  $H_\sigma : (p, q) \in R \times R \rightarrow \langle \sigma | pq \rangle$ , or equivalently, the associated Hankel operator:

$$\begin{aligned} H_\sigma : R &\rightarrow R^* \\ p &\mapsto p \star \sigma \end{aligned}$$

The kernel of  $H_\sigma$  is  $\ker H_\sigma = \{p \in R \mid \forall q \in R, \langle \sigma | pq \rangle = 0\}$ . It is an ideal of  $R$ . Let  $\mathcal{A}_\sigma = R / \ker H_\sigma$  be the associated quotient ring.

The matrix of the bilinear form or the Hankel operator  $H_\sigma$  associated to  $\sigma$  in the monomial basis, and its dual are  $(\langle \Lambda | \mathbf{x}^{\alpha+\beta} \rangle)_{\alpha, \beta \in \mathbb{N}^n}$ . If we restrict them to a space  $V$  spanned by the monomial basis  $(\mathbf{x}^\alpha)_{\alpha \in A}$  for some finite set  $A \subset \mathbb{N}^n$ , we obtain a finite dimensional matrix  $[H_\sigma^{A,A}] = (\langle \Lambda | \mathbf{x}^{\alpha+\beta} \rangle)_{\alpha, \beta \in A}$ , and which is a Hankel matrix.. More generally, for any vector spaces  $V, V' \subset R$ , we define the truncated bilinear form and Hankel operators:  $H_\sigma^{V,V'} : (v, v') \in V \times V' \mapsto \langle \sigma | pq \rangle \in \mathbb{R}$  and  $H_\sigma^{V,V'} : v \in V \mapsto v \star \sigma \in V'^*$ . If  $V$  (resp.  $V'$ ) is spanned by a monomial set  $\mathbf{x}^A$  for  $A \subset \mathbb{N}^n$  (resp.  $\mathbf{x}^B$  for  $B \subset \mathbb{N}^n$ ), the truncated bilinear form and truncated Hankel operator are also denoted by  $H_\sigma^{A,B}$ . The associated Hankel matrix in the monomial basis is then  $[H_\sigma^{A,B}] = (\langle \Lambda | \mathbf{x}^{\alpha+\beta} \rangle)_{\alpha \in A, \beta \in B}$ .

The main property that we will use to characterize a cubature formula is the following (see [23, 20]):

**Proposition 2.1.** *A linear form  $\sigma \in R^*$  can be decomposed as  $\sigma = \sum_{i=1}^r w_i \mathbf{e}_{\zeta_i}$  with  $w_i \in \mathbb{C} \setminus \{0\}$ ,  $\zeta_i \in \mathbb{C}^n$  iff*

- $H_\sigma : p \mapsto p \star \sigma$  is of rank  $r$ ,
- $\ker H_\sigma$  is the ideal of polynomials vanishing at the points  $\{\zeta_1, \dots, \zeta_r\}$ .

This shows that in order to find the points  $\zeta_i$  of a cubature formula, it is sufficient to compute the polynomials  $p \in R$  such that  $\forall q \in R$ ,  $\langle \sigma | pq \rangle = 0$ , and to determine their common zeroes. In section 4 we describe a direct way to recover the points  $\zeta_i$ , and the weights  $\omega_i$  from suboperators of  $H_\sigma$ .

In the case of cubature formulae with real points and positive weights, we already have the following stronger result (see [23, 20]):

**Proposition 2.2.** *Let  $\sigma \in R^*$ .*

$$\sigma = \sum_{i=1}^r w_i \mathbf{e}_{\zeta_i}$$

with  $w_i > 0$ ,  $\zeta_i \in \mathbb{R}^n$  iff  $\text{rank} H_\sigma = r$  and  $H_\sigma \succcurlyeq 0$ .

A linear form  $\sigma = \sum_{i=1}^r w_i \mathbf{e}_{\zeta_i}$  with  $w_i > 0$ ,  $\zeta_i \in \mathbb{R}^n$  is called a  $r$ -atomic measure since it coincides with the weighted sum of the  $r$  Dirac measures at the points  $\zeta_i$ .

Therefore, the problem of constructing a cubature formula  $\sigma$  for  $I$  exact on  $V \subset R$  can be reformulated as follows: Construct a linear form  $\sigma \in R^*$  such that

- $\text{rank} H_\sigma = r < \infty$  and  $H_\sigma \succcurlyeq 0$ .
- $\forall v \in V$ ,  $I[v] = \langle \sigma | v \rangle$ .

The rank  $r$  of  $H_\sigma$  is given by the number of points of the cubature formula, which is expected to be small or even minimal.

The following result states that a cubature formula with  $\dim(V)$  points, always exist.

**Theorem 2.3.** [43, 4] *If a sequence  $(\sigma_\alpha)_{\alpha \in \mathbb{N}^n, |\alpha| \leq t}$  is the truncated moment sequence of a measure  $\mu$  (i.e.  $\sigma_\alpha = \int \mathbf{x}^\alpha d\mu$  for  $|\alpha| \leq t$ ) then it can also be represented by an  $r$ -atomic measure: for  $|\alpha| \leq t$ ,  $\sigma_\alpha = \sum_{i=1}^r w_i \xi_i^\alpha$  where  $r \leq s_t$ ,  $w_i > 0$ ,  $\xi_i \in \text{supp}(\mu)$ .*

This result can be generalized to any set of linearly independent polynomials  $v_1, \dots, v_r \in R$  (see the proof in [4] or Theorem 5.9 in [23]). We deduce that the cubature problem always has a solution with  $\dim(V)$  or less points.

**Definition 2.4.** *Let  $r_c(I)$  be the maximum rank of the bilinear form  $H_I^{W, W'} : (w, w') \in W \times W' \mapsto I[ww']$  where  $W, W' \subset V$  are such that  $\forall w \in W, \forall w' \in W', ww' \in V$ . It is called the Catalecticant rank of  $I$ .*

**Proposition 2.5.** *Any cubature formula for  $I$  exact on  $V$  involves at least  $r_c(I)$  points.*

*Proof.* Suppose that  $\sigma$  is a cubature formula for  $I$  exact on  $V$  with  $r$  points. Let  $W, W' \subset V$  be vector spaces are such that  $\forall w \in W, \forall w' \in W', w w' \in V$ . Since  $H_I^{W, W'}$  coincides with  $H_\sigma^{W, W'}$ , which is the restriction of the bilinear form  $H_\sigma$  to  $W \times W'$ , we deduce that  $r = \text{rank} H_\sigma \geq \text{rank} H_\sigma^{W, W'} = \text{rank}(H_I^{W, W'})$ . Thus  $r \geq r_c(I)$ .  $\square$

**Corollary 2.6.** *Let  $W \subset V$  such that  $\forall w, w' \in W, w w' \in V$ . Then any cubature formula of  $I$  exact on  $V$  involves at least  $\dim(W)$  points.*

*Proof.* As we have  $\forall p \in W, p^2 \in V$  so that  $I(p^2) = 0$  implies  $p = 0$ . Therefore the quadratic form  $H_I^{W, W} : (p, q) \in W \times W \rightarrow I[pq]$  is positive definite of rank  $\dim(W)$ . By Proposition 2.5, a cubature formula of  $I$  exact on  $V$  involves at least  $r_c(I) \geq \dim(W)$  points.  $\square$

In particular, if  $V = R_d$  any cubature formula of  $I$  exact on  $V$  involve at least  $\dim R_{\lfloor \frac{d}{2} \rfloor} = \binom{\lfloor \frac{d}{2} \rfloor + n}{n}$  points.

In [26], this lower bound is improved for cubature problems in two variables.

### 3. FLAT EXTENSIONS

In order to reduce the extension problem to a finite-dimensional problem, we consider hereafter only truncated Hankel operators. Given two subspaces  $W, W'$  of  $R$  and a linear form  $\sigma$  defined on  $W \cdot W'$  (i.e.  $\sigma \in \langle W \cdot W' \rangle^*$ ), we define

$$\begin{aligned} H_\sigma^{W, W'} : W \times W' &\rightarrow \mathbb{R} \\ (w, w') &\mapsto \langle \sigma | ww' \rangle. \end{aligned}$$

If  $\mathbf{w}$  (resp.  $\mathbf{w}'$ ) is a basis of  $W$  (resp.  $W'$ ), then we will also denote  $H_\sigma^{\mathbf{w}, \mathbf{w}'} := H_\sigma^{W, W'}$ . The matrix of  $H_\sigma^{\mathbf{w}, \mathbf{w}'}$  in the basis  $\mathbf{w} = \{w_1, \dots, w_s\}$ ,  $\mathbf{w}' = \{w'_1, \dots, w'_{s'}\}$  is  $[\langle \sigma | w_i w_j \rangle]_{1 \leq i \leq s, 1 \leq j \leq s'}$ .

**Definition 3.1.** *Let  $W \subset V, W' \subset V'$  be subvector spaces of  $R$  and  $\sigma \in \langle V \cdot V' \rangle^*$ . We say that  $H_\sigma^{V, V'}$  is a flat extension of  $H_\sigma^{W, W'}$  if  $\text{rank} H_\sigma^{V, V'} = \text{rank} H_\sigma^{W, W'}$ .*

A set  $B$  of monomials of  $R$  is *connected to 1* if it contains 1 and if for any  $m \neq 1 \in B$ , there exists  $1 \leq i \leq n$  and  $m' \in B$  such that  $m = x_i m'$ .

As a quotient  $R/\ker \tilde{Q}$  has a monomial basis connected to 1, so in the first step we take for  $\mathbf{w}, \mathbf{w}'$ , monomial sets that are connected to 1

For a set  $B$  of monomials in  $R$ , let us define  $B^+ = B \cup x_1 B \cup \dots \cup x_n B$  and  $\partial B = B^+ \setminus B$ .

The next theorem gives a characterization of flat extensions for Hankel operators defined on monomial sets connected to 1. It is a generalized form of the Curto-Fialkow theorem [13].

**Theorem 3.2.** [24, 6, 5]. *Let  $B \subset C, B' \subset C'$  be sets of monomials connected to 1 such that  $|B| = |B'| = r$  and  $C \cdot C'$  contains  $B^+ \cdot B'^+$ . If  $\sigma \in \langle C \cdot C' \rangle^*$  is such that  $\text{rank } H_\sigma^{B, B'} = \text{rank } H_\sigma^{C, C'} = r$ , then  $H_\sigma^{C, C'}$  has a unique flat extension  $H_{\tilde{\sigma}}$  for some  $\tilde{\sigma} \in R^*$ . Moreover, we have  $\ker H_{\tilde{\sigma}} = (\ker H_\sigma^{C, C'})$  and  $R = \langle B \rangle \oplus \ker H_{\tilde{\sigma}} = \langle B' \rangle \oplus \ker H_{\tilde{\sigma}}$ . In the case where  $B' = B$ , if  $H_\sigma^{B, B} \succneq 0$ , then  $H_{\tilde{\sigma}} \succneq 0$ .*

Based on this theorem, in order to find a flat extension of  $H_\sigma^{B, B'}$ , it suffices to construct an extension  $H_\sigma^{B^+, B'^+}$  of the same rank  $r$ .

**Corollary 3.3.** *Let  $V \subset R$  be a finite dimensional vector space. If there exist a set  $B$  of monomials connected to 1 such that  $V \subset \langle B^+ \cdot B^+ \rangle$  and  $\sigma \in \langle B^+ \cdot B^+ \rangle^*$  such that  $\forall v \in V, \langle \sigma | v \rangle = I[v]$  and  $\text{rank } H_\sigma^{B, B} = \text{rank } H_\sigma^{B^+, B^+} = |B| = r$ , then there exists  $w_i > 0, \zeta_i \in \mathbb{R}^n$  such that  $\forall v \in V$ ,*

$$I[v] = \sum_{i=1}^r w_i v(\xi_i).$$

This characterization leads to equations which are at most of degree 2 in a set of variables related to unknown moments and relation coefficients as described by the following proposition:

**Proposition 3.4.** *Let  $B$  and  $B'$  be two sets of monomials of Rof size  $r$ , connected to 1 and  $\sigma$  be a linear form on  $\langle B^+ \cdot B^+ \rangle$ . Then,  $H_\sigma^{B^+, B'^+}$  admits a flat extension  $H_{\tilde{\sigma}}$  such that  $H_{\tilde{\sigma}}$  is of rank  $r$  and  $B$  (resp.  $B'$ ) a basis of  $R / \ker H_{\tilde{\sigma}}$  iff*

$$(2) \quad [H_\sigma^{B^+, B'^+}] = \begin{pmatrix} \mathbb{Q} & \mathbb{M}' \\ \mathbb{M}^t & \mathbb{N} \end{pmatrix},$$

with  $\mathbb{Q} = [H_\sigma^{B, B'}]$ ,  $\mathbb{M}' = [H_\sigma^{B, \partial B'}]$ ,  $\mathbb{M}^t = [H_\sigma^{\partial B, B'}]$ ,  $\mathbb{N} = [H_\sigma^{\partial B, \partial B'}]$  is such that  $\mathbb{Q}$  is invertible and

$$(3) \quad \mathbb{M} = \mathbb{Q}^t \mathbb{P}, \mathbb{M}' = \mathbb{Q} \mathbb{P}', \mathbb{N} = \mathbb{P}^t \mathbb{Q} \mathbb{P}',$$

for some matrices  $\mathbb{P} \in \mathbb{C}^{B \times \partial B'}$ ,  $\mathbb{P}' \in \mathbb{C}^{B' \times \partial B}$ .

*Proof.* If we have  $\mathbb{M} = \mathbb{Q}^t \mathbb{P}, \mathbb{M}' = \mathbb{Q} \mathbb{P}', \mathbb{N} = \mathbb{P}^t \mathbb{Q} \mathbb{P}'$ , then

$$[H_\sigma^{B^+, B'^+}] = \begin{pmatrix} \mathbb{Q} & \mathbb{Q} \mathbb{P}' \\ \mathbb{P}^t \mathbb{Q} & \mathbb{P}^t \mathbb{Q} \mathbb{P}' \end{pmatrix}$$

has clearly the same rank as  $\mathbb{Q}=[H_\sigma^{B,B'}]$ . According to Theorem 3.2,  $H_\sigma^{B^+,B'^+}$  admits a flat extension  $H_{\tilde{\sigma}}$  with  $\tilde{\sigma} \in R^*$  such that  $B$  and  $B'$  are bases of  $\mathcal{A}_{\tilde{\sigma}} = R/\ker H_{\tilde{\sigma}}$ .

Conversely, if  $H_{\tilde{\sigma}}$  is a flat extension of  $H_\sigma^{B^+,B'^+}$  with  $B$  and  $B'$  bases of  $\mathcal{A}_{\tilde{\sigma}} = R/\ker H_{\tilde{\sigma}}$ , then  $[H_{\tilde{\sigma}}^{B,B'}] = [H_\sigma^{B,B'}] = \mathbb{Q}$  is invertible and of size  $r = |B| = |B'|$ . As  $H_{\tilde{\sigma}}$  is of rank  $r$ ,  $H_\sigma^{B^+,B'^+}$  is also of rank  $r$ . Thus, there exists  $\mathbb{P}' \in \mathbb{C}^{B' \times \partial B}$  ( $\mathbb{P}' = \mathbb{Q}^{-1}\mathbb{M}'$ ) such that  $\mathbb{M}' = \mathbb{Q}\mathbb{P}'$ . Similarly, there exists  $\mathbb{P} \in \mathbb{C}^{B \times \partial B'}$  such that  $\mathbb{M} = \mathbb{Q}^t \mathbb{P}$ . Thus, the kernel of  $[H_\sigma^{B^+,B'^+}]$  (resp.  $[H_\sigma^{B^+,B'^+}] = [H_\sigma^{B^+,B'^+}]^t$ ) is the image of  $\begin{pmatrix} -\mathbb{P}' \\ \mathbb{I} \end{pmatrix}$  (resp.  $\begin{pmatrix} -\mathbb{P} \\ \mathbb{I} \end{pmatrix}$ ). We deduce that  $\mathbb{N} = \mathbb{M}'^t \mathbb{P}' = \mathbb{P}^t \mathbb{Q} \mathbb{P}'$ .  $\square$

**Remark 3.5.** *A basis of the kernel of  $H_\sigma^{B^+,B'^+}$  is given by the columns of  $\begin{pmatrix} -\mathbb{P}' \\ \mathbb{I} \end{pmatrix}$ , which represent polynomials of the form*

$$p_\alpha = \mathbf{x}^\alpha - \sum_{\beta \in B} p_{\alpha,\beta} \mathbf{x}^\beta$$

for  $\alpha \in \partial B$ . These polynomials are border relations which project the monomials  $\mathbf{x}^\alpha$  of  $\partial B$  on the vector space spanned by the monomials  $B$ , modulo  $\ker H_\sigma^{B^+,B'^+}$ . It is proved in [6] that they form a border basis of the ideal  $\ker H_{\tilde{\sigma}}$  when  $H_\sigma^{B^+,B'^+}$  is a flat extension and  $H_\sigma^{B,B'}$  is invertible.

**Remark 3.6.** *Let  $A \subset \mathbb{N}^n$  be a set of monomials such that  $\langle \sigma | \mathbf{x}^\alpha \rangle = I[\mathbf{x}^\alpha]$ . Considering the entries of  $\mathbb{P}, \mathbb{P}'$  and the entries  $\sigma_\alpha$  of  $\mathbb{Q}$  with  $\alpha \notin A$  as variables, the constraints (3) are multilinear equations in these variables of total degree at most 3 if  $\mathbb{Q}$  contains unknown entries and 2 otherwise.*

**Example 3.7.** *We consider here  $V = R_{2k}$  for  $k > 0$ . By Proposition 3.4, any cubature formula for  $I$  exact on  $V$  has at least  $r_k := \dim R_k$  points. Let us take  $B$  to be all the monomials of degree  $\leq k$  so that  $B^+$  is the set of monomials of degree  $\leq k+1$ . If a cubature formula for  $I$  is exact on  $R_{2k}$  and has  $r_k$  points, then  $H_\sigma^{B^+,B^+}$  is a flat extension of  $H_\sigma^{B,B}$  of rank  $r_k$ . Consider a decomposition of  $H_\sigma^{B^+,B^+}$  as in (2). By Proposition 3.4, we have the relations*

$$(4) \quad \mathbb{M} = \mathbb{Q}\mathbb{P}, \mathbb{N} = \mathbb{P}'^t \mathbb{Q}\mathbb{P},$$

where

$$\bullet \quad \mathbb{Q} = (I[\mathbf{x}^{\beta+\beta'}])_{\beta, \beta' \in B},$$

- $\mathbb{M} = (\langle \sigma | \mathbf{x}^{\beta+\beta'} \rangle)_{\beta \in B, \beta' \in \partial B}$  with  $\langle \sigma | \mathbf{x}^{\beta+\beta'} \rangle = I[\mathbf{x}^{\beta+\beta'}]$  when  $|\beta + \beta'| \leq 2k$ ,
- $\mathbb{N} = (\langle \sigma | \mathbf{x}^{\beta+\beta'} \rangle)_{\beta, \beta' \in \partial B}$
- $\mathbb{P} = (p_{\beta, \alpha})_{\beta \in B, \alpha \in \partial B}$ .

The equations (4) are quadratic in the variables  $\mathbb{P}$  and linear in the variables in  $\mathbb{M}$ . Solving these equations yields a flat extension  $H_\sigma^{B^+, B^+}$  of  $H_\sigma^{B, B}$ . As  $H_\sigma^{B, B} \succneq 0$ , any real solution of this system of equations corresponds to a cubature for  $I$  on exact  $R_{2k}$  of the form  $\sigma = \sum_{i=1}^{r_k} w_i \mathbf{e}_{\zeta_i}$  with  $w_i > 0$ ,  $\zeta_i \in \mathbb{R}^n$ .

We illustrate the approach with  $R = \mathbb{R}[x_1, x_2]$ ,  $V = R_4$ ,  $B = \{1, x_1, x_2, x_1^2, x_1x_2, x_2^2\}$ ,  $B^+ = \{1, x_1, x_2, x_1^2, x_1x_2, x_2^2, x_1^3, x_1^2x_2, x_1x_2^2, x_2^3\}$ . Let

$$\begin{aligned} \sigma = & 8 + 17z_2 - 4z_1 + 15z_2^2 + 14z_1z_2 - 16z_1^2 + 47z_2^3 - 6z_1z_2^2 + 34z_1^2z_2 - 52z_1^3 \\ & + 51z_2^4 + 38z_1z_2^3 - 18z_1^2z_2^2 + 86z_1^3z_2 - 160z_1^4 \end{aligned}$$

be the series truncated in degree 4, corresponding to the first moments (not necessarily given by an integral).

$$H_\sigma^{B^+, B^+} = \begin{bmatrix} 8 & -4 & 17 & -16 & 14 & 15 & -52 & 34 & -6 & 47 \\ -4 & -16 & 14 & -52 & 34 & -6 & -160 & 86 & -18 & 38 \\ 17 & 14 & 15 & 34 & -6 & 47 & 86 & -18 & 38 & 51 \\ -16 & -52 & 34 & -160 & 86 & -18 & \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 \\ 14 & 34 & -6 & 86 & -18 & 38 & \sigma_2 & \sigma_3 & \sigma_4 & \sigma_5 \\ 15 & -6 & 47 & -18 & 38 & 51 & \sigma_3 & \sigma_4 & \sigma_5 & \sigma_6 \\ -52 & -160 & 86 & \sigma_1 & \sigma_2 & \sigma_3 & \sigma_7 & \sigma_8 & \sigma_9 & \sigma_{10} \\ 34 & 86 & -18 & \sigma_2 & \sigma_3 & \sigma_4 & \sigma_8 & \sigma_9 & \sigma_{10} & \sigma_{11} \\ -6 & -18 & 38 & \sigma_3 & \sigma_4 & \sigma_5 & \sigma_9 & \sigma_{10} & \sigma_{11} & \sigma_{12} \\ 47 & 38 & 51 & \sigma_4 & \sigma_5 & \sigma_6 & \sigma_{10} & \sigma_{11} & \sigma_{12} & \sigma_{13} \end{bmatrix}$$

where  $\sigma_1 = \sigma_{5,0}$ ,  $\sigma_2 = \sigma_{4,1}$ ,  $\sigma_2 = \sigma_{3,2}$ ,  $\sigma_4 = \sigma_{2,3}$ ,  $\sigma_5 = \sigma_{1,4}$ ,  $\sigma_6 = \sigma_{0,5}$ ,  $\sigma_7 = \sigma_{6,0}$ ,  $\sigma_8 = \sigma_{5,1}$ ,  $\sigma_9 = \sigma_{4,2}$ ,  $\sigma_{10} = \sigma_{3,3}$ ,  $\sigma_{11} = \sigma_{2,4}$ ,  $\sigma_{12} = \sigma_{1,5}$ ,  $\sigma_{13} = \sigma_{0,6}$ .

The first  $6 \times 6$  diagonal block  $H_\sigma^{B, B}$  is invertible. To have a flat extension  $H_\sigma^{B^+, B^+}$ , we impose the condition that the sixteen  $7 \times 7$  minors of  $H_\sigma^{B^+, B^+}$ , which contains the first 6 rows and columns, must vanish. This yields the following system of quadratic equations:

$$\left\{ \begin{array}{l}
 -814592 \sigma_1^2 - 1351680 \sigma_1 \sigma_2 - 476864 \sigma_1 \sigma_3 - 599040 \sigma_2^2 - 301440 \sigma_2 \sigma_3 - 35072 \sigma_3^2 \\
 \quad - 520892032 \sigma_1 - 396821760 \sigma_2 - 164529152 \sigma_3 + 1693440 \sigma_7 - 86394672128 = 0 \\
 -814592 \sigma_2^2 - 1351680 \sigma_2 \sigma_3 - 476864 \sigma_2 \sigma_4 - 599040 \sigma_3^2 - 301440 \sigma_3 \sigma_4 - 35072 \sigma_4^2 \\
 \quad + 335275392 \sigma_2 + 257276160 \sigma_3 + 96277632 \sigma_4 + 1693440 \sigma_9 - 34904464128 = 0 \\
 -814592 \sigma_3^2 - 1351680 \sigma_3 \sigma_4 - 476864 \sigma_3 \sigma_5 - 599040 \sigma_4^2 - 301440 \sigma_4 \sigma_5 - 35072 \sigma_5^2 \\
 \quad + 13226880 \sigma_3 + 13282560 \sigma_4 - 8227200 \sigma_5 + 1693440 \sigma_{11} + 31714560 = 0 \\
 -814592 \sigma_4^2 - 1351680 \sigma_4 \sigma_5 - 476864 \sigma_4 \sigma_6 - 599040 \sigma_5^2 - 301440 \sigma_5 \sigma_6 - 35072 \sigma_6^2 \\
 \quad + 212860736 \sigma_4 + 162698880 \sigma_5 + 51427456 \sigma_6 + 1693440 \sigma_{13} - 13356881792 = 0 \\
 -814592 \sigma_1 \sigma_2 - 675840 \sigma_1 \sigma_3 - 238432 \sigma_1 \sigma_4 - 675840 \sigma_2^2 \\
 \quad - 837472 \sigma_2 \sigma_3 - 150720 \sigma_2 \sigma_4 - 150720 \sigma_3^2 - 35072 \sigma_3 \sigma_4 + 167637696 \sigma_1 - 131807936 \sigma_2 \\
 \quad - 150272064 \sigma_3 - 82264576 \sigma_4 + 1693440 \sigma_8 + 54990043008 = 0 \\
 -814592 \sigma_2 \sigma_3 - 675840 \sigma_2 \sigma_4 - 238432 \sigma_2 \sigma_5 - 675840 \sigma_3^2 - 837472 \sigma_3 \sigma_4 - 150720 \sigma_3 \sigma_5 - 150720 \sigma_4^2 \\
 \quad - 35072 \sigma_4 \sigma_5 + 6613440 \sigma_2 + 174278976 \sigma_3 + 124524480 \sigma_4 + 48138816 \sigma_5 + 1693440 \sigma_{10} - 746438400 = 0 \\
 -814592 \sigma_3 \sigma_4 - 675840 \sigma_3 \sigma_5 - 238432 \sigma_3 \sigma_6 - 675840 \sigma_4^2 - 837472 \sigma_4 \sigma_5 - 150720 \sigma_4 \sigma_6 - 150720 \sigma_5^2 \\
 \quad - 35072 \sigma_5 \sigma_6 + 106430368 \sigma_3 + 87962880 \sigma_4 + 32355008 \sigma_5 - 4113600 \sigma_6 + 1693440 \sigma_{12} - 183201600 = 0 \\
 -814592 \sigma_2 \sigma_4 - 675840 \sigma_2 \sigma_5 - 238432 \sigma_2 \sigma_6 - 675840 \sigma_3 \sigma_4 - 599040 \sigma_3 \sigma_5 - 150720 \sigma_3 \sigma_6 - 238432 \sigma_4^2 \\
 \quad - 150720 \sigma_4 \sigma_5 - 35072 \sigma_4 \sigma_6 + 106430368 \sigma_2 + 81349440 \sigma_3 + 193351424 \sigma_4 + 128638080 \sigma_5 + 48138816 \sigma_6 \\
 \quad + 1693440 \sigma_{11} - 21721986624 = 0 \\
 -814592 \sigma_1 \sigma_3 - 675840 \sigma_1 \sigma_4 - 238432 \sigma_1 \sigma_5 - 675840 \sigma_2 \sigma_3 - 599040 \sigma_2 \sigma_4 - 150720 \sigma_2 \sigma_5 - 238432 \sigma_3^2 \\
 \quad - 150720 \sigma_3 \sigma_4 - 35072 \sigma_3 \sigma_5 + 6613440 \sigma_1 + 6641280 \sigma_2 - 264559616 \sigma_3 - 198410880 \sigma_4 - 82264576 \sigma_5 \\
 \quad + 1693440 \sigma_9 + 1312368000 = 0 \\
 -814592 \sigma_1 \sigma_4 - 675840 \sigma_1 \sigma_5 - 238432 \sigma_1 \sigma_6 - 675840 \sigma_2 \sigma_4 - 599040 \sigma_2 \sigma_5 - 150720 \sigma_2 \sigma_6 - 238432 \sigma_3 \sigma_4 \\
 \quad - 150720 \sigma_3 \sigma_5 - 35072 \sigma_3 \sigma_6 + 106430368 \sigma_1 + 81349440 \sigma_2 + 25713728 \sigma_3 - 260446016 \sigma_4 \\
 \quad - 198410880 \sigma_5 - 82264576 \sigma_6 + 1693440 \sigma_{10} + 34550702464 = 0
 \end{array} \right.$$

The set of solutions of this system is an algebraic variety of dimension 3 and degree 52. A solution is

$$\sigma_1 = -484, \sigma_2 = 226, \sigma_3 = -54, \sigma_4 = 82, \sigma_5 = -6, \sigma_6 = 167, \sigma_7 = -1456, \sigma_8 = 614, \\
 \sigma_9 = -162, \sigma_{10} = 182, \sigma_{11} = -18, \sigma_{12} = 134, \sigma_{13} = 195.$$

**3.1. Computing an orthogonal basis of  $\mathcal{A}_\sigma$ .** In this section, we describe a new method to construct a basis  $B$  of  $\mathcal{A}_\sigma$  and to detect flat extensions, from the knowledge of the moments  $\sigma_\alpha$  of  $\sigma(\mathbf{z})$ . We are going to inductively construct a family  $P$  of polynomials, orthogonal for the inner product

$$(p, q) \mapsto \langle p, q \rangle_\sigma := \langle \sigma \mid pq \rangle,$$

and a monomial set  $B$  connected to 1 such that  $\langle B \rangle = \langle P \rangle$ .

We start with  $B = \{1\}$ ,  $P = \{1\} \subset R$ . As  $\langle 1, 1 \rangle_\sigma = \langle \sigma \mid 1 \rangle \neq 0$ , the family  $P$  is orthogonal for  $\sigma$  and  $\langle B \rangle = \langle P \rangle$ .

We now describe the induction step. Assume that we have a set  $B = \{m_1, \dots, m_s\}$  and  $P = \{p_1, \dots, p_s\}$  such that

- $\langle B \rangle = \langle P \rangle$
- $\langle p_i, p_j \rangle_\sigma \neq 0$  if  $i = j$  and 0 otherwise.

To construct the next orthogonal polynomials, we consider the monomials in  $\partial B = \{m'_1, \dots, m'_l\}$  and project them on  $\langle P \rangle$ :

$$p'_i = m'_i - \sum_{j=1}^s \frac{\langle m'_i, p_j \rangle_\sigma}{\langle p_j, p_j \rangle_\sigma} p_j, i = 1, \dots, l.$$

By construction,  $\langle p'_i, p_j \rangle_\sigma = 0$  and  $\langle p_1, \dots, p_s, p'_i \rangle = \langle m_1, \dots, m_s, m'_i \rangle$ . We extend  $B$  by choosing a subset of monomials  $B' = \{m'_{i_1}, \dots, m'_{i_k}\}$  such that the matrix

$$[\langle p'_{i_j}, p'_{i_{j'}} \rangle_\sigma]_{1 \leq j, j' \leq k}$$

is invertible. The family  $P$  is then extended by adding an orthogonal family of polynomials  $\{p_{s+1}, \dots, p_{s+k}\}$  constructed from  $\{p'_{i_1}, \dots, p'_{i_k}\}$ . If all the polynomials  $p'_i$  are such that  $\langle p'_i, p'_j \rangle_\sigma = 0$ , the process stops.

This leads to the following algorithm:

**Algorithm 1.**

**Input:** the coefficients  $\sigma_\alpha$  of a series  $\sigma \in \mathbb{C}[[\mathbf{z}]]$  for  $\alpha \in A \subset \mathbb{N}^n$  connected to 1 with  $\sigma_0 \neq 0$ .

- Let  $B := \{1\}$ ;  $P = \{1\}$ ;  $r := 1$ ;  $E = \langle \mathbf{z}^\alpha \rangle_{\alpha \in A}$ ;
- While  $s > 0$  and  $B^+ \cdot B^+ \subset E$  do
  - Compute  $\partial B = \{m'_1, \dots, m'_l\}$  and  $p'_i = m'_i - \sum_{j=1}^s \frac{\langle m'_i, p_j \rangle_\sigma}{\langle p_j, p_j \rangle_\sigma} p_j$ ;
  - Compute a (maximal) subset  $B' = \{m'_{i_1}, \dots, m'_{i_k}\}$  of  $\partial B$  such that  $[\langle p'_{i_j}, p'_{i_{j'}} \rangle_\sigma]_{1 \leq j, j' \leq k}$  is invertible.
  - Compute an orthogonal family of polynomials  $\{p_{s+1}, \dots, p_{s+k}\}$  from  $\{p'_{i_1}, \dots, p'_{i_k}\}$ .
  - $B := B \cup B'$ ,  $P := P \cup \{p_{s+1}, \dots, p_{s+k}\}$ ;  $r += k$ ;
- If  $B^+ \cdot B^+ \not\subset E$  then return *failed*.

**Output:** *failed* or *success* with

- a set of monomials  $B = \{m_1, \dots, m_r\}$  connected to 1, and non-degenerate for  $\langle \cdot, \cdot \rangle_\sigma$ .
- a set of polynomials  $P = \{p_1, \dots, p_r\}$  orthogonal for  $\sigma$  and such that  $\langle B \rangle = \langle P \rangle$ .
- the relations  $\rho_i := m'_i - \sum_{j=1}^s \frac{\langle m'_i, p_j \rangle_\sigma}{\langle p_j, p_j \rangle_\sigma} p_j$  for the monomials  $m'_i$  in  $\partial B = \{m'_1, \dots, m'_l\}$ .

The above algorithm is a Gramm-Schmidt-type orthogonalization method, where, at each step, new monomials are taken in  $\partial B$  and projected onto the space spanned by the previous monomial set  $B$ . Notice that if the polynomials  $p_i$  are of degree at most  $d' < d$ , then only the moments of  $\sigma$  of degree  $\leq 2d' + 1$  are involved in this computation.

**Proposition 3.8.** *If Algorithm 1 outputs with success a set  $B = \{m_1, \dots, m_r\}$  and the relations  $\rho_i := m'_i - \sum_{j=1}^s \frac{\langle m'_i, p_j \rangle_\sigma}{\langle p_j, p_j \rangle_\sigma} p_j$ , for  $m'_i$  in  $\partial B = \{m'_1, \dots, m'_l\}$ , then  $\sigma$  coincides on  $\langle B^+ \cdot B^+ \rangle$  with the series  $\tilde{\sigma}$  such that*

- $\text{rank} H_{\tilde{\sigma}} = r$ ;
- $B$  and  $P$  are basis of  $\mathcal{A}_{\tilde{\sigma}}$  for the inner product  $\langle \cdot, \cdot \rangle_{\tilde{\sigma}}$ ;
- The ideal  $I_{\tilde{\sigma}} = \ker H_{\tilde{\sigma}}$  is generated by  $(\rho_i)_{i=1, \dots, l}$ ;
- The matrix of multiplication by  $x_k$  in the basis  $P$  of  $\mathcal{A}_{\tilde{\sigma}}$  is

$$M_k = \left( \frac{\langle \sigma \mid x_k p_i p_j \rangle}{\langle \sigma \mid p_j^2 \rangle} \right)_{1 \leq i, j \leq r}.$$

*Proof.* By construction,  $B$  is connected to 1. A basis  $B'$  of  $\langle B^+ \rangle$  is formed by the elements of  $B$  and the polynomials  $\rho_i, i = 1, \dots, l$ . Since Algorithm 1 stops with success, we have  $\forall i, j \in [1, l], \forall b \in \langle B \rangle, \langle \rho_i, b \rangle_\sigma = \langle \rho_i, \rho_j \rangle_\sigma = 0$  and  $\rho_1, \dots, \rho_l \in \ker H_{\sigma}^{B^+, B^+}$ . As  $\langle B^+ \rangle = \langle B \rangle \oplus \langle \rho_1, \dots, \rho_l \rangle$ ,  $\text{rank} H_{\sigma}^{B^+, B^+} = \text{rank} H_{\sigma}^{B, B}$  and  $H_{\sigma}^{B^+, B^+}$  is a flat extension of  $H_{\sigma}^{B, B}$ . By construction,  $P$  is an orthogonal basis of  $\langle B \rangle$  and the matrix of  $H_{\sigma}^{B, B}$  in this basis is diagonal with non-zero entries on the diagonal. Thus  $H_{\sigma}^{B, B}$  is of rank  $r$ .

By Theorem 3.2,  $\sigma$  coincides on  $\langle B^+ \cdot B^+ \rangle$  with a series  $\tilde{\sigma} \in R^*$  such that  $B$  is a basis of  $\mathcal{A}_{\tilde{\sigma}} = R/I_{\tilde{\sigma}}$  and  $I_{\tilde{\sigma}} = (\ker H_{\tilde{\sigma}}^{B^+, B^+}) = (\rho_1, \dots, \rho_l)$ .

As  $\langle B^+ \rangle = \langle B \rangle \oplus \langle \rho_1, \dots, \rho_l \rangle = \langle P \rangle \oplus \langle \rho_1, \dots, \rho_l \rangle$  and  $P$  is an orthogonal basis of  $\mathcal{A}_{\tilde{\sigma}}$ , which is orthogonal to  $\langle \rho_1, \dots, \rho_l \rangle$ , we have

$$x_k p_i = \sum_{j=1}^r \frac{\langle \sigma \mid x_k p_i p_j \rangle}{\langle \sigma \mid p_j^2 \rangle} p_j + \rho$$

with  $\rho \in \langle \rho_1, \dots, \rho_l \rangle$ . This shows that the matrix of the multiplication by  $x_k$  modulo  $I_{\tilde{\sigma}} = (\rho_1, \dots, \rho_l)$ , in the basis  $P = \{p_1, \dots, p_r\}$  is  $M_k = \left( \frac{\langle \sigma \mid x_k p_i p_j \rangle}{\langle \sigma \mid p_j^2 \rangle} \right)_{1 \leq i, j \leq r}$ .  $\square$

**Remark 3.9.** *It can be shown that the polynomials  $(\rho_i)_{i=1, \dots, l}$  are a border basis of  $I_{\tilde{\sigma}}$  for the basis  $B$  [24, 6, 28, 29].*

**Remark 3.10.** *If  $H_{\sigma}^{B, B} \succcurlyeq 0$ , then by Proposition 2.2, the common roots  $\zeta_1, \dots, \zeta_r$  of the polynomials  $\rho_1, \dots, \rho_l$  are simple and real  $\in \mathbb{R}^n$ . They are the cubature points:*

$$\tilde{\sigma} = \sum_{i=1}^r w_j \mathbf{e}_{\zeta_j},$$

with  $w_j > 0$ .

## 4. THE CUBATURE FORMULA FROM THE MOMENT MATRIX

We now describe how to recover the cubature formula, from the moment matrix  $H_\sigma^{B^+, B'^+}$ . We assume that the flat extension condition is satisfied:

$$(5) \quad \text{rank } H_\sigma^{B^+, B'^+} = H_\sigma^{B, B'} = |B| = |B'|.$$

**Theorem 4.1.** *Let  $B$  and  $B'$  be monomial subsets of  $R$  of size  $r$  connected to 1 and  $\sigma \in \langle B^+ \cdot B'^+ \rangle^*$ . Suppose that  $\text{rank } H_\sigma^{B^+, B'^+} = \text{rank } H_\sigma^{B, B'} = |B| = |B'|$ . Let  $M_i = [H_\sigma^{B, B'}]^{-1} [H_\sigma^{x_i B, B'}]$  and  $(M'_i)^t = [H_\sigma^{B, x_i B'}] [H_\sigma^{B, B'}]^{-1}$ . Then,*

- (1)  $B$  and  $B'$  are bases of  $\mathcal{A}_\sigma = R / \ker H_\sigma$ ,
- (2)  $M_i$  (resp.  $M'_i$ ) is the matrix of multiplication by  $x_i$  in the basis  $B$  (resp.  $B'$ ) of  $\mathcal{A}_\sigma$ ,

*Proof.* By the flat extension theorem 3.2, there exists  $\bar{\sigma} \in R^*$  such that  $H_{\bar{\sigma}}$  is a flat extension of  $H_\sigma^{B^+, B'^+}$  of rank  $r = |B| = |B'|$  and  $\ker H_{\bar{\sigma}} = (\ker H_\sigma^{B^+, B'^+})$ . As  $R = \langle B \rangle \oplus \ker H_{\bar{\sigma}}$  and  $\text{rank } H_{\bar{\sigma}} = r$ ,  $\mathcal{A}_{\bar{\sigma}} = R / \ker H_{\bar{\sigma}}$  is of dimension  $r$  and generated by  $B$ . Thus  $B$  is a basis of  $\mathcal{A}_{\bar{\sigma}}$ . A similar argument shows that  $B'$  is also a basis of  $\mathcal{A}_{\bar{\sigma}}$ . We denote by  $\pi : \mathcal{A}_{\bar{\sigma}} \rightarrow \langle B \rangle$  and  $\pi' : \mathcal{A}_{\bar{\sigma}} \rightarrow \langle B' \rangle$  the isomorphism associated to these basis representations.

The matrix  $[H_\sigma^{B, B'}]$  is the matrix of the Hankel operator

$$\begin{aligned} \bar{H}_{\bar{\sigma}} : \mathcal{A}_{\bar{\sigma}} &\rightarrow \mathcal{A}_{\bar{\sigma}}^* \\ a &\mapsto a \star \bar{\sigma} \end{aligned}$$

in the basis  $B$  and the dual basis of  $B'$ . Similarly,  $[H_\sigma^{x_i B, B'}]$  is the matrix of

$$\begin{aligned} \bar{H}_{x_i \star \bar{\sigma}} : \mathcal{A}_{\bar{\sigma}} &\rightarrow \mathcal{A}_{\bar{\sigma}}^* \\ a &\mapsto a \star x_i \star \bar{\sigma} \end{aligned}$$

in the same bases. As  $x_i \star \bar{\sigma} = \bar{\sigma} \circ M_i$  where  $M_i : \mathcal{A}_{\bar{\sigma}} \rightarrow \mathcal{A}_{\bar{\sigma}}$  is the multiplication by  $x_i$  in  $\mathcal{A}_{\bar{\sigma}}$ , we deduce that  $\bar{H}_{x_i \star \bar{\sigma}} = \bar{H}_{\bar{\sigma}} \circ M_i$  and  $[H_\sigma^{B, B'}]^{-1} [H_\sigma^{x_i B, B'}]$  is the matrix of multiplication by  $x_i$  in the basis  $B$  of  $\mathcal{A}_{\bar{\sigma}}$ . By exchanging the role of  $B$  and  $B'$  and by transposition ( $[H_\sigma^{B, B'}]^t = [H_\sigma^{B', B}]$ ), we obtain that  $[H_\sigma^{B, x_i B'}] [H_\sigma^{B, B'}]^{-1}$  is the transpose of the matrix of multiplication by  $x_i$  in the basis  $B'$  of  $\mathcal{A}_{\bar{\sigma}}$ .  $\square$

**Theorem 4.2.** *Let  $B$  be a monomial subset of  $R$  of size  $r$  connected to 1 and  $\sigma \in \langle B^+ \cdot B^+ \rangle^*$ . Suppose that  $\text{rank } H_\sigma^{B^+, B^+} = \text{rank } H_\sigma^{B, B} = |B| = r$  and that  $H_\sigma^{B, B} \succcurlyeq 0$ . Let  $M_i = [H_\sigma^{B, B'}]^{-1} [H_\sigma^{x_i B, B'}]$ . Then  $\sigma$  can be*

decomposed as

$$\sigma = \sum_{j=1}^r w_j \mathbf{e}_{\zeta_j}$$

with  $w_j > 0$  and  $\zeta_j \in \mathbb{R}^n$  such that  $M_i$  have  $r$  common linearly independent eigenvectors  $\mathbf{u}_j$ ,  $j = 1, \dots, r$  and

- $\zeta_{j,i} = \frac{\langle \sigma | x_i \mathbf{u}_j \rangle}{\langle \sigma | \mathbf{u}_j \rangle}$  for  $1 \leq i \leq n$ ,  $1 \leq j \leq r$ .
- $w_j = \frac{\langle \sigma | \mathbf{u}_j \rangle}{\mathbf{u}_j(\zeta_{j,1}, \dots, \zeta_{j,n})}$ .

*Proof.* By theorem 4.1, the matrix  $M_i$  is the matrix of multiplication by  $x_i$  in the basis  $B$  of  $\mathcal{A}_{\bar{\sigma}}$ . As  $H_{\sigma}^{B,B} \succcurlyeq 0$ , the flat extension theorem 3.2 implies that  $H_{\bar{\sigma}} \succcurlyeq 0$  and that

$$\bar{\sigma} = \sum_{j=1}^r w_j \mathbf{e}_{\zeta_j}$$

where  $w_j > 0$  and  $\zeta_j \in \mathbb{R}^n$  are the simple roots of the ideal  $\ker H_{\bar{\sigma}}$ . Thus the commuting operators  $M_i$  are diagonalizable in a common basis of eigenvectors  $\mathbf{u}_i$ ,  $i = 1, \dots, r$ , which are scalar multiples of the interpolation polynomials at the roots  $\zeta_1, \dots, \zeta_r$ :  $\mathbf{u}_i(\zeta_i) = \lambda_i \neq 0$  and  $\mathbf{u}_i(\zeta_j) = 0$  if  $j \neq i$  (see [15][Chap. 4] or [10]). We deduce that

$$\langle \bar{\sigma} | \mathbf{u}_j \rangle = \sum_{k=1}^r w_k \mathbf{u}_j(\zeta_k) = w_j \lambda_j \text{ and } \langle \bar{\sigma} | x_i \mathbf{u}_j \rangle = \zeta_{j,i} w_j \lambda_j,$$

so that  $\zeta_{j,i} = \frac{\langle \sigma | x_i \mathbf{u}_j \rangle}{\langle \sigma | \mathbf{u}_j \rangle}$ . As  $\mathbf{u}_i(\zeta_i) = \lambda_i$ , we have  $w_j = \frac{\langle \sigma | \mathbf{u}_j \rangle}{\mathbf{u}_j(\zeta_{j,1}, \dots, \zeta_{j,n})}$ .  $\square$

### Algorithm 2.

**Input:**  $B$  is a set of monomials connected to 1,  $\sigma \in \langle B^+ \cdot B^+ \rangle^*$  such that  $H_{\sigma}^{B^+, B^+}$  is a flat extension of  $H_{\sigma}^{B, B}$  of rank  $|B|$ .

- Compute an orthogonal basis  $\{p_1, \dots, p_r\}$  of  $B$  for  $\sigma$ ;
- Compute the matrices  $M_k = \left( \frac{\langle \sigma | x_k p_i p_j \rangle}{\langle \sigma | p_j^2 \rangle} \right)_{1 \leq i, j \leq r}$ ;
- Compute their common eigenvectors  $\mathbf{u}_1, \dots, \mathbf{u}_r$ ;

**Output:** For  $j = 1, \dots, r$ ,

- $\zeta_j = \left( \frac{\langle \sigma | x_1 \mathbf{u}_j \rangle}{\langle \sigma | \mathbf{u}_j \rangle}, \dots, \frac{\langle \sigma | x_n \mathbf{u}_j \rangle}{\langle \sigma | \mathbf{u}_j \rangle} \right)$
- $w_j = \frac{\langle \sigma | \mathbf{u}_j \rangle}{\mathbf{u}_j(\zeta_{j,1}, \dots, \zeta_{j,n})}$

**Remark 4.3.** Since the matrices  $M_k$  commute and are diagonalizable with the same basis, their common eigenvectors can be obtained by computing the eigenvectors of a generic linear combination  $l_1 M_1 + \dots + l_n M_n$ ,  $l_i \in \mathbb{R}$ .

## 5. CUBATURE FORMULA BY CONVEX OPTIMIZATION

As described in the previous section, the computation of cubature formulae reduces to a low rank Hankel matrix completion problem, using the flat extension property. In this section, we describe a new approach which relaxes this problem into a convex optimization problem.

Let  $V \subset R$  be a vector space spanned by monomials  $\mathbf{x}^\alpha$  for  $\alpha \in A \subset \mathbb{N}^n$ . Our aim is to construct a cubature formula for an integral function  $I$  exact on  $V$ . Let  $\mathbf{i} = (I[\mathbf{x}^\alpha])_{\alpha \in A}$  be the sequence of moments given by the integral  $I$ . We also denote  $\mathbf{i} \in V^*$  the associated linear form such that  $\forall v \in V \langle \mathbf{i} | v \rangle = I[v]$ .

For  $k \in \mathbb{N}$ , we denote by

$$\mathcal{H}^k(\mathbf{i}) = \{H_\sigma \mid \sigma \in R_{2k}^*, \sigma_\alpha = \mathbf{i}_\alpha \text{ for } \alpha \in A, H_\sigma \succcurlyeq 0\},$$

the set of semi-definite Hankel operators on  $R_k$  is associated to moment sequences which extend  $\mathbf{i}$ . We can easily check that  $\mathcal{H}^k(\mathbf{i})$  is a convex set. We denote by  $\mathcal{H}_r^k(\mathbf{i})$  the set of elements of  $\mathcal{H}^k(\mathbf{i})$  of rank  $\leq r$ .

A subset of  $\mathcal{H}_r^k(\mathbf{i})$  is the set of Hankel operators associated to cubature formulae of  $r$  points:

$$\mathcal{E}_r^k(\mathbf{i}) = \left\{ H_\sigma \in \mathcal{H}^k(\mathbf{i}) \mid \sigma = \sum_{i=1}^r w_i \mathbf{e}_{\zeta_i}, w_i > 0, \zeta_i \in \mathbb{R}^n \right\}.$$

We can check that  $\mathcal{E}^k(\mathbf{i}) = \cup_{r \in \mathbb{N}} \mathcal{E}_r^k(\mathbf{i})$  is also a convex set.

To impose the cubature points to be in a semialgebraic set  $\mathcal{S}$  defined by equality and inequalities  $\mathcal{S} = \{\mathbf{x} \in \mathbb{R}^n \mid g_1^0(\mathbf{x}) = 0, \dots, g_{n_1}^0(\mathbf{x}) = 0, g_1^+(\mathbf{x}) \geq 0, \dots, g_{n_2}^+(\mathbf{x}) \geq 0\}$ , one can refine the space of  $\mathcal{H}^k(\mathbf{i})$  by imposing that  $\sigma$  is positive on the quadratic module (resp. preordering) associated to the constraints  $\mathbf{g} = \{g_1^0, \dots, g_{n_1}^0; g_1^+, \dots, g_{n_2}^+\}$  [19]. For the sake of simplicity, we don't analyze this case here, which can be done in a similar way.

The Hankel operator  $H_\sigma \in \mathcal{E}_r^k(\mathbf{i})$  associated to a cubature formula of  $r$  points is an element of  $\mathcal{H}_r^k(\mathbf{i})$ . In order to find a cubature formula of minimal rank, we would like to compute a minimizer solution of the following optimization problem:

$$\min_{H \in \mathcal{H}^k(\mathbf{i})} \text{rank}(H)$$

However this problem is NP-hard [16]. We therefore relax it into the minimization of the nuclear norm of the Hankel operators, i.e. the minimization of the sum of the singular values of the Hankel matrix [38]. More precisely, for a generic matrix  $P \in \mathbb{R}^{s_t \times s_t}$ , we consider the

following minimization problem:

$$(6) \quad \min_{H \in \mathcal{H}^k(\mathbf{i})} \text{trace}(P^t H P)$$

Let  $(A, B) \in \mathbb{R}^{s_k \times s_k} \times \mathbb{R}^{s_k \times s_k} \rightarrow \langle A, B \rangle = \text{trace}(AB)$  denote the inner product induced by the trace on the space of  $s_k \times s_k$  matrices. The optimization problem (6) requires minimizing the linear form  $H \rightarrow \text{trace}(H P P^t) = \langle H, P P^t \rangle$  on the convex set  $\mathcal{H}^k(\mathbf{i})$ . As the trace of  $P^t H P$  is bounded by below by 0 when  $H \succcurlyeq 0$ , our optimization problem (6) has a non-negative minimum  $\geq 0$ .

Problem (6) is a Semi-Definite Program (SDP), which can be solved efficiently by interior point methods. See [32]. SDP is an important ingredient of relaxation techniques in polynomial optimization. See [19, 23].

Let  $\Sigma^k = \left\{ p = \sum_{i=1}^l p_i^2 \mid p_i \in R_k \right\}$  be the set of polynomials of degree  $\leq 2k$  which are sums of squares, let  $\mathbf{x}^{(k)}$  be the vector of all monomials in  $\mathbf{x}$  of degree  $\leq k$  and let  $q(\mathbf{x}) = (\mathbf{x}^{(k)})^t P P^t \mathbf{x}^{(k)} \in \Sigma^k$ . Let  $p_i(\mathbf{x})$  ( $1 \leq i \leq s_k$ ) denote the polynomial  $\langle P_i, \mathbf{x}^{(k)} \rangle$  associated to the column  $P_i$  of  $P$ . We have  $q(\mathbf{x}) = \sum_{i=1}^{s_k} p_i(\mathbf{x})^2$  and for any  $\sigma \in R_{2k}$ ,

$$\text{trace}(P^t H_\sigma P) = \langle H_\sigma \mid P P^t \rangle = \langle \sigma \mid q(\mathbf{x}) \rangle.$$

For any  $l \in \mathbb{N}$ , we denote by  $\pi_l : R_l \rightarrow R_l$  the linear map which associates to a polynomial  $p \in R_l$  its homogeneous component of degree  $l$ . We say that  $P$  is a proper matrix if  $\pi_{2k}(q(\mathbf{x})) \neq 0$  for all  $\mathbf{x} \in \mathbb{R}^n$ .

We are thus looking for cubature formula with a small number of points, which correspond to Hankel operators with small rank. The next results describes the structure of truncated Hankel operators, when the degree of truncation is high enough, compared to the rank.

**Theorem 5.1.** *Let  $\sigma \in R_{2k}^*$  and let  $H_\sigma$  be its truncated Hankel operator on  $R_k$ . If  $H_\sigma$  is of rank  $r \leq k$ , then*

$$\sigma \equiv \sum_{i=1}^{r'} \omega_i \mathbf{e}_{\zeta_i} + \sum_{i=r'+1}^r w_i \mathbf{e}_{\zeta_i} \circ \pi_{2k}$$

with  $\omega_i \in \mathbb{C} \setminus \{0\}$ ,  $\zeta_i \in \mathbb{C}^n$  distinct. If moreover  $H_\sigma \succcurlyeq 0$ , then  $\omega_i > 0$  and  $\zeta_i \in \mathbb{R}^n$  for  $i = 1, \dots, r$ .

*Proof.* The substitution  $\tau_0 : S_{[2k]} \rightarrow R_{2k}$  which replaces  $x_0$  by 1 is an isomorphism of  $\mathbb{K}$ -vector spaces. Let  $\tau_0^* : R_{2k}^* \rightarrow S_{[2k]}^*$  be the pull-back map on the dual ( $\tau_0^*(\sigma) = \sigma \circ \tau_0$ ). Let  $\bar{\sigma} = \sigma \circ \tau_0 = \tau_0^*(\sigma) \in S_{[2k]}^*$  be the linear form induced by  $\sigma$  on  $S_{[2k]}$  and let  $H_{\bar{\sigma}} : S_{[k]} \rightarrow S_{[k]}^*$  be the corresponding truncated operator on  $S_{[k]}$ . The kernel  $\bar{K}$  of  $H_{\bar{\sigma}}$  is the

vector space spanned by the homogenization in  $x_0$  of the elements of the kernel  $K$  of  $H_\sigma$ .

Let  $\succ$  be the lexicographic ordering such that  $x_0 \succ \dots \succ x_n$ . By [14][Theorem 15.20, p. 351], after a generic change of coordinates, the initial  $J$  of the homogeneous ideal  $(\bar{K}) \subset S$  is Borel fixed. That is, if  $x_i \mathbf{x}^\alpha \in J$ , then  $x_j \mathbf{x}^\alpha \in J$  for  $j > i$ . Let  $\bar{B}$  be the set of monomials of degree  $k$ , which are not in  $J$ . Note  $J$  is Borel fixed and different from  $S_{[2k]}$ ,  $x_0^k \in \bar{B}$ . Similarly we check that if  $x_0^{\alpha_0} \dots x_n^{\alpha_n} \in J$  with  $\alpha_1 = \dots = \alpha_{l-1} = 0$ , then  $x_0^{\alpha_0+1} x_l^{\alpha_l-1} \dots x_n^{\alpha_n} \in J$ . This shows that  $B = \tau_0(\bar{B})$  is connected to 1.

As  $\langle \bar{B} \rangle \oplus \langle J \rangle = \langle \bar{B} \rangle \oplus \bar{K} = S_{[k]}$  where  $\bar{K} = \ker H_{\bar{\sigma}}$ , we have  $|B| = r$ . As  $B$  is connected to 1,  $\deg(B) < r \leq k$  and  $B^+ \subset R_k$ .

By the substitution  $x_0 = 1$ , we have  $R_k = \langle B \rangle \oplus K$  with  $K = \ker H_\sigma$ . Therefore,  $H_\sigma$  is a flat extension of  $H_\sigma^{B,B}$ . Via the flat extension theorem 3.2, there exists  $\lambda_i \in \mathbb{K} \setminus \{0\}$ ,  $\bar{\zeta}_i = (\zeta_{i,0}, \zeta_{i,1}, \dots, \zeta_{i,n}) \in \mathbb{K}^{n+1}$  distinct for  $i = 1, \dots, r$  such that

$$(7) \quad \bar{\sigma} \equiv \sum_{i=1}^r \lambda_i \mathbf{e}_{\bar{\zeta}_i} \text{ on } S_{[2k]}.$$

Notice that for any  $\lambda \neq 0$ ,  $\mathbf{e}_{\bar{\zeta}_i} = \lambda^{-k} \mathbf{e}_{\lambda \bar{\zeta}_i}$  on  $S_{[2k]}$ .

By an inverse change of coordinates, the points  $\bar{\zeta}_i$  of (7) are transformed into some points  $\zeta_i = (\zeta_{i,0}, \zeta_{i,1}, \dots, \zeta_{i,n}) \in \mathbb{K}^{n+1}$  such that  $\zeta_{i,0} \neq 0$  (say for  $i = 1, \dots, r'$ ) and the remaining  $r - r'$  points with  $\zeta_{i,0} = 0$ . The image by  $\tau_0^*$  of  $\mathbf{e}_{\bar{\zeta}_i} \in S_{[2k]}^*$  with  $\zeta_{i,0} \neq 0$  is

$$\tau_0^*(\mathbf{e}_{\bar{\zeta}_i}) \equiv \zeta_{i,0}^{2k} \mathbf{e}_{\zeta_i}$$

where  $\zeta_i = \frac{1}{\zeta_{i,0}}(\zeta_{i,1}, \dots, \zeta_{i,n})$ . The image by  $\tau_0^*$  of  $\mathbf{e}_{\bar{\zeta}_i} \in S_{[2k]}^*$  with  $\zeta_{i,0} = 0$  is a linear form  $\in R_{2k}^*$ , which vanishes on all the monomials  $\mathbf{x}^\alpha$  with  $|\alpha| < 2k$ , since their homogenization in degree  $2k$  is  $x_0^{2k-|\alpha|} \mathbf{x}^\alpha$  and their evaluation at  $\bar{\zeta}_i = (0, \zeta_{i,1}, \dots, \zeta_{i,n})$  gives 0. The value of  $\tau_0^*(\mathbf{e}_{\bar{\zeta}_i})$  at  $\mathbf{x}^\alpha$  with  $|\alpha| = 2k$   $\zeta_{i,1}^{\alpha_1} \dots \zeta_{i,n}^{\alpha_n} = \mathbf{e}_{\zeta_i}(\mathbf{x}^\alpha)$  where  $\zeta_i = (\zeta_{i,1}, \dots, \zeta_{i,n})$ . We deduce that

$$\tau_0^*(\mathbf{e}_{\bar{\zeta}_i}) \equiv \mathbf{e}_{\zeta_i} \circ \pi_{2k}$$

The flat extension theorem implies that if  $H_\sigma \succcurlyeq 0$  then  $\lambda_i > 0$  and  $\bar{\zeta}_i \in \mathbb{R}^{n+1}$  in the decomposition (7). By dehomogenization, we have  $\omega_i = \lambda_i \zeta_{i,0}^{2k} > 0$ ,  $\zeta_i = \frac{1}{\zeta_{i,0}}(\zeta_{i,1}, \dots, \zeta_{i,n}) \in \mathbb{R}^n$  for  $i = 1, \dots, r'$  and  $\zeta_i = (\zeta_{i,1}, \dots, \zeta_{i,n}) \in \mathbb{R}^n$  for  $i = r' + 1, \dots, n$ .  $\square$

We exploit this structure theorem to show that if the truncation order is sufficiently high, a minimizer of (6) corresponds to a cubature formula.

**Theorem 5.2.** *Let  $P$  be a proper operator and  $k \geq \frac{\deg(V)+1}{2}$ . Assume that there exists  $\sigma^* \in R_{2k}^*$  be such that  $H_{\sigma^*}$  is a minimizer of (6) of rank  $r$  with  $r \leq k$ . Then  $H_{\sigma^*} \in \mathcal{E}_r^k(\mathbf{i})$  i.e. there exists  $\omega_i > 0$  and  $\zeta_i \in \mathbb{R}^n$  such that*

$$\sigma^* \equiv \sum_{i=1}^r \omega_i \mathbf{e}_{\zeta_i}.$$

*Proof.* By Theorem 5.1,

$$\sigma^* \equiv \sum_{i=1}^{r'} \omega_i \mathbf{e}_{\zeta_i} + \sum_{i=r'+1}^r w_i \mathbf{e}_{\zeta_i} \circ \pi_{2k},$$

with  $\omega_i > 0$  and  $\zeta_i \in \mathbb{R}^n$  for  $i = 1, \dots, r$ .

Let us suppose that  $r \neq r'$ . As  $k \geq \frac{\deg(V)+1}{2}$ , the elements of  $V$  are of degree  $< 2k$ , therefore  $\sigma^*$  and  $\sigma' \equiv \sum_{i=1}^{r'} \omega_i \mathbf{e}_{\zeta_i}$  coincide on  $V$  and  $H_{\sigma'} \in \mathcal{H}^k(\mathbf{i})$ . We have the decomposition

$$\text{trace}(PH_{\sigma^*}P) = \langle \sigma^* | q \rangle = \langle \sigma' | q \rangle + \sum_{i=1}^{r'} \omega_i \pi_{2k}(q)(\zeta_i).$$

The homogeneous component of highest degree  $\pi_{2k}(q)$  of  $q(\mathbf{x}) = \sum_{i=1}^{s_k} p_i(\mathbf{x})^2$  is the sum of the squares of the degree- $k$  components of the  $p_i$ :

$$\pi_{2k}(q) = \sum_{i=1}^{s_k} (\pi_k(p_i))^2,$$

so that  $\sum_{i=1}^{r'} \omega_i \pi_{2k}(q)(\zeta_i) \geq 0$ . As  $\text{trace}(PH_{\sigma^*}P)$  is minimal, we must have  $\sum_{i=r'+1}^r \omega_i \pi_{2k}(q)(\zeta_i) = 0$ , which implies that  $\pi_{2k}(q)(\zeta_i) = 0$  for  $i = r'+1, \dots, r$ . However, this is impossible, since  $P$  is proper. We thus deduce that  $r' = r$ , which concludes the proof of the theorem.  $\square$

This theorem shows that an optimal solution of the minimization problem (6) of small rank ( $r \leq k$ ) yields a cubature formula, which is exact on  $V$ . Among such minimizers, we have those of minimal rank as shown in the next proposition.

**Proposition 5.3.** *Let  $k \geq \frac{\deg(V)+1}{2}$  and  $H$  be an element of  $\mathcal{H}^k(\mathbf{i})$  with minimal rank  $r$ . If  $k \geq r$ , then  $H \in \mathcal{E}_r^k(\mathbf{i})$  and it is either an extremal point of  $\mathcal{H}^k(\mathbf{i})$  or on a face of  $\mathcal{H}^k(\mathbf{i})$ , which is included in  $\mathcal{E}_r^k(\mathbf{i})$ .*

*Proof.* Let  $H_\sigma \in \mathcal{E}_r^k(\mathbf{i})$  be of minimal rank  $r$ .

By Theorem (5.1),  $\sigma \equiv \sum_{i=1}^{r'} \omega_i \mathbf{e}_{\zeta_i} + \sum_{i=r'+1}^r w_i \mathbf{e}_{\zeta_i} \circ \pi_{2k}$  with  $\omega_i > 0$  and  $\zeta_i \in \mathbb{R}^n$  for  $i = 1, \dots, r$ . The elements of  $V$  are of degree  $< 2k$ ,

therefore  $\sigma$  and  $\sigma' \equiv \sum_{i=1}^{r'} \omega_i \mathbf{e}_{\zeta_i}$  coincide on  $V$ . We deduce that  $H_{\sigma'} \in \mathcal{H}^k(\mathbf{i})$ .

As  $\text{rank} H_{\sigma'} = r' \leq r$  and  $H_\sigma \in \mathcal{H}^k(\mathbf{i})$  is of minimal rank  $r$ ,  $r = r'$  and  $H_\sigma \in \mathcal{E}_r^k(\mathbf{i})$ .

Let us assume that  $H_\sigma$  is not an extremal point of  $\mathcal{H}^k(\mathbf{i})$ . Then it is in the relative interior of a face  $F$  of  $\mathcal{H}^k(\mathbf{i})$ . For any  $H_{\sigma_1}$  in a sufficiently small ball of  $F$  around  $H_\sigma$ , there exist  $t \in ]0, 1[$  and  $H_{\sigma_2} \in F$  such that

$$H_\sigma = tH_{\sigma_1} + (1-t)H_{\sigma_2}.$$

The kernel of  $H_\sigma$  is the set of polynomials  $p \in R_k$  such that

$$0 = H_\sigma(p, p) = tH_{\sigma_1}(p, p) + (1-t)H_{\sigma_2}(p, p).$$

As  $H_{\sigma_i} \succcurlyeq 0$ , we have  $H_{\sigma_i}(p, p) = 0$  for  $i = 1, 2$ . This implies that  $\ker H_\sigma \subset \ker H_{\sigma_i}$ , for  $i = 1, 2$ . From the inclusion  $\ker H_{\sigma_1} \cap \ker H_{\sigma_2} \subset \ker H_\sigma$ , we deduce that

$$\ker H_\sigma = \ker H_{\sigma_1} \cap \ker H_{\sigma_2}.$$

As  $H_\sigma$  is of minimal rank  $r$ , we have  $\dim \ker H_\sigma \geq \dim \ker H_{\sigma_i}$ . This implies that  $\ker H_\sigma = \ker H_{\sigma_1} = \ker H_{\sigma_2}$ .

As  $r \leq k$ ,  $\mathcal{A}_\sigma$  has a monomial basis  $B$  (connected to 1) in degree  $< k$  and  $R_k = \langle B \rangle \oplus \ker H_\sigma$ . Consequently,  $H_\sigma$  (resp.  $H_{\sigma_i}$ ) is a flat extension of  $H_\sigma^{B,B}$  (resp.  $H_{\sigma_i}^{B,B}$ ) and we have the decomposition

$$\sigma_1 \equiv \sum_{i=1}^r \omega_{i,1} \mathbf{e}_{\zeta_i}, \quad \sigma_2 \equiv \sum_{i=1}^r \omega_{i,2} \mathbf{e}_{\zeta_i},$$

with  $\omega_{i,j} > 0$ ,  $i = 1, \dots, r$ ,  $j = 1, 2$ . We deduce that  $H_{\sigma_1} \in \mathcal{E}_r^k(\mathbf{i})$  and all the elements of the line  $(H_\sigma, H_{\sigma_1})$  which are in  $F$  are also in  $\mathcal{E}_r^k(\mathbf{i})$ . Since  $F$  is convex, we deduce that  $F \subset \mathcal{E}_r^k(\mathbf{i})$ .  $\square$

**Remark 5.4.** *A cubature formula is interpolatory when the weights are uniquely determined from the points. From the previous Theorem and Proposition, we see that if a cubature formula is of minimal rank and interpolatory, then it is an extremal point of  $\mathcal{H}^k(\mathbf{i})$ .*

According to the previous proposition, by minimizing the nuclear norm of a random matrix, we expect to find an element of minimal rank in one of the faces of  $\mathcal{H}^k(\mathbf{i})$ , provided  $k$  is big enough. This yields the following simple algorithm, which solves the SDP problem, and checks the flat extension property using Algorithm 1. Furthermore it computes the decomposition using Algorithm 2 or increases the degree if there is no flat extension:

**Algorithm 3.**

- $k := \left\lceil \frac{\deg(V)}{2} \right\rceil$ ; *notflat* := true;  $P :=$  random  $s_k \times s_k$  matrix;
- While (*notflat*) do
  - Let  $\sigma$  be a solution of the SDP problem:  $\min_{H \in \mathcal{H}^k(i)} \text{trace}(P^t H P)$ ;
  - If  $H_\sigma^k$  is not a flat extension, then  $k := k + 1$ ; else *notflat* := false;
- Compute the decomposition of  $\sigma = \sum_{i=1}^r w_i \mathbf{e}_{\zeta_i}$ ,  $w_i > 0$ ,  $\zeta_i \in \mathbb{R}^n$ .

## 6. EXAMPLES

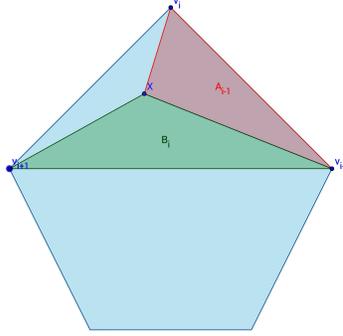
We now illustrate our cubature method on a few explicit examples.

**Example 6.1** (Cubature on a square). *Our first application is a well known case, namely, the square domain  $\Omega = [-1, 1] \times [-1, 1]$ . We solve the SDP problem 6, with a random matrix  $P$  and with no constraint on the support of the points. In following table, we give the degree of the cubature formal (i.e. the degree of the polynomials for which the cubature formula is exact), the number  $N$  of cubature points, the coordinates of the cubature points and the associated weights.*

Degree	N	Points	Weights
3	4	$\pm(0.46503, 0.464462)$ $\pm(0.855875, -0.855943)$	1.545 0.454996
5	7	$\pm(0.673625, 0.692362)$ $\pm(0.40546, -0.878538)$ $\pm(-0.901706, 0.340618)$ (0, 0)	0.595115 0.43343 0.3993 1.14305
7	12	$\pm(0.757951, 0.778815)$ $\pm(0.902107, 0.0795967)$ $\pm(0.04182, 0.9432)$ $\pm(0.36885, 0.19394)$ $\pm(0.875533, -0.873448)$ $\pm(0.589325, -0.54688)$	0.304141 0.203806 0.194607 0.756312 0.0363 0.50478

The cubature points are symmetric with respect to the origin (0, 0). The computed cubature formula involves the minimal number of points, which all lie in the domain  $\Omega$ .

**Example 6.2** (Barycentric Wachspress coordinates on a pentagon). *Here we consider the pentagon  $C$  of vertices  $v_1 = (0, 1)$ ,  $v_2 = (1, 0)$ ,  $v_3 = (-1, 0)$ ,  $v_4 = (-0.5, -1)$ ,  $v_5 = (0.5, -1)$ .*



To this pentagon, we associate (Wachpress) barycentric coordinates [44], which are defined as follows. The weighted function associated to the vertex  $v_i$  is defined as:

$$w_i(\mathbf{x}) = \frac{A_{i-1} - B_i + A_i}{A_{i-1} \cdot A_i}$$

where  $A_i$  is the signed area of the triangle  $(\mathbf{x}, v_{i-1}, v_i)$  and  $B_i$  is the area of  $(\mathbf{x}, v_{i+1}, v_{i-1})$ . The coordinate function associated to  $v_i$  is:

$$\lambda_i(\mathbf{x}) = \frac{w_i(\mathbf{x})}{\sum_{i=1}^5 w_i(\mathbf{x})}.$$

These coordinate functions satisfy:

- $\lambda_i(\mathbf{x}) \geq 0$  for  $\mathbf{x} \in C$
- $\sum_{i=1}^5 \lambda_i = 1$ ,
- $\sum_{i=1}^5 v_i \cdot \lambda_i(\mathbf{x}) = \mathbf{x}$

For all polynomials  $p \in R = \mathbb{R}[u_0, u_1, u_2, u_3, u_4]$ , we consider

$$I[p] = \int_{\mathbf{x} \in \Omega} p \circ \lambda(\mathbf{x}) d\mathbf{x}$$

We look for a cubature formula  $\sigma \in R^*$  of the form:

$$(8) \quad \langle \sigma | p \rangle = \sum_{j=1}^r w_j p(\zeta_j)$$

with  $w_i > 0$ ,  $\zeta_i \in \mathbb{R}^5$ , such that  $I[p] = \langle \sigma | p \rangle$  for all polynomial  $p$  of degree  $\leq 2$ .

The moment matrix  $H_\sigma^{B,B}$  associated to  $B = \{1, u_0, u_1, u_2, u_3, u_4\}$  involves moments of degree  $\leq 2$ :

$$H_\sigma^{B,B} = \begin{pmatrix} 2.5000 & 0.5167 & 0.5666 & 0.5167 & 0.4500 & 0.4500 \\ 0.5167 & 0.2108 & 0.1165 & 0.0461 & 0.0440 & 0.0992 \\ 0.5666 & 0.1165 & 0.2427 & 0.1165 & 0.0454 & 0.0454 \\ 0.5167 & 0.0461 & 0.1165 & 0.2108 & 0.0992 & 0.0440 \\ 0.4500 & 0.0440 & 0.0454 & 0.0992 & 0.1701 & 0.0911 \\ 0.4500 & 0.0992 & 0.0454 & 0.0440 & 0.0911 & 0.1701 \end{pmatrix}$$

Its rank is  $\text{rank}(H_\sigma^{B,B}) = 5$ .

We compute  $H_\sigma^{B^+,B^+}$ . In this matrix there are 105 unknown parameters. We solve the following SDP problem

$$(9) \quad \begin{aligned} \min \quad & \text{trace}(H_\sigma^{B^+,B^+}) \\ \text{s.t.} \quad & H_\sigma^{B^+,B^+} \succeq 0 \end{aligned}$$

which yields a solution with minimal rank 5. Since the rank of the solution matrix is the rank of  $H_\sigma^{B,B}$ , we do have a flat extension. Applying Algorithm 1, we find the orthogonal polynomials  $\rho_i$ , the matrices of the operators of multiplication by a variable, their common eigenvectors, which gives the following cubature points and weights:

Points	Weights
(0.249888, -0.20028, 0.249993, 0.350146, 0.350193)	0.485759
(0.376647, 0.277438, -0.186609, 0.20327, 0.329016)	0.498813
(0.348358, 0.379898, 0.244967, -0.174627, 0.201363)	0.509684
(-0.18472, 0.277593, 0.376188, 0.329316, 0.201622)	0.490663
(0.242468, 0.379314, 0.348244, 0.200593, -0.170579)	0.51508

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