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► **To cite this version:**

Gilles Puy, Mike E. Davies, Rémi Gribonval. Linear embeddings of low-dimensional subsets of a Hilbert space to  $R^m$ . SPARS15 - Signal Processing with Adaptive Sparse Structured Representations, Jul 2015, Cambridge, United Kingdom. hal-01157992

**HAL Id: hal-01157992**

**<https://inria.hal.science/hal-01157992>**

Submitted on 29 May 2015

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# Linear embeddings of low-dimensional subsets of a Hilbert space to $\mathbb{R}^m$

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**Abstract**—We consider the problem of embedding a low-dimensional set,  $\mathcal{M}$ , from an infinite-dimensional Hilbert space to a finite-dimensional space. Defining appropriate random linear projections, we construct a linear map which has the restricted isometry property on the secant set of  $\mathcal{M}$ , with high probability for a number of projections essentially proportional to the intrinsic dimension of  $\mathcal{M}$ .

## I. INTRODUCTION

In compressed sensing (CS), the restricted isometry property (RIP) has been widely used to study the performance of explicit decoders [1], [3] and, more generally, to study the existence of instance optimal decoders for arbitrary models [4]. A matrix  $A \in \mathbb{R}^{m \times n}$  satisfies the restricted isometry property on a general set  $S \subset \mathbb{R}^n$ , if there exists a constant  $0 < \delta < 1$ , such that for all  $\mathbf{x} \in S$ ,

$$(1 - \delta) \|\mathbf{x}\|_2^2 \leq \|A\mathbf{x}\|_2^2 \leq (1 + \delta) \|\mathbf{x}\|_2^2. \quad (1)$$

For example, if  $A$  satisfies the RIP on the set  $S$  of  $2s$ -sparse with constant  $\delta < 1/\sqrt{2}$  then every  $s$ -sparse vector  $\mathbf{x}$  is accurately and stably recovered from its noisy measurements  $\mathbf{z} = A\mathbf{x} + \mathbf{n}$  by solving the Basis Pursuit problem [2]. For more general low-dimensional model  $\Sigma \subset \mathbb{R}^n$ , one needs to show that the matrix  $A$  satisfies the RIP on the secant set  $S(\Sigma)$  to ensure stable recovery [3]–[6].

In this finite dimensional setting, random matrices with independent entries drawn from the centered Gaussian distribution with variance  $m^{-1}$  are examples of matrices that satisfies the RIP with high probability for many different low-dimensional models  $\Sigma$  in  $\mathbb{R}^n$ : sparse signals [1], low-rank matrices [5], or compact Riemannian manifold [6]. In these scenarios, the RIP holds for a number of measurements  $m$  essentially proportional to the dimension of  $\Sigma$ .

In this work, we are interested in constructing a finite-dimensional linear map which has the RIP on low-dimensional signal models  $\mathcal{M}$  in an infinite-dimensional real Hilbert space  $\mathcal{H}$ . These developments are important, *e.g.*, in CS to extend the theory to an analog setting [7] and explore connections with the sampling of signals with finite rate of innovation [8], and also in machine learning to develop efficient methods to compute information-preserving sketches of probability distributions [9].

## II. A LINEAR EMBEDDING OF $\mathcal{M}$ IN $\mathbb{R}^m$

In this section,  $\mathcal{H}$  denotes a real Hilbert space with scalar product  $\langle \cdot, \cdot \rangle$  and associated norm denoted by  $\|\cdot\|$ .

### A. Signal model with finite box-counting dimension

We consider here a signal model  $\mathcal{M} \subset \mathcal{H}$  and our goal is to construct a linear map  $L: \mathcal{H} \rightarrow \mathbb{R}^m$  that satisfies<sup>1</sup>

$$(1 - \delta) \|\mathbf{x}_1 - \mathbf{x}_2\| \leq \|L(\mathbf{x}_1 - \mathbf{x}_2)\|_2 \leq (1 + \delta) \|\mathbf{x}_1 - \mathbf{x}_2\| \quad (2)$$

This work was partly funded by the European Research Council, PLEASE project (ERC-StG-2011-277906).

<sup>1</sup>Remark that the “non-squared” RIP, as in (2), implies the “squared” RIP, as in (1), with a RIP constant multiplied by 3.

for all pairs of vectors  $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{M}$ . This is equivalent to show that  $\sup_{\mathbf{y} \in S(\mathcal{M})} \left| \|L(\mathbf{y})\|_2 - 1 \right| \leq \delta$  where  $S(\mathcal{M})$  is the normalized secant set:

$$S(\mathcal{M}) = \left\{ \mathbf{y} = \frac{\mathbf{x}_1 - \mathbf{x}_2}{\|\mathbf{x}_1 - \mathbf{x}_2\|} \mid \mathbf{x}_1, \mathbf{x}_2 \in \mathcal{M}, \mathbf{x}_1 \neq \mathbf{x}_2 \right\}. \quad (3)$$

We then make the following assumption on the “dimension” of  $S(\mathcal{M})$ .

**Assumption II.1.** *The normalized secant set  $S(\mathcal{M})$  of  $\mathcal{M}$  has finite upper-box counting dimension strictly bounded by  $s > 0$ .*

Assumption II.1 means that there exists  $\epsilon_0 > 0$  such that, for all  $\epsilon \in (0, \epsilon_0)$ ,  $S(\mathcal{M})$  can be covered by at most  $\epsilon^{-s}$  balls of radius  $\epsilon$ . We denote by  $T(\epsilon)$  the set of centers of these balls. We remark that if  $\mathcal{M}$  is the set of  $k$ -sparse signals in an orthonormal basis in finite dimension then  $s = 2k$ . However, our result is not restricted to this example and extends, *e.g.*, to low-dimensional manifolds in  $L^2(\mathbb{R})$ .

Let us highlight that other definitions of dimension exist. However, in an infinite-dimensional Hilbert space, one should be careful with the definition used. Indeed, there are examples of sets with finite dimension (according to some definition) for which no stable embedding exists (see, *e.g.*, [11] for further details).

### B. Construction of a linear map which has the RIP

Fix a resolution  $\epsilon \in (0, \epsilon_0)$ . Let  $\mathcal{V}_\epsilon \subset \mathcal{H}$  be the finite-dimensional linear subspace spanned by  $T(\epsilon)$  and  $(\mathbf{a}_1, \dots, \mathbf{a}_d)$  be an orthonormal basis for  $\mathcal{V}_\epsilon$ . Draw  $m$  independent centered isotropic random gaussian vectors  $\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_m \in \mathbb{R}^d$  and set

$$\mathbf{l}_i = \frac{1}{\sqrt{m}} \sum_{k=1}^m \alpha_{ik} \mathbf{a}_k \in \mathcal{V}_\epsilon, \quad \text{for all } i \in \{1, \dots, m\}, \quad (4)$$

where  $\alpha_{ik}$  is the  $k^{\text{th}}$  entry of  $\boldsymbol{\alpha}_i$ . We define our linear map  $L_\epsilon: \mathcal{H} \rightarrow \mathbb{R}^m$  as  $\mathbf{x} \mapsto L_\epsilon(\mathbf{x}) = (\langle \mathbf{l}_1, \mathbf{x} \rangle, \dots, \langle \mathbf{l}_m, \mathbf{x} \rangle)^\top$ .

**Theorem II.2.** *There exist absolute constants  $D_1, D_2, D_3 > 0$  with  $D_1 < 1$  such that if Assumption II.1 holds, then for any  $0 < \delta < \min(D_1, \epsilon_0)$  and  $\rho \in (0, 1)$ , if  $\epsilon \leq \delta/2$  and*

$$m \geq D_2 \delta^{-2} \max \{s \log(D_3/\delta), \log(6/\rho)\}, \quad (5)$$

*then  $\sup_{\mathbf{y} \in S(\mathcal{M})} \left| \|L_\epsilon(\mathbf{y})\|_2 - 1 \right| \leq \delta$  with probability at least  $1 - \rho$ .*

The proof is based on a chaining argument such as used, *e.g.*, in [6], [10] in a finite-dimensional ambient space. Also inspired by a monograph of Robinson [11], we adapt their technique and the construction of the linear map to handle signal models in an infinite-dimensional Hilbert space [12]. Finally, let us acknowledge the related work of Dirksen [13] where the ambient space is also infinite-dimensional. The linear map  $L$  we propose does not appear in [13] but a similar result to Theorem II.2 could be derived using his generic result. However, our technique can handle structured measurement processes and infinite-dimensional Banach spaces.

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