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Linear embeddings of low-dimensional subsets of a Hilbert space to $\mathbb{R}^m$

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Abstract—We consider the problem of embedding a low-dimensional set, $\mathcal{M}$, from an infinite-dimensional Hilbert space to a finite-dimensional space. Defining appropriate random linear projections, we construct a linear map which has the restricted isometry property on the secant set of $\mathcal{M}$, with high probability for a number of projections essentially proportional to the intrinsic dimension of $\mathcal{M}$.

I. INTRODUCTION

In compressed sensing (CS), the restricted isometry property (RIP) has been widely used to study the performance of explicit decoders [1], [3] and, more generally, to study the existence of instance optimal measurements [11]. Defining appropriate random linear projections, we construct a linear map which has the restricted isometry property on the secant set $\mathcal{M}_\Sigma$ model the Basis Pursuit problem [2]. For more general low-dimensional models, the RIP holds for a number of vectors $\mathcal{M}$ that satisfies the RIP with high probability for many different low-dimensional models [12]. In these scenarios, the RIP holds for a number of vectors $\mathcal{M}$ that satisfies the RIP with high probability for many different low-dimensional models [12]. In these scenarios, the RIP holds for a number of vectors $\mathcal{M}$ that satisfies the RIP with high probability for many different low-dimensional models [12].

In this work, we are interested in constructing a finite-dimensional linear embedding of $\mathcal{M}$ in an infinite-dimensional Hilbert space $\mathcal{H}$. These developments are important, e.g., in CS to extend the theory to an analog setting [7] and explore connections with the sampling of signals with finite rate of innovation [8], and also in machine learning to develop efficient methods to compute information-preserving sketches of probability distributions [9].

II. A LINEAR EMBEDDING OF $\mathcal{M}$ IN $\mathbb{R}^m$

In this section, $\mathcal{H}$ denotes a real Hilbert space with scalar product $\langle \cdot, \cdot \rangle$ and associated norm denoted by $\| \cdot \|$. We consider a signal model $\mathcal{M} \subset \mathcal{H}$ and our goal is to construct a linear map $L : \mathcal{H} \rightarrow \mathbb{R}^m$ that satisfies

\[
(1 - \delta) \| x_1 - x_2 \| \leq \| L(x_1) - L(x_2) \| \leq (1 + \delta) \| x_1 - x_2 \|,
\]

for all pairs of vectors $x_1, x_2 \in \mathcal{M}$. This is equivalent to show that the RIP on the secant set $S(\mathcal{M})$ of $\mathcal{M}$.

We then make the following assumptions on the “dimension” of $S(\mathcal{M})$.

Assumption II.1. The normalized secant set $S(\mathcal{M})$ of $\mathcal{M}$ has finite upper-box counting dimension strictly bounded by $s > 0$.

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Remark that the “non-squared” RIP, as in [3], implies the “squared” RIP, as in [4], with a RIP constant multiplied by $\sqrt{2}$.
REFERENCES


[12] Puy et al., “Linear embeddings of low-dimensional subsets of a hilbert space to $\mathbb{R}^m$,” https://hal.inria.fr/hal-01116153, 2015.