

# COMPUTING INDIVIDUAL DISCRETE LOGARITHMS FASTER IN $\text{GF}(p^n)$

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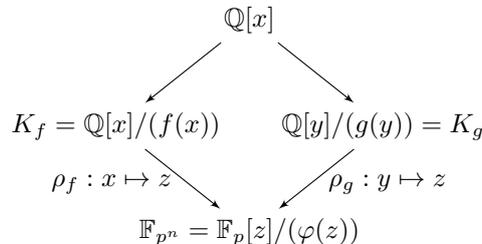
**ABSTRACT.** The Number Field Sieve (NFS) algorithm is the best known method to compute discrete logarithms (DL) in large characteristic finite fields  $\mathbb{F}_{p^n}$ , with  $p$  large and  $n \geq 1$  small. This algorithm comprises four steps: polynomial selection, relation collection, linear algebra and finally, individual logarithm computation. The first step outputs two numbers fields equipped with a map to  $\mathbb{F}_{p^n}$ . After the relation collection and linear algebra phases, the (virtual) logarithm of a subset of elements in each number field is known. The fourth step computes a preimage in one number field of the target element in  $\mathbb{F}_{p^n}$ . If one can write the target preimage as a product of elements of known (virtual) logarithm, then one can deduce the discrete logarithm of the target.

The traditional approach for the individual logarithm step can be extremely slow, and it is too slow especially for  $n$  greater than 3. Its asymptotic complexity is  $L_Q[1/3, c]$  with  $c \geq 1.44$ . We present a new preimage computation that provides a dramatic improvement for individual logarithm computations for small  $n$ , both in practice and in asymptotic running-time: we have  $L_Q[1/3, c]$  with  $c = 1.14$  for  $n = 2, 4$ ,  $c = 1.26$  for  $n = 3, 6$  and  $c = 1.34$  for  $n = 5$ . Our method generalizes to any  $n$ ; in particular  $c < 1.44$  for the two state-of-the-art variants of NFS for extension fields.

**Keywords:** Discrete logarithm, finite field, number field sieve, individual logarithm.

## 1. INTRODUCTION

**1.1. The Number Field Sieve Algorithm for Discrete Logarithms in Finite Fields.** We recall that the NFS algorithm is made of four steps: polynomial selection, relation collection, linear algebra and finally, individual logarithm computation. This last step is mandatory to break any given instance of a discrete logarithm problem. The polynomial selection outputs two numbers fields, each equipped with a map  $\rho_f, \rho_g$  to  $\mathbb{F}_{p^n}$ , as shown in the following diagram. Moreover, the monic polynomial defining the finite field is  $\varphi = \text{gcd}(f, g) \bmod p$ , of degree  $n$ .




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In the remaining of this paper, we will only use  $\rho = \rho_f$  and  $K = K_f$ . After the relation collection and linear algebra phases, the (virtual) logarithm of a subset of elements in each number field is known. The individual logarithm step computes a preimage in one number field of the target element in  $\mathbb{F}_{p^n}$ . If one can write the target preimage as a product of elements of known (virtual) logarithm, then one can deduce the individual logarithm of the target. The key point of individual logarithm computation is finding a smooth enough decomposition of the target preimage.

The asymptotic running time of NFS algorithm steps are estimated with the  $L$ -function:

$$L_Q[\alpha, c] = \exp\left((c + o(1))(\log Q)^\alpha (\log \log Q)^{1-\alpha}\right) \quad \text{with } \alpha \in [0, 1] \text{ and } c > 0.$$

The  $\alpha$  parameter measures the gap between polynomial time ( $\alpha = 0$ ) and exponential time ( $\alpha = 1$ ). When  $c$  is implicit, or obvious from the context, we simply write  $L_Q[\alpha]$ . When the complexity relates to an algorithm for a prime field  $\mathbb{F}_p$ , we write  $L_p[\alpha, c]$ .

**1.2. Previous Work on Individual Logarithm.** Many improvements for computing discrete logarithms first concerned prime fields. In 1993, Gordon [9] proposed the first version of NFS–DL algorithm for prime fields  $\mathbb{F}_p$  with asymptotic complexity  $L_p[1/3, 9^{1/3}]$ . The previous state-of-the-art for computing discrete logarithms in prime fields was the Coppersmith, Odlyzko and Schroepel (COS) algorithm, in  $L_p[1/2, 1]$ . Gordon’s  $L_p[1/3]$  algorithm is interesting for very large values of  $p$ , that were not yet targets for discrete logarithm computations in the nineties. Buhler, H. Lenstra and Pomerance [5] estimated the crossover point between 100 and 150 decimal digits, i.e. between 330 and 500 bits. However, with the  $L_p[1/3]$  algorithm there is a new difficulty: the individual logarithm phase. The COS algorithm computes the logarithms of many “small” elements. Writing a relation between any given target element and the “small” elements of known logarithms can be done quite fast. But in the  $L_p[1/3]$  algorithm, many fewer logarithms are known, because the relation collection is shorter, explaining the  $L_p[1/3]$  running time instead of  $L_p[1/2]$ . As a drawback, the individual logarithm phase becomes quite time consuming. Since some non-small elements in the decomposition of the target have an unknown logarithm, a dedicated sieving and linear algebra phase is done for each of them. Gordon estimated the running-time of individual logarithm computation to be  $L_p[1/3, 3^{2/3}]$ . In 1998, Weber [17, §6] compared the NFS–DL algorithm to the COS algorithm for a 85 decimal digit prime and made the same observation about individual logarithm cost. In 2003, Joux and Lercier [11] were the first to dissociate relation collection plus linear algebra on one side and individual logarithm on the other side. They used the *special-q* technique to find the logarithm of medium-sized elements in the target decomposition. In 2006, Commeine and Semaev [7] analyzed the Joux–Lercier method. They obtained an asymptotic complexity of  $L_p[1/3, 3^{1/3}]$  for computing individual logarithms, independent of the relation collection and linear algebra phases. In 2013, Barbulescu [2, §4, §7.3] gave a tight analysis of the individual logarithm computation for prime fields, decomposed in three steps: smoothness step, descent step and final combination of logarithms. The smoothness step has an asymptotic complexity of  $L_p[1/3, 1.23]$  and the descent step of  $L_p[1/3, 1.21]$ . The final computation has a negligible cost.

In 2006, Joux, Lercier, Smart and Vercauteren [12] computed a discrete logarithm in a cubic extension of a prime field. They used the *special-q* descent technique again. They generalized the *rational reconstruction* of the target to extension fields. For discrete logarithms in prime fields, the target is an integer modulo  $p$ . The rational reconstruction method outputs two integers of half size compared to  $p$ , such that their quotient is equal to the target element modulo  $p$ . Finding a smooth

decomposition of the target modulo  $p$  becomes equivalent to finding a (simultaneous) smooth decomposition of two elements, each of half the size. We explain their method (that we call the JLSV fraction method in the following) for extension fields in Section 2.3. They used for their record in [12] another polynomial selection method (that we call the JLSV<sub>0</sub> method in the following), whose first polynomial has very small coefficients, and the second one has coefficients of size  $O(p)$ . Thanks to the very small coefficients of the first polynomial, their technique was useful. Their polynomial selection technique is now superseded by the JLSV<sub>1</sub> method [12, §2.3] for larger value of  $p$  and more recently, by the gJL and Conjugation methods [4]. As noted in [12, §3.2], the technique is useful in some practical cases, for small  $n$ . But for the JLSV<sub>1</sub> method and  $n \geq 3$ , this is already too slow. In 2008, Zając [18] implemented the NFS-DL algorithm for computing discrete logarithms in  $\mathbb{F}_{p^6}$  with  $p$  of 40 bits (12 decimal digits (dd), i.e.  $\mathbb{F}_{p^6}$  of 240 bits or 74 dd). He used the methods described in [12], with a first polynomial of very small coefficients and a second one of coefficients in  $O(p)$ . In this case, individual logarithm was possible (see the well-documented [18, §8.4.5]). In 2013, Hayasaka, Aoki, Kobayashi and Takagi [10] computed a discrete logarithm in  $\mathbb{F}_{p^{12}}$  with  $p = 122663$ . They also selected the two polynomials with the JLSV<sub>0</sub> method. We noted that all these records used the JLSV<sub>0</sub> polynomial selection method, so that one of them has very small coefficients (e.g.  $f = x^3 + x^2 - 2x - 1$ ) whereas the second one has  $O(p)$  coefficients. Despite the JLSV<sub>1</sub> method was the state-of-the-art for polynomial selection since 2006, it was never used in practice. JLSV<sub>0</sub> was used.

In 2009, Joux, Lercier, Naccache and Thomé [13] proposed an attack of discrete logarithm problem in a protocol context. The relation collection is speeded-up with queries to an oracle. They wrote in [13, §B] an extended analysis of individual logarithm computation. In their case, the individual logarithm phase of the NFS-DL algorithm has a running-time of  $L_Q[1/3, c]$  where  $c = 1.44$  in the large characteristic case, and  $c = 1.62$  in the medium characteristic case. In 2014, Barbulescu and Pierrot [1] presented a multiple number field sieve variant (MNFS) for extension fields, based on Coppersmith's ideas [8]. The individual logarithm is studied in [1, §A]. They also used a *descent* technique, for a global estimated running time in  $L_Q[1/3, (9/2)^{1/3}]$ , with a constant  $c \approx 1.65$ . Recently in 2014, Barbulescu, Gaudry, Guillevic and Morain [3, 4] announced a 160 and a 180 decimal digit discrete logarithm record in quadratic fields. They also used a technique derived from the JLSV fraction method and a special- $\mathbf{q}$  descent technique. However their technique does not scale well for larger  $n$ .

**1.3. Our Contributions.** An integer is said to be  $B$ -smooth if all its prime divisors are less than  $B$ . An ideal in a number field is said to be  $B$ -smooth if it factors into prime ideals whose norm is bounded by  $B$ . For NFS-DL in a prime field  $\mathbb{F}_p$ , the norm of the target preimage in the number field is bounded by  $p$ . This bound gives the running time of the individual logarithm step. Finding a smooth decomposition of the preimage and computing the individual logarithm has complexity  $L_p[1/3, 3^{1/3}]$  [7] (with  $3^{1/3} \simeq 1.44$ ), which is smaller than the complexity of relation collection and linear algebra, which is  $L_p[1/3, (64/9)^{1/3}]$  (with  $c = 1.92$ ). For NFS-DL in  $\mathbb{F}_{p^n}$ , the norm bound is in fact much higher. Without any improvements in preimage computation, the individual logarithm step are in  $L_Q[1/3, c]$ , with  $c = (9/2)^{1/3} \simeq 1.65$ ,  $3^{1/3} \simeq 1.44$ ,  $6^{1/3} \simeq 1.82$  respectively, for the three polynomial selection methods available (JLSV<sub>1</sub> [12], generalized Joux-Lercier (gJL) [16, 4], Conjugation [4]). Applying the JLSV fraction method lowers the norm bound to  $O(Q)$  for the gJL and the Conjugation methods. The individual logarithm in these case has complexity  $L_Q[1/3, 3^{1/3}]$  as for prime fields (without the improvements of [2, §4]). However, this method is not suited for number fields generated with the

JLSV<sub>1</sub> method, for  $n \geq 3$ . In practice, we realized that this method which seems interesting and enough because of the  $O(Q)$  bound, is neither enough for gJL and Conjugation method, for  $n$  greater than 3. The preimage norm is much too large, so finding a smooth factorization is too slow by an order of magnitude. This contradicts the common idea that computing the individual logarithm of an element of norm bounded by  $O(Q)$  is clearly at hand once one was able to do the relation collection and linear algebra phases.

Firstly we prove a general theorem on the running-time needed to find a smooth decomposition of the norm preimage. Note that the smoothness bound  $B = L_Q[2/3, \gamma]$  here is not the same as for the relation collection step, where the smoothness bound is  $B_0 = L_Q[1/3, \beta_0]$ .

**Theorem 1.1** (Running-time of  $B$ -smooth decomposition). *Let  $s$  be an element of a number field  $K$ . Assume that the norm  $S$  of  $s$  in  $K$  is bounded by  $Q^e = L_Q[1, e]$ . Let  $B = L_Q[\alpha_B, \gamma]$ . Then the optimal bound of running time for finding a  $B$ -smooth decomposition of  $S$  is  $L_Q[1/3, (3e)^{1/3}]$ , obtained with  $\alpha_B = 2/3$  and  $\gamma = (e^2/3)^{1/3}$ .*

Secondly we propose a way to lift the target from the finite field to the number field, such that the norm is strictly smaller than  $O(Q)$  for the gJL and Conjugation methods:

**Theorem 1.2.** *Let  $s \in \mathbb{F}_{p^n}^*$  a random element (not in a proper subfield of  $\mathbb{F}_{p^n}$ ). We want to compute its discrete logarithm modulo  $\ell$ , where  $\ell \mid \Phi_n(p)$ . Let  $K$  be a number field given by a polynomial selection method.*

*Then there exists a preimage  $\bar{r}$  in  $K$  of an  $r \in \mathbb{F}_{p^n}^*$ , such that  $\log \rho(\bar{r}) \equiv \log s \pmod{\ell}$  and whose norm in  $K$  is bounded by  $O(Q^e)$ , where  $Q^e$  equal*

- (1)  $Q^{1-1/n}$  for the gJL and Conjugation methods;
- (2)  $Q^{\frac{3}{2}-\frac{3}{2n}}$  for the JLSV<sub>1</sub> method;
- (3)  $Q^{1-2/n}$  for the Conjugation method, if the number field has a well-chosen quadratic subfield satisfying the conditions of Lemma 4.1;
- (4)  $Q^{\frac{3}{2}-\frac{5}{2n}}$  for the JLSV<sub>1</sub> method, if the number field has a well-chosen quadratic subfield satisfying the conditions of Lemma 4.1.

Our method reaches the optimal bound of  $Q^{\varphi(n)/n}$ , with  $\varphi(n)$  the Euler totient function, for  $n = 2, 3, 4, 5$  and the gJL or the Conjugation method. This provides a dramatic improvement for individual logarithm computation for small  $n$ : the running-time of the first step (the smoothing step) is  $L_Q[1/3, c]$  with  $c = 1.14$  for  $n = 2, 4$ ,  $c = 1.26$  for  $n = 3, 6$  and  $c = 1.34$  for  $n = 5$ . It generalizes to any  $n$ , so that  $c < 1.44$  for the two state-of-the-art variants of NFS for extension fields.

**1.4. Outline.** We recall in Section 2 the three polynomial selection methods involved for NFS-DL in extension fields. We also give a commonly used bound on the norm of an element in a number field. We also present a generalization of the JLSV fraction method for the two other polynomial selection methods studied in this paper. We present our main idea in Sections 3 and 4 to reduce the norm of the preimage in the number field. We give a complexity analysis of the individual logarithm phase with our method in Section 5.

## 2. PRELIMINARIES

We recall the three polynomial selection methods we will study along this paper in Section 2.1. We give a common simple upper bound on the norm of an element in a number field in Section 2.2. We will need this formula to estimate a bound on the norm target and the corresponding asymptotic running-time of individual logarithm computation.

We recall now an important property of the LLL algorithm [15]. Given a lattice of  $\mathbb{Z}^n$  defined by an  $n \times n$  matrix  $L$ , the LLL algorithm outputs a *short* vector of the lattice, whose coefficients are bounded by

$$C \det(L)^{1/n},$$

where  $C$  is the LLL constant. In the remaining of this paper, we will denote by  $C$  this LLL constant.

**2.1. Polynomial Selection Methods.** Three polynomial selection methods are competitive for the initialization step of the NFS algorithm:

- (1) the Joux–Lercier–Smart–Vercauteren (JLSV<sub>1</sub>) method [12, §2.3];
- (2) the generalized Joux–Lercier (gJL) method [16, 4, §2, §3.2];
- (3) the Conjugation method [4, §3.3].

The gJL method has the best asymptotic running-time in the large characteristic case, while the Conjugation method holds the best one in the medium characteristic case. However for a record computation in  $\mathbb{F}_{p^2}$ , the Conjugation method was used [4]. Since the use in practice of each method is not fixed, we study and compare the three methods for the individual logarithm step of NFS. We recall now the construction and properties of these three methods.

**2.1.1. Joux–Lercier–Smart–Vercauteren (JLSV<sub>1</sub>) Method.** This method was introduced in 2006. We describe it in Algorithm 1. The two polynomials  $f, g$  have degree  $n$  and coefficient size  $O(p^{1/2})$ . We set  $\varphi = \gcd(f, g) \bmod p$  monic of degree  $n$ . We will use  $\varphi$  to represent the finite field extension  $\mathbb{F}_{p^n} = \mathbb{F}_p[x]/(\varphi(x))$ .

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**Algorithm 1:** Polynomial selection with the JLSV<sub>1</sub> method [12, §2.3]

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**Input:**  $p$  prime and  $n$  integer

**Output:**  $f, g, \varphi$  with  $f, g \in \mathbb{Z}[x]$  irreducible and  $\varphi = \gcd(f \bmod p, g \bmod p)$  in  $\mathbb{F}_p[x]$  irreducible of degree  $n$

- 1 Select  $f_1(x), f_0(x)$ , two polynomials with small integer coefficients,  $\deg f_1 < \deg f_0 = n$
  - 2 **repeat**
  - 3 | choose  $y \geq \lceil \sqrt{p} \rceil$
  - 4 **until**  $f = f_0 + yf_1$  is irreducible in  $\mathbb{F}_p[x]$
  - 5  $(u, v) \leftarrow$  a rational reconstruction of  $y$  modulo  $p$
  - 6  $g \leftarrow vf_0 + uf_1$
  - 7 **return**  $(f, g, \varphi = f \bmod p)$
- 

**2.1.2. Generalized Joux–Lercier (gJL) Method.** This method was presented in Barulescu’s PhD thesis [2, §8.3] and published in [4]. An earlier publication by Matyukhin in Russian [16, §2] presents the same method. This is a generalization of the Joux–Lercier method [11] for prime fields. We sketch this method in Algorithm 2. The coefficients of  $g$  have size  $O(Q^{1/(d+1)})$  and those of  $f$  have size  $O(\log p)$ , with  $\deg g = d \geq n$  and  $\deg f = d + 1$ .

**2.1.3. Conjugation Method.** This method was published in [4] and used for the discrete logarithm record in  $\mathbb{F}_{p^2}$ , with  $f = x^4 + 1$ . The coefficient size of  $f$  is in  $O(\log p)$  and the coefficient size of  $g$  is in  $O(p^{1/2})$ . It provides the best asymptotic complexity of NFS algorithm in the medium-characteristic case. We describe it in Algorithm 3.

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**Algorithm 2:** Polynomial selection with the generalized Joux–Lercier method (gJL) ([4, §3.2] and [16, §2])

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**Input:**  $p$  prime,  $n$  integer and  $d \geq n$  integer

**Output:**  $f, g, \varphi$  with  $f, g \in \mathbb{Z}[x]$  irreducible and  $\varphi = \gcd(f \bmod p, g \bmod p)$  in  $\mathbb{F}_p[x]$  irreducible of degree  $n$

- 1 Choose a polynomial  $f(x)$  of degree  $d + 1$  with small integer coefficients which has a monic irreducible factor  $\varphi(x) = \varphi_0 + \varphi_1 x + \cdots + x^n$  of degree  $n$  modulo  $p$
- 2 Reduce the following matrix using LLL

$$M = \left[ \begin{array}{cccc} p & & & \\ & \ddots & & \\ & & p & \\ \varphi_0 & \varphi_1 & \cdots & 1 \\ & \ddots & \ddots & \ddots \\ & & \varphi_0 & \varphi_1 & \cdots & 1 \end{array} \right] \left. \begin{array}{l} \\ \\ \\ \\ \\ \end{array} \right\} \begin{array}{l} \text{deg } \varphi = n \\ \\ \\ d + 1 - n \end{array}$$

to get

$$\text{LLL}(M) = \left[ \begin{array}{cccc} g_0 & g_1 & \cdots & g_d \\ & & & \\ & & * & \\ & & & \end{array} \right] .$$

**return**  $(f, g = g_0 + g_1 x + \cdots + g_d x^d, \varphi)$

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**Algorithm 3:** Polynomial selection with the Conjugation method [4, §3.3]

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**Input:**  $p$  prime and  $n$  integer

**Output:**  $f, g, \varphi$  with  $f, g \in \mathbb{Z}[x]$  irreducible and  $\varphi = \gcd(f \bmod p, g \bmod p)$  in  $\mathbb{F}_p[x]$  irreducible of degree  $n$

- 1 **repeat**
  - 2     Select  $g_1(x), g_0(x)$ , two polynomials with small integer coefficients,  $\deg g_1 < \deg g_0 = n$
  - 3     Select  $P_y(Y)$  a quadratic, monic, irreducible polynomial over  $\mathbb{Z}$  with small coefficients
  - 4 **until**  $P_y(Y)$  has a root  $y$  in  $\mathbb{F}_p$  and  $\varphi(x) = g_0(x) + y g_1(x)$  is irreducible in  $\mathbb{F}_p[x]$
  - 5  $f \leftarrow \text{Res}_Y(P_y(Y), g_0(x) + Y g_1(x))$
  - 6  $(u, v) \leftarrow$  a rational reconstruction of  $y$
  - 7  $g \leftarrow v g_0 + u g_1$
  - 8 **return**  $(f, g, \varphi)$
- 

2.1.4. *Comparison.* We summarize in Table 1 the polynomial selection properties of these three methods.

2.2. **Norm Upper Bound in a Number Field.** In Section 3 we will compute the norm of an element  $s$  in a number field  $K_f$ . We will need an upper bound of this norm. Let  $f$  be an irreducible polynomial over  $\mathbb{Q}$  and let  $K_f = \mathbb{Q}[x]/(f(x))$  a number field. Write  $s \in K_f$  as a polynomial in  $x$ :  $s = \sum_{i=0}^{\deg f - 1} s_i x^i$ . The norm is

TABLE 1. Properties: degree and coefficient size of the three polynomial selection methods for NFS-DL in  $\mathbb{F}_{p^n}$ . The coefficient sizes are in  $O(X)$ . To lighten the notations, we simply write the  $X$  term.

method	$\deg f$	$\deg g$	$\ f\ _\infty$	$\ g\ _\infty$
JLSV <sub>1</sub>	$n$	$n$	$Q^{1/2n}$	$Q^{1/2n}$
gJL	$d+1 > n$	$d \geq n$	$O(\log p)$	$Q^{1/(d+1)}$
Conjugation	$2n$	$n$	$O(\log p)$	$Q^{1/2n}$

defined by a resultant computation:

$$(1) \quad N_{K_f/\mathbb{Q}}(s) = \text{Res}(f, s) .$$

We use Kalkbrener's bound [14, Corollary 2] for an upper bound:

$$(2) \quad |\text{Res}(f, s)| \leq \kappa(\deg f, \deg s) \|f\|_\infty^{\deg s} \|s\|_\infty^{\deg f} ,$$

where  $\kappa(n, m) = \binom{n+m}{n} \binom{n+m-1}{n}$  and  $\|f\|_\infty = \max_{0 \leq i \leq \deg f} |f_i|$  the absolute value of the greatest coefficient. An upper bound for  $\kappa(n, m)$  is  $(n+m)!$ . We will use the following bound in Section 3:

$$(3) \quad N_{K_f/\mathbb{Q}}(s) \leq (\deg f + \deg s)! \|f\|_\infty^{\deg s} \|s\|_\infty^{\deg f} .$$

### 2.3. Joux–Lercier–Smart–Vercauteren Fraction Method.

**Notation 2.1.** Row and column indices. In the following, we will define matrices of size  $d \times d$ , with  $d \geq n$ . For ease of notation, we will index the rows and columns from 0 to  $d-1$  instead of 1 to  $d$ , so that the  $(i+1)$ -th row at index  $i$ ,  $L_i = [L_{ij}]_{0 \leq j \leq d-1}$ , can be written in polynomial form  $\sum_{j=0}^{d-1} L_{ij} x^j$ , and the column index  $j$  coincides with the degree  $j$  of  $x^j$ .

Computing a preimage of small norm in the number field is a major point for efficient individual logarithm. In 2006 was proposed in [12] a method to generalize to non-prime fields the rational reconstruction method used for prime fields. For discrete logarithms in prime fields, the target is an integer modulo  $p$ . The rational reconstruction method outputs two integers of half size compared to  $p$  and such that their quotient is equal to the target element modulo  $p$ . Finding a smooth decomposition of the target modulo  $p$  becomes equivalent to finding at the same time a smooth decomposition of two elements of half size each. This is actually much faster, see [2, §4].

A generalization to extension fields is to write the target preimage as a quotient of two number field elements, each with norm of half size of the original preimage. For explaining that, we introduce some notation. We denote by  $s$  the target in the finite field  $\mathbb{F}_{p^n}$  of degree  $n$  and by  $\bar{s}$  a preimage (or lift) in the number field  $K$  defined by a polynomial  $f$ . Here is a first very simple preimage choice. Let  $s = \sum_{i=0}^{\deg s} s_i x^i \in \mathbb{F}_{p^n}$ , with  $\deg s < n$  and the finite field extension defined with the polynomial  $\varphi$  as  $\mathbb{F}_{p^n} = \mathbb{F}_p[x]/(\varphi(x))$ . We lift the coefficients  $s_i \in \mathbb{F}_p$  to  $\bar{s}_i \in \mathbb{Z}$  then we set a preimage of  $s$  in the number field  $K$  to be

$$\bar{s} = \sum_{i=0}^{\deg s} \bar{s}_i X^i ,$$

with  $X$  such that  $K = \mathbb{Q}[X]/(f(X))$ . (We can also write  $\bar{s} = \sum_{i=0}^{\deg s} \bar{s}_i \alpha^i$ , with  $\alpha$  a root of  $f$  in the number field:  $K = \mathbb{Q}[\alpha]$ ).







It is straightforward, using Inequality (3), to deduce that

$$N_{K_f/\mathbb{Q}}(\bar{r}) = O(p^{n-1}) = O(Q^{1-1/n}) .$$

Here we obtain a bound that is *always strictly smaller* than  $Q$  for any  $n$ . In the next section we show how to improve this bound to  $O(Q^{1-2/n})$  when  $n$  is even and the number field defined by  $\varphi$  has a well-suited quadratic subfield.

#### 4. PREIMAGES OF SMALLER NORM WITH QUADRATIC SUBFIELDS

Reducing the degree of  $s$  can reduce the norm size in the number field for the JLSV<sub>1</sub> polynomial construction. We present a way to do this when  $n$  is even and the finite field  $\mathbb{F}_{p^n}$  can be expressed as a degree- $n/2$  extension of a quadratic extension defined by a polynomial of a certain form. We will use this to reduce the degree for the number fields in the JLSV<sub>1</sub> method and to reduce the coefficient size of the preimage in all three cases.

**4.1. Smaller Preimage Degree.** In this section, we prove that when  $n$  is even and  $\mathbb{F}_{p^n} = \mathbb{F}_p[x]/(\varphi(x))$  has a quadratic base field  $\mathbb{F}_{p^2}$  of a certain form, from a random element  $s \in \mathbb{F}_{p^n}$  with  $s_{n-1} \neq 0$ , we can compute an element  $r \in \mathbb{F}_{p^n}$  with  $r_{n-1} = 0$ , and  $s = u \cdot r$  with  $u \in \mathbb{F}_{p^2}$ . Then, using Lemma 3.1, we will conclude that  $\log r \equiv \log s \pmod{\ell}$ .

**Lemma 4.1.** *Let  $\varphi(X)$  be a monic irreducible polynomial of  $\mathbb{F}_p[X]$  of even degree  $n$  with a quadratic subfield defined by the polynomial  $P_y = Y^2 + y_1Y + y_0$ . Moreover, assume that  $\varphi$  splits over  $\mathbb{F}_{p^2} = \mathbb{F}_p[Y]/(P_y(Y))$  as*

$$\begin{aligned} \varphi(X) &= (P_z(X) - Y)(P_z(X) - Y^p) \\ \text{or } \varphi(X) &= (P_z(X) - YX)(P_z(X) - Y^pX) \end{aligned}$$

with  $P_z$  monic, of degree  $n/2$  and coefficients in  $\mathbb{F}_p$ . Let  $s \in \mathbb{F}_p[X]/(\varphi(X))$  a random element,  $s = \sum_{i=0}^{n-1} s_i X^i$ .

Then there exists an  $r \in \mathbb{F}_{p^n}$  monic and of degree  $n-2$  in  $X$ , and  $u \in \mathbb{F}_{p^2}$ , such that  $s = u \cdot r$  in  $\mathbb{F}_{p^n}$ .

We first give an example for  $s \in \mathbb{F}_{p^4}$  then present a constructive proof.

**Example 4.2.** Let  $P_y = Y^2 + y_1Y + y_0$  be a monic irreducible polynomial over  $\mathbb{F}_p$  and set  $\mathbb{F}_{p^2} = \mathbb{F}_p[Y]/(P_y(Y))$ . Assume that  $Z^2 - YZ + 1$  is irreducible over  $\mathbb{F}_{p^2}$  and set  $\mathbb{F}_{p^4} = \mathbb{F}_{p^2}[Z]/(Z^2 - YZ + 1)$ . Let  $\varphi = X^4 + y_1X^3 + (y_0 + 2)X^2 + y_1X + 1$  be a monic reciprocal polynomial. By construction,  $\varphi$  factors over  $\mathbb{F}_{p^2}$  into  $(X^2 - YX + 1)(X^2 - Y^pX + 1)$  and  $\mathbb{F}_p[X]/(\varphi(X))$  defines a quartic extension  $\mathbb{F}_{p^4}$  of  $\mathbb{F}_p$ . We have these two representations for  $\mathbb{F}_{p^4}$ :

$$\begin{array}{c} \mathbb{F}_{p^4} = \mathbb{F}_{p^2}[Z]/(Z^2 - YZ + 1) \\ \quad \downarrow \\ \mathbb{F}_{p^2} = \mathbb{F}_p[Y]/(Y^2 + y_1Y + y_0) \\ \quad \downarrow \\ \mathbb{F}_p \end{array}$$

and

$$\begin{array}{c} \mathbb{F}_{p^4} = \mathbb{F}_p[X]/(X^4 + y_1X^3 + (y_0 + 2)X^2 + y_1X + 1) \\ \quad \downarrow \\ \mathbb{F}_p \end{array}$$

of Lemma 4.1. Two possible extension field towers are:

$$\begin{array}{ccc} \mathbb{F}_{p^n} = \mathbb{F}_{p^2}[Z]/(P_z(Z) - Y) & & \mathbb{F}_{p^n} = \mathbb{F}_{p^2}[Z]/(P_z(Z) - YZ) \\ | & & | \\ \mathbb{F}_{p^2} = \mathbb{F}_p[Y]/(P_y(Y)) & \text{and} & \mathbb{F}_{p^2} = \mathbb{F}_p[Y]/(P_y(Y)) \\ | & & | \\ \mathbb{F}_p & & \mathbb{F}_p \end{array}$$

We write  $s$  in the following representation to emphasize the subfield structure:

$$s = \sum_{i=0}^{n/2-1} (a_{i0} + a_{i1}Y)Z^i \text{ with } a_{ij} \in \mathbb{F}_p .$$

- (1) If  $\varphi = P_z(Z) - Y$  then we can divide  $s$  by  $u_{LT} = a_{n/2,0} + a_{n/2,1}Y \in \mathbb{F}_{p^2}$  (the leading term in  $Z$ , i.e. the coefficient of  $Z^{n/2}$ ) to make  $s$  monic in  $Z$  up to a subfield cofactor  $u_{LT}$ :

$$\frac{s}{u_{LT}} = \sum_{i=0}^{n/2-2} (b_{i0} + b_{i1}Y)Z^i + Z^{n/2-1} ,$$

with the coefficients  $b_{ij}$  in the base field  $\mathbb{F}_p$ ,  $b_{i0} + b_{i1}Y = (a_{i0} + a_{i1}Y)/u_{LT}$ . Since  $P_z(Z) = Y$  and  $Z = X$  in  $\mathbb{F}_{p^n}$  by construction, we replace  $Y$  by  $P_z(Z)$  and  $Z$  by  $X$  to get an expression for  $s$  in  $X$ :

$$\frac{s}{u_{LT}} = \sum_{i=0}^{n/2-2} (b_{i0} + b_{i1}P_z(X))X^i + X^{n/2-1} = r(X) .$$

The degree in  $X$  of  $r$  is  $\deg r = \deg P_z(X)X^{n/2-2} = n - 2$  instead of  $\deg s = n - 1$ . We set  $u = 1/u_{LT}$ . By construction,  $u \in \mathbb{F}_{p^2}$ . We conclude that  $s = ur \in \mathbb{F}_{p^n}$ , with  $\deg r = n - 2$  and  $u \in \mathbb{F}_{p^2}$ .

- (2) If  $\varphi = P_z(Z) - YZ$  then we can divide  $s$  by  $u_{CT} = a_{00} + a_{01}Y \in \mathbb{F}_{p^2}$  (the constant term in  $Z$ ) to make the constant coefficient of  $s$  to be 1:

$$\frac{s}{u_{CT}} = 1 + \sum_{i=1}^{n/2-1} (b_{i0} + b_{i1}Y)X^i$$

with  $b_{ij} \in \mathbb{F}_p$ . Since  $P_z(Z) = YZ$  and  $Z = X$  in  $\mathbb{F}_{p^n}$  by construction, we replace  $YZ$  by  $P_z(Z)$  and  $Z$  by  $X$  to get

$$\frac{s}{u_{CT}} = 1 + \sum_{i=1}^{n/2-1} (b_{i0}X^i + b_{i1}P_z(X)X^{i-1}) = r(X) .$$

The degree in  $X$  of  $r$  is  $\deg r = \deg P_z(X)X^{n/2-1-1} = n - 2$  instead of  $\deg s = n - 1$ . We set  $u = 1/u_{CT}$ . By construction,  $u \in \mathbb{F}_{p^2}$ . We conclude that  $s = ur \in \mathbb{F}_{p^n}$ , with  $\deg r = n - 2$  and  $u \in \mathbb{F}_{p^2}$ . □

Now we apply the technique described in Section 3.1 to reduce the coefficient size of  $r$  in the JLSV<sub>1</sub> construction. We have  $r_{n-1} = 0$  and we assume that  $r_{n-2} = 1$ . We define the lattice by the  $(n - 1) \times (n - 1)$  matrix

$$L = \left[ \begin{array}{cccc} p & & & \\ & \ddots & & \\ & & p & \\ \bar{r}_0 & \dots & \bar{r}_{n-3} & 1 \end{array} \right]_{n-1 \times n-1} \left. \begin{array}{l} 0 \\ \vdots \\ n-3 \\ n-2 \end{array} \right\} \begin{array}{l} n-2 \text{ rows} \\ \text{row } n-2 \text{ with } \bar{r} \text{ coeffs} \end{array}$$



TABLE 2. Norm bound of the reduced element  $\bar{r}\bar{r}$  in the number field  $K$  when  $\varphi$  satisfies the conditions of Lemma 4.1.

method	$\ \bar{r}\bar{r}\ _\infty$	$N_{K_f/\mathbb{Q}}(\bar{r}\bar{r})$
JLSV <sub>1</sub>	$p^{\frac{n-2}{n}}$	$Q^{\frac{3}{2} - \frac{5}{2n}}$
gJL	$p^{\frac{n-2}{d+1}}$	$Q^{1 - \frac{2}{n}}$
Conjugation	$p^{\frac{n-2}{2n}}$	$Q^{1 - \frac{2}{n}}$

4.3.1. *Example for  $n = 2$ , Conjugation Method.* We take the parameters of the record in [4]:  $p$  is a 90 decimal digit (300 bit) prime number, and  $f, \varphi$  are computed with the Conjugation method. We choose a target  $s$  from the decimal digits of  $\exp(1)$ .

$$\begin{aligned}
p &= 314159265358979323846264338327950288419716939937510582097494459230781640628620899877709223 \\
f &= x^4 + 1 \\
\varphi &= x^2 + 107781513095823018666989883102244394809412297643895349097410632508049455376698784691699593x + 1 \\
s &= 271828182845904523536028747135319858432320810108854154561922281807332337576949857498874314x \\
&\quad + 9588806625076732632114201657575319902272235411526548684808440973949208471194724618090692
\end{aligned}$$

We first compute  $s' = \frac{1}{s_0} s$  then reduce

$$L = \begin{bmatrix} p & 0 & 0 & 0 \\ s'_0 & 1 & 0 & 0 \\ 1 & \varphi_1 & 1 & 0 \\ 0 & 1 & \varphi_1 & 1 \end{bmatrix}$$

then LLL( $L$ ) produces  $\bar{r}$  of degree 3 and coefficient size  $O(p^{1/4})$ . Actually LLL outputs four short vectors, hence we get four small candidates for  $\bar{r}$ , each of norm  $N_{K_f/\mathbb{Q}}(\bar{r}) = O(p) = O(Q^{1/2}) = O(Q^{\varphi(n)/n})$ , i.e. 90 dd. To slightly improve the smoothness search time, we can compute linear combinations of these four reduced preimages.

$$\begin{aligned}
&3603397286457205828471x^3 + 13679035553643009711078x^2 + 5577462470851948956594x + 856176942703613067714 \\
&9219461324482190814893x^3 - 4498175796333854926013x^2 + 8957750025494673822198x + 1117888241691130060409 \\
&28268390944624183141702x^3 + 5699666741226225385259x^2 - 17801940403216866332911x + 5448432247710482696848 \\
&3352162792941463140060x^3 + 3212585012235692902287x^2 - 5570636518084759125513x + 46926508290544662542327
\end{aligned}$$

The norm of the first element is

$$N_{K_f/\mathbb{Q}}(\bar{r}) = 2139882802952016861116904528030242843486696657097075761337598070760485340948677800162921$$

of 90 decimal digits, as expected.

4.3.2. *Example for  $n = 3$ , gJL Method.* We take  $p$  of 60 dd (200 bits) so that  $\mathbb{F}_{p^3}$  has size 180 dd (600 bits) as above. We took  $p$  a prime made of the 60 first decimal digits of  $\pi$ . We constructed  $f, \varphi, g$  with the gJL method described in [4].

$$\begin{aligned}
p &= 314159265358979323846264338327950288419716939937510582723487 \\
f &= x^4 - x + 1 \\
\varphi &= x^3 + 227138144243642333129902287795664772043667053260089299478579x^2 \\
&\quad + 126798022201426805402186761110440110121157863791585328913565x \\
&\quad + 86398309157441443539791899517788388184853963071847115552638 \\
g &= 2877670889871354566080333172463852249908214391x^3 + 6099516524325575060821841620140470618863403881x^2 \\
&\quad - 10123533234834473316053289623165756437267298403x + 2029073371791914965976041284208208450267120556 \\
s &= 271828182845904523536028747135319858432320810108854154561922x^2 \\
&\quad + 281807332337576949857498874314095888066250767326321142016575x \\
&\quad + 7531990227223541152654868480858951626493739297259139859875
\end{aligned}$$

We set  $s' = \frac{1}{s_2}s$ . The lattice to be reduced is

$$L = \begin{bmatrix} p & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ s'_0 & s'_1 & 1 & 0 \\ \varphi_0 & \varphi_1 & \varphi_2 & 1 \end{bmatrix}$$

then  $\text{LLL}(L)$  computes four short vectors  $\bar{r}$  of degree 3, of coefficient size  $O(p^{1/2})$ , and of norm size  $N_{K_f/\mathbb{Q}}(\bar{r}) = O(p^2) = O(Q^{2/3}) = O(Q^{\varphi(n)/n})$ .

$$\begin{aligned} & 159774930637505900093909307018x^3 + 165819631832105094449987774814x^2 + 177828199322419553601266354904x - 159912786936943488400590389195 \\ & 136583029354520905232412941048x^3 - 521269847225531188433352927453x^2 + 322722415562853671586868492721x + 255238068915917937217884608875 \\ & 118289007598934068726663000266x^3 + 499013489972894059858543976363x^2 - 105084220861844155797015713666x + 535978811382585906107397024241 \\ & 411603890054539500131474313773x^3 - 240161030577722451131067159670x^2 - 373289346204280810310169575030x - 389720783049275894296185820094 \end{aligned}$$

The norm of the first element is

$$N_{K_f/\mathbb{Q}}(\bar{r}) = 997840136509677868374734441582077227769466501519927620849763845265357390584602475858356409809239812991892769866071779$$

of 117 decimal digits (note that  $\frac{2}{3}180 = 120$  dd).

#### 4.3.3. Example for $n = 4$ , $JLSV_1$ Method.

$$\begin{aligned} p &= 314159265358979323846264338327950288419980011 \\ \ell &= 49348022005446793094172454999380755676651143247932834802731698819521755649884772819780061 \\ f = \varphi &= x^4 + x^3 + 70898154036220641093162x^2 + x + 1 \\ g &= 101916096427067171567872x^4 + 101916096427067171567872x^3 + 220806328874049898551011x^2 \\ &\quad + 101916096427067171567872x + 101916096427067171567872 \\ s &= 271828182845904523536028747135319858432320810x^3 + 108854154561922281807332337576949857498874314x^2 \\ &\quad + 958880662507673263211420165753199022772235x + 41152654868480844097394920847127588391952018 \end{aligned}$$

We set  $s' = \frac{1}{s_3}s$ . The subfield simplification for  $s$  gives

$$r = x^2 + 134969122397263102979743226915282355400161911x + 104642440649937756368545765334741049207121011 .$$

We reduce the lattice defined by

$$L = \begin{bmatrix} p & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ r_0 & r_1 & 1 & 0 \\ s'_0 & s'_1 & s'_2 & 1 \end{bmatrix}$$

then  $\text{LLL}(L)$  produces these four short vectors of degree 3, coefficient size  $O(p^{1/2})$ , and norm  $N_{K_f/\mathbb{Q}}(\bar{r}\bar{r}) = O(p^{7/2}) = O(Q^{7/8})$  (smaller than  $O(Q)$ ).

$$\begin{aligned} & 5842961997149263751946x^3 + 290736827330861011376x^2 - 5618779793817086743792x + 1092494800287557029045 \\ & 1640842643903161175359x^3 + 15552590269131889589575x^2 - 4425488394163838271378x - 5734086421794811858814 \\ & 6450686906504525374853x^3 + 13768771242650957399419x^2 + 10617583944234090880579x + 16261617079167797580912 \\ & 16929135804139878865391x^3 + 698185571704810258344x^2 + 12799300411012246114079x - 22787282698718065284157 \end{aligned}$$

The norm of the first element is

$$N_{K_f/\mathbb{Q}}(\bar{r}\bar{r}) = 14521439292172711151668611104133579982787299949310242601944218977645007049527 \setminus \\ 012365602178307413694530274906757675751698466464799004360546745210214642178285$$

of 155 decimal digits (with  $\frac{7}{8}180 = 157.5$ ).

## 5. ASYMPTOTIC COMPLEXITIES

In this section, we prove Theorem 1.1. First, we need a result on smoothness probability. We recall the definition of  $B$ -smoothness already stated in Section 1.3: An integer  $S$  is  $B$ -smooth if and only if all its prime divisors are less than or equal to  $B$ . We also recall that the  $L$ -notation widely used for sub-exponential asymptotic complexities:

$$L_Q[\alpha, c] = \exp\left((c + o(1))(\log Q)^\alpha (\log \log Q)^{1-\alpha}\right) \quad \text{with } \alpha \in [0, 1] \text{ and } c > 0 .$$

The Canfield–Erdős–Pomerance [6] theorem provides a useful result to measure smoothness probability:

**Theorem 5.1** ( *$B$ -smoothness probability*). *For an integer  $S$  bounded by  $L_Q[\alpha_S, \sigma]$  and a smoothness bound  $B = L_Q[\alpha_B, \beta]$ , the probability that  $S$  is  $B$ -smooth is*

$$(7) \quad \Pr(S \text{ is } B\text{-smooth}) = L_Q \left[ \alpha_S - \alpha_B, -(\alpha_S - \alpha_B) \frac{\sigma}{\beta} \right] .$$

We prove now the Theorem 1.1 that states the running-time of individual logarithm when the norm of the target in a number field is bounded by  $O(Q^e)$ .

*of Theorem 1.1.* From Theorem 5.1, the probability that  $S$  bounded by  $Q^e = L_Q[1, e]$  is  $B$ -smooth with  $B = L_Q[\alpha_B, \gamma]$  is  $\Pr(S \text{ is } B\text{-smooth}) = L_Q \left[ 1 - \alpha_B, -(1 - \alpha_B) \frac{e}{\gamma} \right]$ . We assume that a  $B$ -smoothness test with ECM takes time  $L_B[1/2, 2^{1/2}] = L_Q \left[ \frac{\alpha_B}{2}, (2\gamma\alpha_B)^{1/2} \right]$ . The running-time for finding a  $B$ -smooth decomposition of  $S$  is the ratio of the time per test (ECM cost) to the  $B$ -smoothness probability of  $S$ :

$$L_Q \left[ \frac{\alpha_B}{2}, (2\gamma\alpha_B)^{1/2} \right] L_Q \left[ 1 - \alpha_B, (1 - \alpha_B) \frac{e}{\gamma} \right] .$$

We optimize first the  $\alpha$  value, so that  $\alpha \leq 1/3$  (that is, not exceeding the  $\alpha$  of the two previous steps of the NFS algorithm):

$$\max(\alpha_B/2, 1 - \alpha_B) \leq \frac{1}{3} .$$

This gives the system

$$\begin{cases} \alpha_B \leq 2/3 \\ \alpha_B \geq 2/3 \end{cases}$$

So we conclude that

$$(8) \quad \alpha_B = \frac{2}{3} .$$

The running-time for finding a  $B$ -smooth decomposition of  $S$  is therefore

$$(9) \quad L_Q \left[ 1/3, \left( \frac{4}{3} \gamma \right)^{1/2} + \frac{e}{3\gamma} \right] .$$

The minimum of the function  $(\frac{4}{3}\gamma)^{1/2} + \frac{e}{3\gamma}$  is  $(3e)^{1/3}$ , corresponding to  $\gamma = (e^2/3)^{1/3}$ , which yields our optimal running time:

$$L_Q \left[ 1/3, (3e)^{1/3} \right] .$$

□

This result should be compared with the running time of  $L_Q[\frac{1}{3}, (\frac{9}{2})^{1/3}]$  for the JLSV<sub>1</sub> method ( $e = 3/2$ ) computed by Barbulescu and Pierrot [1, Appendix A], and the running time of  $L_Q[\frac{1}{3}, 3^{1/3}]$  for prime fields ( $e = 1$ ) computed by Commeine and Semaev [7, §4.1].

TABLE 3. Running-time to compute a  $B$ -smooth decomposition of  $S$  bounded by  $Q^e$ , for some values of  $e$  in  $[1/3, 3]$  and  $B = L_Q[2/3, (e^2/3)^{1/3}]$ .

Norm bound	$Q^{1/3}$	$Q^{1/2}$	$Q^{2/3}$	$Q^{4/5}$	$Q$	$Q^{3/2}$	$Q^2$	$Q^3$
running-time constant $c$ in $L_Q[\frac{1}{3}, c]$	1	$(\frac{3}{2})^{1/3}$	$2^{1/3}$	$\frac{12^{1/3}}{5}$	$3^{1/3}$	$\frac{9^{1/3}}{2}$	$6^{1/3}$	$9^{1/3}$
constant $c = (3e)^{1/3}$	1.00	1.14	1.26	1.34	1.44	1.65	1.82	2.08

**5.1. Running-Time of Special- $q$  Descent.** The second step of the individual logarithm computation is the *special- $q$  descent*. This consists in computing the logarithms of the medium-sized elements in the factorization of the target in the number field. This step is very technical and we refer to Joux–Lercier–Naccache–Thomé [13, §B], Commeine and Semaev [7, §4.3], Barbulescu and Pierrot [1, §A], and Barbulescu [2, §7.3] for an analysis. They claim that this is possible to tune parameters so that the running time is in  $L_Q[1/3, \delta]$  with  $\delta$  strictly less than the constant involved in the running time of finding a  $B$ -smooth decomposition of the target. Since we considerably lowered this constant to  $c = (3e)^{1/3}$ , more work is needed to find whether we still have  $\delta < c$ . The third and final step of individual logarithm computation is very fast. It combines all the logarithms computed before, to get the final discrete logarithm of the target. It seems that now the special- $q$  descent is clearly the bottleneck of the individual logarithm computation.

## 6. CONCLUSION

We give in Table 4 an upper bound for the norm of  $s$  in a number field  $K_f$  for three polynomial selection methods: the JLSV<sub>1</sub> method, the generalized Joux–Lercier method and the Conjugation method, and the complexity of finding a  $B$ -smooth decomposition of  $\mathbb{N}_{K_f/\mathbb{Q}}(s)$ .

TABLE 4. Properties of polynomials and norm estimate for the three polynomial selection methods, with  $s = \sum_{i=0}^{\deg s} s_i x^i \in \mathbb{F}_p^*$  and assuming that  $\deg s = n - 1$ .

polynomial select.	JLSV <sub>1</sub> [12, §2.3]	gJL[16, 4]	Conjugation[4]
$\deg f$	$n$	$d + 1 \geq n + 1$	$2n$
$\ f\ _\infty$	$O(p^{1/2})$	$O(\log p)$	$O(\log p)$
$\text{Norm}_{K_f/\mathbb{Q}}(s)$	$p^{\deg s/2} \ s\ _\infty^n$	$\ s\ _\infty^{d+1}$	$\ s\ _\infty^{2n}$
nothing	$Q^{3/2-1/(2n)}$	$Q^{1+1/n}$	$Q^2$
[12] and §2.3	$Q^2$	$Q$	$Q$
This work, 3.1, 3.2	$Q^{3/2-3/(2n)}$	$Q^{1-1/n}$	$Q^{1-1/n}$
This work, 4.2	$Q^{3/2-5/(2n)}$	$Q^{1-2/n}$	$Q^{1-2/n}$

Finally, we give our practical results for small  $n$  in Table 5, where there are the most dramatic improvements. We obtain the optimal norm size of  $Q^{\varphi(n)/n}$  for  $n = 2, 3, 5$  with the gJL method and also for  $n = 4$  with the Conjugation method.

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TABLE 5.

polynomial select.	JLSV <sub>1</sub> [12, §2.3]	gJL[16, 4]	Conjugation[4]
$n = 2$			
Norm bound	$Q^{3/4}$ , § 3.1	$Q^{1/2}$ , § 3.2	
running time, $c$	$\left(\frac{9}{4}\right)^{1/3} = 1.31$	$\left(\frac{3}{2}\right)^{1/3} = 1.14$	
$n = 3$			
Norm bound	$Q$ § 3.1	$Q^{2/3}$ , § 3.2	
running time, $c$	$3^{1/3} = 1.44$	$2^{1/3} = 1.26$	
$n = 4$			
Norm bound	$Q^{7/8}$ , § 4.2	$Q^{3/4}$ , § 3.2	$Q^{1/2}$ , § 4.2
running time, $c$	$\left(\frac{21}{8}\right)^{1/3} = 1.38$	$\left(\frac{9}{4}\right)^{1/3} = 1.31$	$\left(\frac{3}{2}\right)^{1/3} = 1.14$
$n = 5$			
Norm bound	$Q^{6/5}$ , § 3.1	$Q^{4/5}$ , § 3.2	
running time, $c$	$\left(\frac{18}{5}\right)^{1/3} = 1.53$	$\left(\frac{12}{5}\right)^{1/3} = 1.34$	
$n = 6$			
Norm bound	$Q^{13/12}$ , § 4.2	$Q^{5/6}$ , § 3.2	$Q^{2/3}$ , § 4.2
running time, $c$	$\left(\frac{13}{4}\right)^{1/3} = 1.48$	$\left(\frac{5}{2}\right)^{1/3} = 1.36$	$2^{1/3} = 1.26$

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