

Stationary Graph Signals using an Isometric Graph Translation

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Signal Processing over Graphs

$G = (V, E)$ a symmetric graph

Adjacency matrix: $A_{ij} = w_{ij}, ij \in E$

Degree matrix: $D = \text{diag}(d_1, \dots, d_N), d_i = \sum_j w_{ij}$

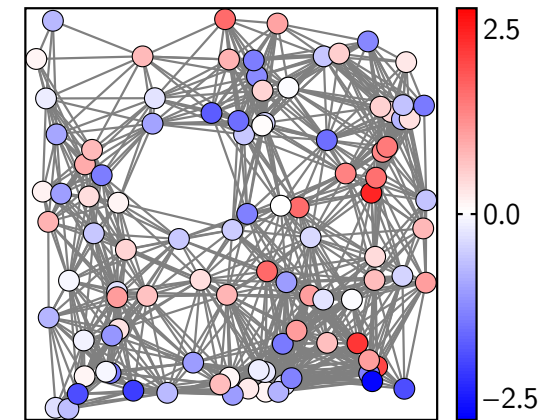
Laplacian matrix: $L = D - A$

Normalized Laplacian: $\mathcal{L} = D^{-1/2} L D^{-1/2}$

$X : V \rightarrow \mathbb{R}$ or \mathbb{C} a graph signal

Fourier: $F = [\chi_0 \cdots \chi_{N-1}]^*$, $\hat{X} = FX$

Example: $L = F^* \Lambda F$



Some successes (non-exhaustive growing list):

- Wavelets [Hammond et al. 2011]
- Filter Banks [Narang & Ortega 2012-2013]
- Graph Shift [Sandryhaila & Moura 2013-2014]
- Vertex-Frequency [Shuman et al. 2014]
- Multiscale Community Mining [Tremblay & Borgnat 2015]

Why (Non-)Stationarity?

- Stochastic signals
- Toy Example: Denoising
⇒ Noise statistical invariance to an absolute time
- Non-stationarity: **phase changes**
⇒ Ruptures, anomalies, failure diagnosis, etc...

Which tools for (non-)stationary graph signals?

Stationary Temporal Signals

- Stationary: Statistical invariance to an absolute time

$$\mathbb{P}[x(t) = x] = \mathbb{P}[x(t - \tau) = x]$$

and for higher orders:

$$\mathbb{P}[x(t_1) = x_1 \text{ and } \dots \text{ and } x(t_k) = x_k] = \\ \mathbb{P}[x(t_1 - \tau) = x_1 \text{ and } \dots \text{ and } x(t_k - \tau) = x_k].$$

⇒ **Strict-Sense Stationary (SSS)** signal x

- Time shift $T_\tau\{x\}(t) = x(t - \tau)$ invariance:

$x \text{ and } T_\tau\{x\} \text{ are statistically equal}$

Stationary Graph Signals?

- Straightforward definition:

x and $T_\tau\{x\}$ are statistically equal

- Question:

What is T_τ ?

- $T_{\tau_i} = \Delta_i$, the Generalized Translation? [Shuman et al. 2013]
- $T_\tau = A^\tau$, the Graph Shift? [Sandryhaila et al. 2013]

\Rightarrow These operators are not **isometric**: $\|AX\|_2 \neq \|X\|_2 \neq \|\Delta_i X\|_2$

Definition (Graph Translation)

New isometric operator:

$$T_{\mathcal{G}}^\tau = e^{-i\tau\sqrt{L}}$$

Remarks

- Coherence with the Time Shift:

$$T_\tau\{e_\omega\} = e^{-i\tau\omega} e_\omega \quad \Rightarrow \quad T_{\mathcal{G}}^\tau \chi_l = e^{-i\tau\sqrt{\lambda_l}} \chi_l$$

$$\widehat{T}_{\mathcal{G}}^\tau = \text{diag}(e^{-i\tau\sqrt{\lambda_0}}, \dots, e^{-i\tau\sqrt{\lambda_{N-1}}})$$

- Supports other graph Fourier transform:

- ▶ $T_{\mathcal{G}}^\tau = e^{-i\tau\sqrt{\mathcal{L}}}$

- Complex operator

- ▶ Not a real problem for stationarity
 - ▶ Analytical signals?



$T_{\mathcal{G}}$ is NOT an operator *shifting* energy from one vertex to another.

Definition of Stationary Graph Signals

Definition (SSS Graph Signal)

A stochastic graph signal X verifying $\forall \tau, X \stackrel{d}{=} T_{\mathcal{G}}^{\tau} X$.

Definition (WSS Graph Signal)

A stochastic graph signal X is Wide Sense Stationary iff:

$$\begin{cases} \mu_X = \mathbb{E}[X] = \mathbb{E}[T_{\mathcal{G}} X] = \mu_{T_{\mathcal{G}} X} \\ R_X = \mathbb{E}[XX^*] = \mathbb{E}[(T_{\mathcal{G}} X)(T_{\mathcal{G}} X)^*] = R_{T_{\mathcal{G}} X} \end{cases}$$

Spectral Characterisation

Theorem

A stochastic graph signal X is WSS iff

$$\forall l, \lambda_l \neq 0 \Rightarrow \mathbb{E}[\hat{X}(l)] = 0 \quad (\text{i})$$

$$\forall l, \lambda_l \neq \lambda_k \Rightarrow \mathbb{E}[\hat{X}(l)\hat{X}^*(k)] = 0 \quad (\text{ii})$$

Remark on (i) (First Moment)

First moment condition: $\mathbb{E}[X] \propto \chi_0$

- $L = F^* \Lambda F \Rightarrow \chi_0 = [1, \dots, 1]^T$
- $\mathcal{L} = F^* \Lambda F \Rightarrow \chi_0 = [1/\sqrt{d_1}, \dots, 1/\sqrt{d_N}]^T$

Remark on (ii) (Second Moment)

if $\forall l \neq k, \lambda_l \neq \lambda_k$, then (ii) $\Leftrightarrow S = \mathbb{E}[\hat{X}\hat{X}^*]$ diagonal

Discussion

- **Complex Operator:** But real WSS signals exists
- **R Non Toeplitz:** $R_{i,j} = \mathbb{E}[X_i X_j^*] \neq \mathbb{E}[X_{i+k} X_{j+k}^*] = R_{i+k,j+k}$
- **Alternative Definition of Translation?**
 \Rightarrow Yes, as long as $T = e^{-i\Omega}$, and $T\chi_0 = \chi_0$

Summary before Illustrations

Graph Translation

$$T_{\mathcal{G}}^{\tau} = e^{-i\tau\sqrt{L}}$$

Wide-Sense Stationary Graph Signal

X and $T_{\mathcal{G}}$ have equal first and second moments:

$$\begin{aligned}\mathbb{E}[X] &= \mu_X = \mu_{T_{\mathcal{G}}X} \\ \mathbb{E}[XX^*] &= R_X = R_{T_{\mathcal{G}}X}\end{aligned}$$

Spectral Characterisation

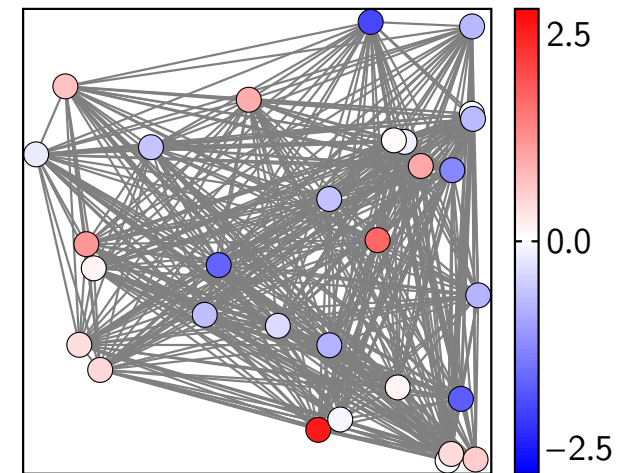
$$\mathbb{E}[\hat{X}(l)] = \begin{cases} \mathbb{E}[\|X\|] & \text{if } l = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbb{E}[\hat{X}\hat{X}^*] = S_X \text{ diagonal (if } \forall l \neq k, \lambda_l \neq \lambda_k)$$

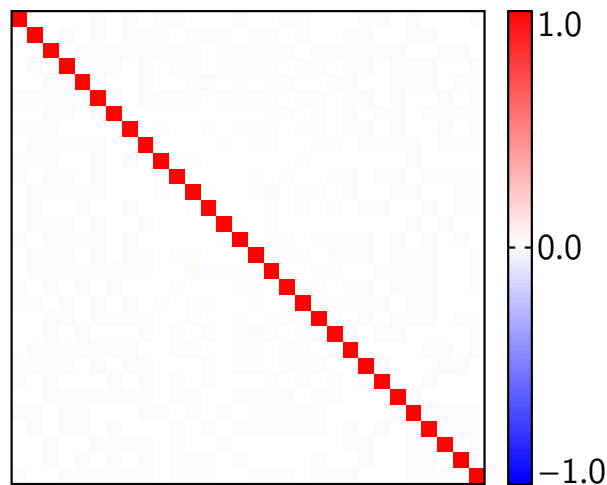
Stationary White Noise on Graph

Signal with flat power spectrum:

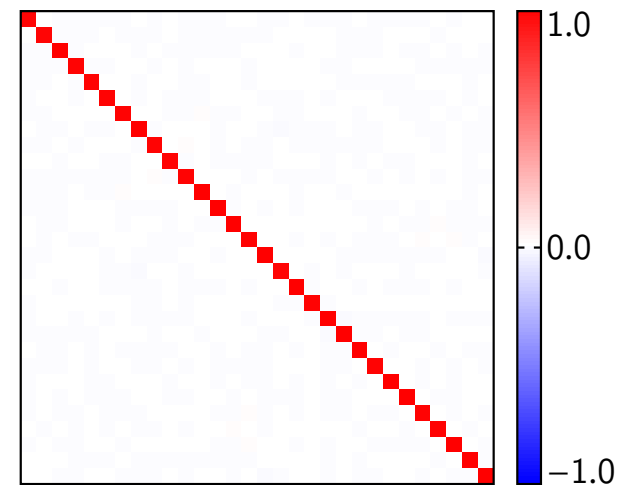
$$S = \mathbb{E}[\hat{X}\hat{X}^*] = \sigma^2 I$$



A realisation



Empirical R (50k realisations)

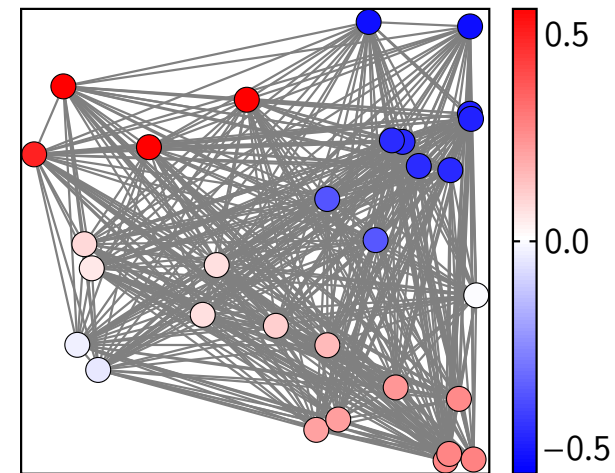


Empirical S (50k realisations)

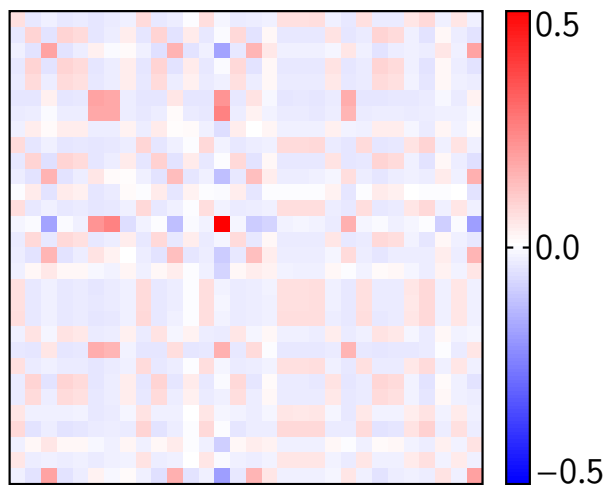
WSS Graph Signal

Low pass WSS graph signal:

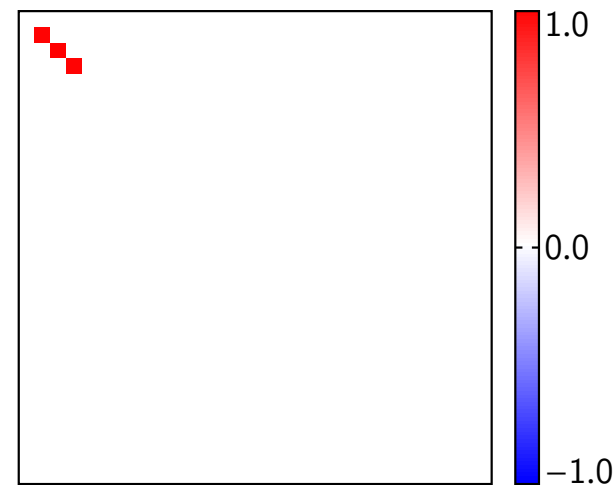
$$S_{ll} = \mathbb{E}[|\hat{X}_l|^2] = \begin{cases} 1 & \text{if } l \in \{1, 2, 3\} \\ 0 & \text{otherwise} \end{cases}$$



A realisation



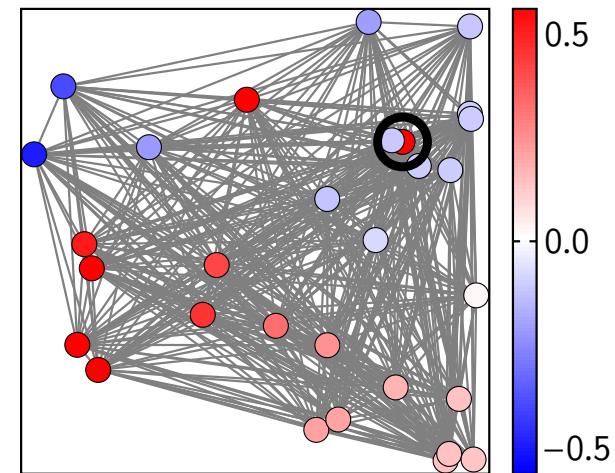
Empirical R (50k realisations)



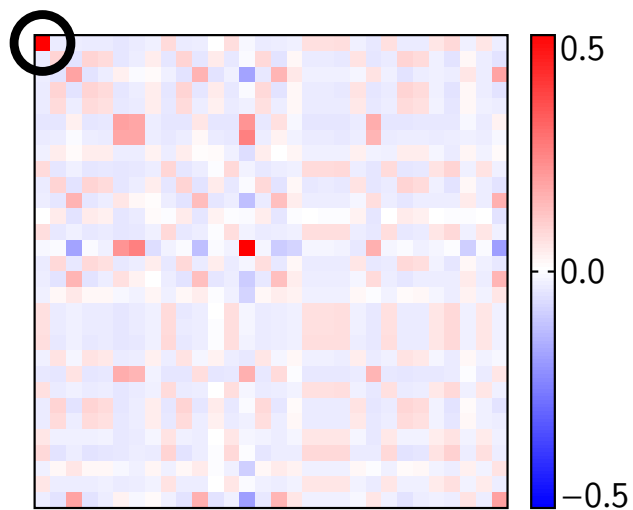
Empirical S (50k realisations)

Non Stationary Graph Signal

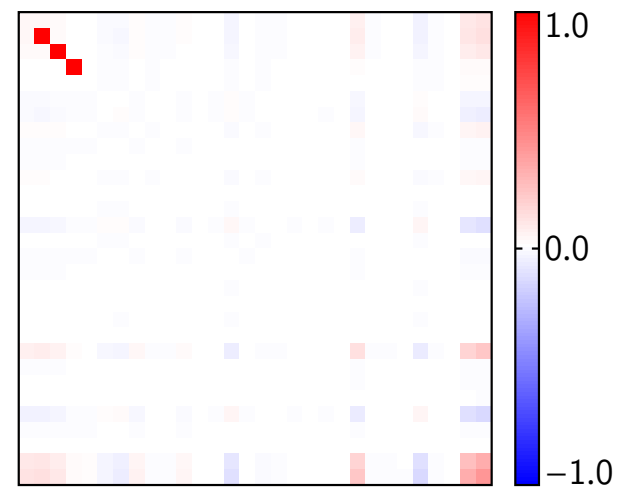
Same as previously, with additive noise on $X(1)$ with distribution $\mathcal{N}(0,1)$.



A realisation



Empirical R (50k realisations)



Empirical S (50k realisations)

Perspectives

Ongoing work

- Application to weather reports
⇒ Next week at Grets2015, Lyon, FRANCE
- Statistical test for stationarity
- Classes of interesting non-stationary signals

Open Questions

- Meaning of a translated signal?
- Analytic signal?
- Alternative invariance operator?