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Enumeration Reducibility in Closure Spaces with Applications to Logic and Algebra

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Abstract

In many instances in first order logic or computable algebra, classical theorems show that many problems are undecidable for general structures, but become decidable if some rigidity is imposed on the structure. For example, the set of theorems in many finitely axiomatisable theories is nonrecursive, but the set of theorems for any finitely axiomatisable *complete* theory is recursive. Finitely presented groups might have an nonrecursive word problem, but finitely presented *simple* groups have a recursive word problem.

In this article we introduce a topological framework based on closure spaces to show that many of these proofs can be obtained in a similar setting. We will show in particular that these statements can be generalized to cover arbitrary structures, with no finite or recursive presentation/axiomatization. This generalizes in particular work by Kuznetsov and others.

Examples from first order logic and symbolic dynamics will be discussed at length.

Introduction

When one deals with sufficiently complicated algebraic structures, it is customary to see uncomputability results for many questions: A finitely presented semigroup with an undecidable word problem was presented by Post [35]. The same result was obtained for groups by Novikov [32] and Boone [7, 8], and for division rings by MacIntyre [29].

If we add however some hypothesis on the structure, we are sometimes able to prove that the problem becomes decidable. This is the case for instance of finitely presented *simple* groups [9]. More generally, Kuznetsov [27] proved that (using the vocabulary of Maltsev [30, Theorem 4.2.2]), every simple finitely presented algebra is constructive.

There are however other structures that are not algebras, where the exact same situation happens. As an example, there are unsolvable finitely axioma-

tisable first-order theories (for instance Robinson's Q [36]). By contrast, any *complete* finitely axiomatisable theory is decidable (Folklore).

A similar situation happens in multidimensional symbolic dynamics, i.e. the study of tilings of the plane: Some subshifts of finite type have an uncomputable language in dimension 2 [37]. However *minimal* subshifts of finite type have a computable language [21, 3].

The goal of this paper is to provide an unifying framework in which all the decidable results will be seen as an instance of the same theorem. We will use the vocabulary of topology rather than algebras. The main reason to do this is that using topology is already sufficient to obtain the main theorems, and we are able this way to obtain results on structures which cannot be made easily into algebras (in the sense of Maltsev).

The main idea is to put a structure on the set of all theories in a given language, or on the set of all groups with a given generating set, or on the set of all subshifts over a given alphabet. This structure will be called a *quasivariety* by analogy with universal algebra. In the case of finitely generated groups, this structure is known as the space of marked groups [18].

For this structure, finitely presented groups, finitely axiomatizable theories, subshifts of finite type will correspond to the same objects, that we call *finitely presented* points, using the vocabulary from algebra.

In this structure, simple groups, complete theories and minimal subshifts all correspond to the same objects, *maximal* points.

Our main theorem states that finitely presented maximal points in a given quasivariety are computable, thereby generalizing all previous theorems.

By doing this abstractly we will see that it is actually possible to drop the hypothesis that the points are finitely presented, and obtain a more general statement when the objects in our quasivariety are not supposed to have a finite presentation, or even a recursive presentation. Of course in this case, the points will be unlikely to be computable. Our main result is that, in a maximal point, false statements are *enumeration-reducible* to true statements: Given an enumeration of all statements that are true in the structure, we are able to deduce all statements that are not true: we obtain negative information about the structure from positive information.

Roughly speaking, a set A is enumeration reducible [17, 34] to a set B, if an enumeration of A can be obtained in some effective way from any enumeration of B. It is not surprising that this concept has an important role here, as it already has been applied successfully, in particular in the context of groups. We know for example that for every enumeration degree d, there exists a finitely generated group for which the word problem has enumeration degree d (the reduction is actually stronger, Dobritsa [5, Theorem 2.4], see also [43]) or that enumeration reducibility characterizes, given a group H, when a group G can be embedded into a group that is finitely presented over H (C.F. Miller, see [20, Chapter 6]). This notion has also been used in a more general context by Belegradek [6], and our first easy propositions about presentations are reflected

in this article.

We note in passing that various other reductions have been used in conjunction with algebraic objects, in particular quasi-reducibility [15] and Ziegler-reducibility [44]. However many of our theorems have converses (see in particular Theorem 6) which suggest enumeration-reducibility is indeed the right notion in our context.

The article is organized as follows: We first define the main concept of a quasivariety (and its associated closure space) in the first section. In the next section, we will be interested in computability properties of all possible presentations of a point. The remaining sections are concerned with the main theorem, namely the concept of a maximal points, and computability properties of maximal points, and generalizations of maximal points. The article ends with a discussion on which other properties one might try to capture in this framework.

There are three main examples used in this paper: first-order logic, symbolic dynamics and finitely generated groups. We will focus on the first two, and the results for finitely generated groups will be given in appendix. As our results are proven in a full generality, it is quite likely that stronger computability statements can be proven in some particular examples. This happens for first-order logic, where we can obtain a much stronger result for the main theorem. For the quasivariety of subshifts, various recent results show however that our general results are actually tight.

Related work

While our approach is based on closure spaces, it is of course not the only way to obtain an unifying framework for different structures.

One possibility would be to use universal algebra. Many of the results presented here for finitely presented algebras can be found in the work of Maltsev and Kuznetsov [30, 27]. While recursive properties have been investigated from the start, the concept of enumeration-reducibility only appear in Belegradek [6], and is not used in its full generality. Universal algebra however suffer from the fact that many theories we are interested here do not fit easily in the setting. This is the case for first order theory, and for subshifts. Without going into much details, one of the problem with the theory of subshifts, that we don't solve here, is how to obtain a good notation of the restriction of a subshift over an alphabet A to a subshift over an alphabet $B \subseteq A$. In terms of universal algebra, if a subshift is generated by three generators a, b, c, it is not clear what is the object generated only by a and b, and in particular if it is also a subshift.

Other approaches using category theory or model theory suffer from the same problems.

One possible solution is to use Stone Duality, and to associate to each object (group, subshift, theory, etc.) a topological space and/or a boolean algebra. We do now know if it is possible to use this connection to obtain our theorems here.

1 Definitions

1.1 Quasivarieties

We will assume rudimentary notions of computability theory, in particular the notion of a recursive set of integers, a recursive function, and the concept of a recursively enumerable set. See [33] for details.

Let I be an infinite recursive set, that we identify with the set of integers. In applications, I will be the set of finite words over a given finite alphabet, or the set of formulas in a given finite signature.

In this article, we will always identify a subset $X \subseteq I$ and a point $x \in \{0,1\}^I$.

We are interested in subsets $X \subseteq I$ that can be defined by some Horn formulas , i.e. by axioms of the type:

$$a \in X \wedge b \in X \wedge .. \wedge c \in X \rightarrow z \in X$$

In the vocabulary of Higman, these are called *identical implications*.

In what follows, we will be given such a collection of formulas, and we will look at the set of all X that satisfy all formulas of the collection.

Definition 1.1. *Let S be a recursively enumerable set of finite sequences of elements of I.*

A word $x \in \{0,1\}^I$ satisfies S if for all $(n_0, n_1, \dots n_k) \in S$,

$$x_{n_1} = 1 \wedge x_{n_2} = 1 \wedge \dots x_{n_k} = 1 \rightarrow x_{n_0} = 1$$

Equivalently, a set $X \subseteq I$ satisfies S if for all $(n_0, n_1, \dots n_k) \in S$:

$$n_1 \in X \wedge n_2 \in X \wedge n_k \in X \rightarrow n_0 \in X$$

The quasivariety *V defined by S is the set of all words x (or all subsets $X \subseteq I$) that satisfy S.*

The fact that S is recursively enumerable is not mandatory, but happens in all interesting examples. This assumption can be dropped in almost all theorems, to obtain relativized versions of the theorems, by replacing all statements of the form $X \leq_e Y$ by $X \leq_e Y \oplus S$, where $\oplus$ is the disjoint union: $A \oplus B = \{(0,x), x \in A\} \cup \{(1,x), x \in B\}$.

Before giving more properties of quasivariety, we will give a few alternate definitions.

First it is easy to see from the definitions that V can be given the structure of a topological space, by inheriting the natural (product/Tychonoff) topology on $\{0,1\}^I$. For this topology V is topologically closed, and even compact. As the set S of formulas that define V is recursively enumerable, V is actually effectively closed (i.e. V is a Π^0_1 class) [11, 12].

Definition 1.2. *A set $X \subseteq \{0,1\}^I$ is effectively closed if there exists a recursively enumerable set $\mathcal{F} = \{f^i\}_{i \in \mathbb{N}}$ of partial finite maps (i.e. $f^i \in \{0,1\}^{F_i}$ with F_i finite), s.t. X are exactly the points of $\{0,1\}^I$ that disagree with every element of $\mathcal{F}$:*

$$x \in X \iff \forall i \in \mathbb{N}, \exists j \in F_i, x_j \neq f^i_j$$

Fact 1. *V is effectively closed.*

Proof. For each $(n_0, n_1 \dots n_k) \in S$, consider the map f defined by $f_{n_1} = f_{n_2} = \dots f_{n_k} = 1$ and $f_{n_0} = 0$. As S is recursively enumerable, the set $\mathcal{F}$ of all functions we obtain this way is recursively enumerable, and proves that V is effectively closed. □

The quasivariety V has also the structure of a complete semi-lattice, as evidenced by the following easy facts:

Fact 2. *Let V be a quasivariety. Then any intersection of elements of V is again in V. In particular:*

- *V contains a minimal element, the intersection of all elements of V.*
- *V contains a maximal element, the set I itself.*
- *For any set $Y \subseteq I$ there exists a smallest element X of V that contains Y.*

Note that a complete semi-lattice is also a complete lattice, where we define the meet of X and Y to be the smallest element of V that contains $X \cup Y$.

This is a characterization in the following sense:

Theorem 1 (Alternate definition 1)**.** *An effectively closed set $S \subseteq \{0,1\}^I$ is a quasivariety iff it contains I and is closed under (finite) intersections (if $X \in S$ and $Y \in S$ then $X \cap Y \in S$).*

We defer the proof of this theorem to the appendix, to not deviate from the narrative.

Note however that this characterization does not mean that we are actually investigating complete lattices that are effectively closed when seen as subsets of $\{0,1\}^I$. Indeed we are not interested in the lattice V itself but in the pair $(\{0,1\}^I, V)$, i.e. in how V behaves as a subset of the surrounding set $\{0,1\}^I$.

1.2 The closure operator

We will now give another characterization from the point of view of deductive systems.

Definition 1.3. *For a quasivariety V over a set I, the closure operator $\mathcal{C}$ associated to V is the map from $\{0,1\}^I$ to V that sends a set R to the smallest set in V containing R.*

This is well defined due to Fact 2. Intuitively, if we see R as a set of axioms, then $\mathcal{C}(R)$ is the set of all consequences of R.

The vocabulary "closure operator" comes from the fact that this indeed makes I a closure space:

Definition 1.4 ([31])**.** *A pair (I, C), where C is a map from $\{0,1\}^I$ to $\{0,1\}^I$ is a closure space if*

- *For all $R \subseteq I$, $R \subseteq C(R)$.*
- *C is idempotent: For all $R \subseteq I$, $C(C(R)) = C(R)$.*
- *C is monotone: For all $A, B \subseteq I$, if $A \subseteq B$ then $C(A) \subseteq C(B)$.*

A closure space is a Tarski space[41, 40] if additionally I is countable and C is finitary: For all $R \subseteq I$, if $x \in C(R)$ then there exists a finite $R' \subseteq R$ s.t. $x \in C(R')$.

To be accurate Tarski [41] assumes further properties from the space, in particular that the set I itself is finitely presented (see below for what it means). All of our examples satisfy this assumption, and many, but not all, of our theorems, have it as a hypothesis.

It is customary in logic to write $X \models y$ instead of $y \in C(X)$ and we will use this notation in a few proofs, in particular for the quasivariety of theories.

Fact 3. *Let V be a quasivariety, and $\mathcal{C}$ its closure operator. Then $(I, \mathcal{C})$ is a Tarski space.*

This gives another definition:

Theorem 2 (Alternate Definition 2)**.** *Let (I, C) be a Tarski space and V the image of C. V is a quasivariety iff I is recursive and V is effectively closed.*

Proof. One direction is easy. Now suppose I is recursive and V is effectively closed. First, by idempotency, V is exactly the set of points X s.t. $C(X) = X$.

By the first axiom of closure space, $C(I) \supseteq I$ and therefore $C(I) = I$ and $I \in V$.

Let $X, Y \in V$, i.e. $C(X) = X$ and $C(Y) = Y$. Then by monotonicity, $C(X \cap Y) \subseteq C(X) = X$, and similarly $C(X \cap Y) \subseteq Y$. We therefore deduce $C(X \cap Y) \subseteq X \cap Y$ and finally $C(X \cap Y) = X \cap Y$, therefore $X \cap Y \in V$.

The result then follows from the previous theorem.

□

Note that the map $\mathcal{C}$ is usually not recursive, but as we will see in the next section, it is given by a enumeration operator.

This is the last time we mention the notion of closure space, and we will use vocabulary relevant to algebra rather than topology in the following. Table 1 gives a correspondence between the vocabularies.

2 Examples

Before proceeding to the main definitions and theorems, we will give a few running examples. The first three examples are central.They are actually examples of families of quasivarieties, as each of them is parametrized by some finite set (a signature, or a finite alphabet).

Quasivarieties V	Tarski spaces
X is a point of V	X is a (deductive) system
R is a presentation of $X \neq I$	R is consistent
$X \in V$ is maximal, $X \neq I$	X is maximally consistent
X is finitely presented	X is finitely axiomatizable/X is compact
I is finitely presented	(I, V) is compact

Table 1: Dictionary between quasivarieties (as defined in this paper) and Tarski spaces

2.1 The quasivariety $V_{FO,\tau}$ of first-order theories

Let τ be a finite signature. The set of all first order theories over the signature τ can be given a structure of a quasivariety $V_{FO,\tau}$. (Here we define a theory as the set of all logical consequences of some (possibly empty) set of axioms).

Indeed, let I be the set of all formulas over the signature τ, and let S be the set of all formulas $(\phi_0, \phi_1, \dots \phi_k)$ s.t. ϕ_0 is a consequence of $\phi_1 \dots \phi_k$. This set S is indeed recursively enumerable. This defines a quasivariety $V_{FO,\tau}$.

Any theory is immediately a point of the quasivariety $V_{FO,\tau}$, and by Gödel completeness theorem for first order logic, all points of the quasivariety V are indeed theories (closed under logical consequence): if ϕ is a consequence of the formulas of $X \in V_{FO,\tau}$, then ϕ is a consequence of finitely many formulas of X, and therefore in X itself.

The quasivariety $V_{FO,\tau}$ of theories contains two particular points: the point X consisting of all tautologies and the point $X = I$ of all formulas (i.e. the inconsistent theory).

Notice that we are defining different quasivarieties $V_{FO,\tau}$ depending on the signature τ. As we discuss computability, we assume here that τ is finite. The case where τ is countably infinite can be handled with some care.

If $\mathcal{F}$ is a fragment of the first-order language (for example if $\mathcal{F}$ is the set of all universal sentences), it is also possible to restrict the quasivariety to $V_{\mathcal{F},\tau}$ by restricting to formulae in $\mathcal{F}$. This example may be useful later on.

2.2 The quasivariety V_{sym} of subshifts

Let Σ be a finite alphabet. A subset Y of $\Sigma^{\mathbb{Z}}$ is called a subshift [28] if it is topologically closed and invariant under translation. Y is entirely characterized by the set X of all *forbidden* words, i.e. finite words that do not appear in any word of Y. For example if $\Sigma = \{0, 1\}$ and $Y = \{\dots 000 \dots, \dots 00100 \dots\}$ is the set of all words with at most one occurence of the symbol 1, then the set of forbidden words X of Y is exactly the set of all words that contains at least two occurences of the symbol 1.

The quasivariety of all subshifts over Σ will not be given with subshifts as points, but with forbidden languages as points. This is of course equivalent.

A set X of words over Σ is the forbidden language of a subshift if it is extensible and factorial, that is:

- For any letter a, if $w \in X$ then $aw \in X$
- For any letter a, if $w \in X$ then $wa \in X$
- If $wa \in X$ for all letters a, then $w \in X$.
- If $aw \in X$ for all letters a, then $w \in X$.

By taking all these Horn formulas as our set S, we see that the set of of subshifts over Σ is a quasivariety V_{sym} defined over the set $I = \Sigma^\star$, the set of all finite words.

The quasivariety V_{sym} of subshifts contains two particular points: the point $X = \emptyset$ (which corresponds to the subshift $Y = \Sigma^{\mathbb{Z}}$) and the point $X = \Sigma^\star$ (which corresponds to the subshift $Y = \emptyset$).

Similar definitions can be given for higher dimensional subshifts (i.e. subsets of $\Sigma^{\mathbb{Z}^d}$), which are again characterized by the set of finite patterns that do not appear in them.

2.3 The quasivariety V_{grp} of finitely generated groups

Let n be an integer. The set of all groups with n generators may be seen as a quasivariety. Indeed, a group G with n generators can be seen (up to isomorphism) as a quotient of the free group $\mathbb{F}_n$, or equivalently as a normal subgroup R of $\mathbb{F}_n$ (the subgroup R corresponds to the word problem of G, i.e. all combinations of generators of G that are equal to the identity).

Indeed, a set $X \subseteq \mathbb{F}_n$ is a normal subgroup of $\mathbb{F}_n$ (i.e. codes a group) if:

- If (nothing) then $1 \in X$.
- If $g \in X$ then $g^{-1} \in X$.
- If $g \in X, h \in X$ then $gh \in X$.
- For any h, if $g \in X$ then $hgh^{-1} \in X$.

For example, for $\mathbb{Z}^2 = \{a, b | ab = ba\}$, we have $a \notin X$, $b \notin X$ but $aba^{-1}b^{-1} \in X$, $aaba^{-1}b^{-1}a^{-1} \in X$, and more generally, X is exactly the set of words of the free group for which the number of occurences of a is equal to the number of occurences of a^{-1}, and the same for b and b^{-1}.

By taking all previous Horn formulas as our set S, we see that the set of of groups with n generators is a quasivariety V_{grp} defined over the set $I = \mathbb{F}_n$

The quasivariety contains two particular points: the point $X = \{1\}$ (which corresponds to the group $G = \mathbb{F}_n$) and the point $X = \mathbb{F}_n$ (which corresponds to the one-element group).

This particular quasivariety is usually called the space of *marked groups*, see Grigorchuck [18].

2.4 The quasivariety $V_{\mathbb{R}}$ of closed subsets of $\mathbb{R}$

As quasivarieties can be defined by closure operators, it is of course natural that topologies can be made into quasivarieties in our sense. The astute reader might for example see how to twist the definitions of the quasivariety of subshifts to obtain the quasivarieties of all closed sets of the cantor space $\{0,1\}^{\mathbb{N}}$.

We will look at another example, the set of all closed subsets of $\mathbb{R}$. Similarly to the example of subshifts, it will be more natural to see a closed subset by the set of intervals that do not intersect it.

More precisely let $I = \{(p,q) \in \mathbb{Q} \times \mathbb{Q}, p < q\}$. Elements (p,q) of I are to be understood as open intervals $]p,q[$.

Let K be a closed subset of $\mathbb{R}$. Then X is the set of all intervals that do not intersect K if and only if:

- If $]p,q[\in X$ and $r \in]p,q[$, then $]p,r[\in X$ and $]r,q[\in X$
- If $]p,r[\in X$ and $]s,q[\in X$ and $p \leq s < r$ then $]p,q[\in X$

The first property states that if $D_1 \subseteq D_2$ are intervals and D_2 does not intersect K, then D_1 does not intersect K. The second property states that if D_1 and D_2 are two overlapping intervals that do not intersect K, then $D_1 \cup D_2$ does not intersect K.

By taking all previous Horn formulas as our set S, we see that the set of all closed subsets of $\mathbb{R}$ is a quasivariety $V_{\mathbb{R}}$.

The quasivariety contains two particular points: the point $X = \emptyset$ (which corresponds to the closed set $\mathbb{R}$) and the point $X = I$ (which corresponds to the closed set $\emptyset$)

We note that this construction is quite general, and will work in many topological spaces as long as they are computable in some sense (TODO), by taking for I a suitable countable basis. We note however that the closed subsets of the Baire Space $\mathbb{N}^{\mathbb{N}}$ cannot be put in the natural way into a quasivariety, due to $\mathbb{N}^{\mathbb{N}}$ not being relatively compact.

3 Presentations

Definition 3.1. *A* presentation *of $X \in V$ is a set Y so that $X = \mathcal{C}(Y)$.*

X is finitely presented *if it admits a finite presentation. X is* recursively presented *if it admits a recursively enumerable presentation.*

The vocabulary of presentations comes from algebra. From the point of view of logic, we could see Y as an axiomatization of the theory X.

Example 3.1.

- *A finitely presented theory is usually called a finitely axiomatisable theory: It is a theory that can be given by finitely many axioms.*

  A recursively presented theory is usually called a recursively axiomatisable theory.

- *A finitely presented subshift is usually called a subshift of finite type: It is a subshift given by a finite list of forbidden words (or forbidden patterns, in higher dimensions). As an example the set Y of all biinfinite words over the alphabet $\{0,1\}$ where every symbol 0 is preceded by a symbol 1 is a subshift of finite type and given by the set of forbidden words $\{00\}$. The set Y of all biinfinite words with at most one symbol 1 is* not *of finite type.*

  A recursively presented subshift is usually called an effectively closed subshift.

- *Our vocabulary coincides with the vocabulary from group theory, so that finitely (resp. recursively) presented groups in the quasivariety of groups are exactly the finitely (resp. recursively) presented groups*

- *Finitely presented closed subsets of $\mathbb{R}$ are complements of finite union of open (finite) intervals with rational coefficients. Equivalently finitely presented closed subsets of $\mathbb{R}$ are of the form $]-\infty, r] \cup K \cup [s, +\infty[$ where K is a finite union of closed intervals (with rational coefficients) and $r, s \in \mathbb{Q}$. In particular the empty set is not finitely presented.*

  Recursively presented closed subsets of $\mathbb{R}$ are usually called Π^0_1 subsets of $\mathbb{R}$, or effectively closed sets.

In what follows, we are interested in what computability properties on a presentation of X transfer to X. It is well known for example that the set of true formulas of a finitely axiomatisable (or recursively axiomatisable) theory is recursively enumerable, but what happens if our theory is not finitely axiomatisable, or even worse, not recursively axiomatisable?

It turns out that the good notion to use in such a context is the notion of enumeration reducibility. While the formal definition is a bit cumbersome, the intuitive idea is as follows: A set A is enumeration-reducible to a set B if there is some procedure that can enumerate the elements of A given *any* enumeration of the elements of B. By an enumeration of B we mean any possible (typically non recursive) way to give elements of B one at a time. It is not required that the elements of A are enumerated in a specific order, and this order will actually usually depend on the order in which the elements of B are given.

Definition 3.2 ([17, 34])**.** *Let A, B two subsets of I.*

We say that A is enumeration reducible to B, written $A \leq_e B$, if there exists a partial recursive function $f : \mathbb{N} \times I \to P_f(I)$, where $P_f(I)$ is (a recursive encoding of) the finite subsets of I, so that:

$$x \in A \iff \exists n, f(n,x) \text{ is defined and } f(n,x) \subseteq B$$

When we want to emphasize the function f, we will say that A is enumeration-reducible via f, written $A \leq_e^f B$. (Note that A is uniquely defined given f and B).

To simplify notations, we will write "$f(n,x) \subseteq B$" as a shortcut for "$f(n,x)$ is defined and $f(n,x) \subseteq B$."

The intuition is that, for each n, $f(n,x)$ is a (finite) set that witnesses that $x \in A$. If $f(n,x)$ is defined for no value of n, there is no possible witness that $x \in A$.

Intuitively, if we are given the elements of B one at a time, we can determine if $x \in A$ by enumerating all (possibly infinitely many) possible sets of witness $f(n,x)$ and see if one of them contains only elements that we know are in B. If it is not the case, we wait until we get more elements of B, or more possible sets of witnesses.

In particular if $A \leq_e B$ with B recursively enumerable, then A is recursively enumerable.

Notice that, at any instant in an enumeration of B, we obtain some information that some elements are in B, but no information on which elements are not in B. Somehow, $A \leq_e B$ means that we are able to enumerate A given only *positive information* on B. In particular the complement $\overline{B}$ of B is usually not enumeration reducible to B. Similarly $A \leq_e B$ does not imply that $\overline{A} \leq_e \overline{B}$.

Proposition 1. *Let V be a quasivariety and $\mathcal{C}$ its corresponding closure operator.*

Then $\mathcal{C}(Y) \leq_e Y$ for any $Y \subseteq I$. More precisely there exists a partial recursive function f (depending only on V) so that $\mathcal{C}(Y) \leq_e^f Y$ for any $Y \subseteq I$.

In particular, a point X is enumeration-reducible to any of its presentations. In particular a finitely/recursively presented point is recursively enumerable (as a subset of I).

From the point of view of logic system, this means we can enumerate the set of all consequences of Y from an enumeration of Y.

In particular it means that the set X itself is the smallest presentation of the point X in terms of information content, as any other presentation can compute this particular presentation.

Example 3.2. *The set of valid formulas in finitely and recursively axiomatisable theories is recursively enumerable.*

Subshifts of finite type and effectively closed subshifts have a recursively enumerable set of forbidden words.

Proof. Let V be defined by a set S, and S' be the set of all statements of the form

$$a \in X \wedge b \in X \wedge .. \wedge c \in X \to z \in X$$

that are logical consequences of (finitely many) statements of S.

As S is recursively enumerable, it is easy to see that S' is also recursively enumerable, and of course define the same quasivariety[1]. Note that S' contains in particular all implications of the form "$a \in X \to a \in X$".

[1]It is easy to see that (if we drop the hypothesis for the base set S to be recursively enumerated) the set of all quasivarieties V over a set I may be given itself the structure of a quasivariety $\mathcal{V}$, where each quasivariety $V \in \mathcal{V}$ is identified with the set $S'(V)$ of all implications true in any point of the quasivariety. The map $S \to S'$ is of course the closure operator in this new quasivariety.

On input $a \in I$, consider all finite sets F s.t. the statement "$\bigwedge_{i \in F} i \in X \to a \in X$" is a statement of S'. As S' is recursively enumerable, we can recursively enumerate all such statements, i.e. there is a partial recursive function $f : I \times \mathbb{N} \to P_f(I)$ such that $f(a, \mathbb{N})$ is exactly the set of all possible F.

Then it is clear that $A \leq_e^f B$ iff $A = \mathcal{C}(B)$. □

Therefore there is a procedure f that, given any enumeration of any presentation of $X \in V$, gives an enumeration of X. This means that the closure operator, while not computable in a traditional sense, is computable as an enumeration operator.

With this theorem, we get a new definition of a quasivariety

Theorem 3 (Alternate Definition 3)**.** *Let I be a recursive set and $(I, \mathcal{C})$ is a Tarski space.*

Then the image of $\mathcal{C}$ is a quasivariety iff there exists f s.t. $\mathcal{C}(Y) \leq_e^f Y$ for all $Y \subseteq I$.

So quasivariety are closure spaces where the closure operator is computable as an enumeration operator.

Proof. By definition, recall that $A \leq_e^f B$ means that $x \in A \iff \exists n, f(n, x) \subseteq B$.

Consider the set S of all Horn formulas

$$\bigwedge_{j \in f(n,i)} j \in X \to i \in X$$

for all n, i whenever $f(n, i)$ is defined. S is clearly recursively enumerable and defines a quasivariety V. It remains to show that V coincides with the image of $\mathcal{C}$.

Suppose that X is in the image of $\mathcal{C}$, i.e. $X = \mathcal{C}(X)$ and therefore $i \in X \iff \exists n, f(n, i) \subseteq X$. Keeping only one direction of this equivalence, we get that for all n and all i, $[f(n, i) \subseteq X \Rightarrow i \in X]$. Therefore $X \in V$ by definition of the quasivariety V.

Conversely suppose that $X \in V$ and let $Y = \mathcal{C}(X)$. As $Y \leq_e^f X$ we know that $i \in Y \iff \exists n, f(n, i) \subseteq X$. If $i \in Y$, then there exists n s.t. $f(n, i) \subseteq X$ (in particular $f(n, i)$ is defined) which implies by definition of V that $i \in X$. We conclude $Y \subseteq X$, i.e. $\mathcal{C}(X) \subseteq X$ and therefore $\mathcal{C}(X) = X$ and X is in the image of $\mathcal{C}$. □

3.1 Craig's theorem

Craig's theorem is a classical theorem that states that recursively axiomatisable theories admit recursive presentations. This theorem is also valid for subshifts and groups, and can be obtained in the following way:

Definition 3.3. *A quasivariety V has redundant axioms if for any finite set S, there exist a set T disjoint from S s.t. $\mathcal{C}(S) = \mathcal{S}(T)$*

The definition implies that no finite set of axioms is necessary, as it can always be replaced by another equivalent set of axioms. This second set can be taken finite:

Proposition 2. *Suppose that $\mathcal{C}(S) = \mathcal{S}(T)$ for some finite set S. Then there exists a finite subset T' of T s.t. $\mathcal{C}(S) = \mathcal{C}(T)$*

Proof. Let $S = \{s_1 \dots s_n\}$. As the closure operator is finitary, every element s_i of S is in the closure of a finite subset T_i of T.

We now take $T' = \cup T_i$.

□

Example 3.3. *In the following examples, we will give, for any axiom $p \in I$, a infinite collection of disjoint finite sets, each of them being equivalent ot p. This obviously proves the result.*

- *The quasivariety of theories has redundant axioms. Indeed we can always replace an axiom ϕ by any axiom of the form $\overbrace{\phi \wedge \phi \wedge \dots \phi}^{n}$*
- *The quasivariety of groups has redundant axioms. Indeed we can always replace an axiom u by any axiom of the form vuv^{-1} for some v*
- *The quasivariety of subshifts has redundant axioms. Indeed we can always replace an axiom u by any set of axioms $\{uv, v \in \Sigma^n\}$ for some n.*
- *The quasivariety of closed subsets has redundant axioms. Indeed we can always replace an axiom $]p, q[$ by the two axioms $]p, r[,]s, q[$ for any $r \leq q$ and $s \geq p$*

Proposition 3. *Suppose that V is a quasivariety with redundant axioms. There exists a total recursive function g that given an axiom $p \in I$ and a finite set S outputs a set T s.t. $C(T) = C(p)$ and T is disjoint from S.*

Proof. Such a T exists by hypothesis. Notice that $EQ = \{(T, T')|C(T) = C(T')\}$ is recursively

enumerable. Therefore the recursive function g on input p and S only has to look at all possibles sets T that are disjoint from S until it finds one that is equivalent to p. □

Theorem 4 (Craig's theorem for quasivariety with redundant axioms)**.** *Suppose that V is a quasivariety with redundant axioms. If X is recursively presented ($X = C(R)$ for some recursively enumerable set R), then there exists a recursive set R' s.t. $X = C(R')$.*

Proof. Without loss of generality, we identify the core set I with the set of integers $\mathbb{N}$.

Suppose $R \neq \emptyset$ is the image of a total recursive function $R = \{f(n), n \in \mathbb{N}\}$. We inductively build a sequence of finite sets T_i by $T_{i+1} = g(f(i+1), [0, \max T_i])$.

That is, T_{i+1} is a finite set equivalent to $f(i+1)$ that uses only integers greater than any seen previously.

By construction $\bigcup T_i$ is a presentation of X. Furthermore, $\bigcup T_i$ is not only recursively enumerable, but recursively enumerable in an increasing order, which makes it recursive. $\square$

The trivial quasivariety (given by no Horn formulas) show this theorem is not true in general without any assumptions.

4 Maximal elements

We will show how some structural properties on some particular points of a quasivariety translate into computability properties. We start with maximality.

Definition 4.1. *An element X of a quasivariety V is maximal if $X \subseteq Y$, with $Y \in V$, implies $Y = X$ or $Y = I$.*

The case $X = I$ is a degenerate case that is usually not interesting in the applications.

Example 4.1. *In the quasivariety V of theories $X \subseteq Y$ means that X has less theorems than Y, i.e that Y is an extension of X. Therefore maximal points in the quasivariety V of theories are exactly the* complete *theories (plus the inconsistent theory).*

In the quasivariety V of subshifts, if X and Y are respectively the words that do not appear in subshifts S and T, then $X \subset Y$ means that $T \subseteq S$. Therefore maximal points in the quasivariety V of subshifts are called minimal *subshifts (plus the empty subshift).*

In the quasivariety V of f.g. groups with n elements, if X and Y are respectively the normal subgroups that code F and G then $X \subseteq Y$ means that G is a quotient of F. Therefore maximal points in this quasivariety are exactly the simple groups (plus the trivial group).

The case of the quasivariety V of closed subsets of $\mathbb{R}$ is similar to the case of subshifts, and we get that maximal points in this quasivariety are exactly the singletons $\{x\}$ (plus the empty set).

Our first theorem generalizes the theorem of Kuznetsov, without any hypothesis on the computability of the structure:

Theorem 5. *Let V be a quasivariety. If $X \in V$ is maximal then $\overline{X} \leq_e X$.*

Recall that $\overline{X}$ is the complement (in I) of X. A set X s.t. $\overline{X} \leq_e X$ is sometimes called a *total* set.

Before going to the (easy) proof, let us explain the significance of the theorem, and why it generalizes previous theorems. $\overline{X} \leq_e X$ means that, from any enumeration of X, one can compute some enumeration of $\overline{X}$: Given positive information on X (which elements are in X) we can get negative information on X (which elements are not in X).

This does not happen quite often. For an example, suppose that X is recursively enumerable: there is an algorithm that produces an enumeration of the elements of X. If $\overline{X} \leq_e X$, we get that $\overline{X}$ is also recursively enumerable, and therefore X is recursive. So the only recursively enumerable sets X for which $\overline{X} \leq_e X$ are the recursive sets. In fact, using a well known theorem of Selman[39], $\overline{X} \leq_e X$ can be reformulated as "For every oracle A, if X is recursively enumerable in A, then X is recursive in A".

Notice that existing theorems [27, 21, 9], when rephrased in our vocabulary are usually of the form "Assume that X is finitely presented. Then if X is maximal, it is computable". Our theorem is more general, as we do not have any assumption about the presentation of X (in particular X might not be recursively enumerable).

We will now give a brief idea of the proof using the language of logic. Suppose that X is a maximal set of consistent formulas. By definition, $\phi \notin X$ if adding ϕ to X is inconsistent. This is equivalent to saying that there exists a finite $Y \subseteq X$ s.t. $\{\phi\} \cup Y$ is inconsistent. To know whether $\phi \notin X$, it is therefore sufficient to list all possible Y s.t. $\{\phi\} \cup Y$ is inconsistent, and see if one of them is included in X. The proof mimics closely this idea.

Proof. If $X = I$ the result is obvious by taking $f(n, x)$ to be nowhere defined. Otherwise, let $a \notin X$.

Then $x \in \overline{X}$ iff the smallest point containing both X and x contains a.

Recall there is a function f so that $A \leq_e^f B$ iff B is a presentation of A.

Thus $x \in \overline{X}$ iff there exists n so that $f(n, a) \subseteq X \cup \{x\}$.

Let g be the partial recursive function defined by $g(n, x) = f(n, a) \setminus \{x\}$ whenever $f(n, a)$ is defined.

Then $x \in \overline{X}$ iff there exists n so that $g(n, x) \subseteq X$. □

Corollary 1. *Let V be a quasivariety. If X is finitely (or recursively) presented and maximal, then X is recursive.*

Example 4.2. *Let S be a complete theory. Then the set of formulas that are invalid in S is enumeration-reducible to the set of formulas that are valid in S. In particular, if S is finitely axiomatisable (or recursively axiomatisable), then the set of formulas that are valid in S is recursive [24].*

Let S be a minimal subshift. Then the set of words that appear in S is enumeration-reducible to the set of words that do not appear in S. In particular if S is a minimal subshift of finite type (or a minimal effectively closed subshift), the set of forbidden words of S is recursive. This theorem was first proven in [21, 3].

Let G be a f.g. simple groups . Then the complement of the word problem of G is is enumeration-reducible to the word problem of G. In particular if G is a finitely presented simple group, the word problem of G is recursive. This theorem was first proven in [9].

Let $x \in \mathbb{R}$. Then the set of open rational intervals $]p, q[$ that contains x can be enumerated from the set of open rational intervals that do not contain x.

In particular, if x is recursively presented, i.e. if $\{x\}$ is a Π^0_1 class, then x is recursive. See for example [10].

The astute reader may realise that in the case of first order logic there is actually an easier proof of the theorem: Indeed $\phi \notin X$ iff $\neg\phi \in X$. This means we have a stronger reduction: $\overline{X}$ is many-one reducible to X, and in this particular case, the reverse is also true: X is many-one reducible to $\overline{X}$. This does not hold in general. To see why, let's look at a variant of the quasivariety V_{FO}. If we look only at $\forall\exists$ formulas, we get the following: If T is a complete theory which is $\forall\exists$-axiomatizable, then we can enumerate the $\forall\exists$ formulas that are false from any enumeration of the $\forall\exists$ formulas that are true. This cannot be proven by using the $\neg$ operator as a magic wand, as the negation of a $\forall\exists$ formula is not a $\forall\exists$ formula.

In fact our theorem is best possible: If X is a maximal point in a quasivariety, the fact that $\overline{X} \leq_e X$ is the strongest statement we can prove in full generality on X. Indeed there is a converse: If a set X satisfies $\overline{X} \leq_e X$, then it is a maximal point in a suitable quasivariety:

Theorem 6. *Let $A \subseteq I$ with I recursive.*

Let $\overline{A} \leq_e A$. Then there exists a quasivariety V s.t. $A \in V$ and A is maximal.

Note that recent (unpublished) results of Ethan McCarthy show that we can take V to be the quasivariety V_{sym} of subshifts, in the sense that any set $\overline{A}$ s.t. $\overline{A} \leq_e A$ is enumeration-equivalent to (the language of) a minimal subshift.

Proof. Let $\overline{A} \leq_e^f A$. Thus $x \in \overline{A}$ iff there exists n s.t. $f(n,x) \subseteq A$.

Let V be the quasivariety defined by all axioms

$$x \in X \wedge \bigwedge_{i \in f(n,x)} i \in X \to j \in X.$$

for all $n \in \mathbb{N}, x, j \in I$ whenever $f(n,x)$ is defined.

A is in this quasivariety: Indeed, there is no $x \in A$ and n so that $f(n,x) \subseteq A$, thus all premises are false.

It is clearly maximal: Let $A \subseteq A'$ and $a \in A' \setminus A$. Then $a \in \overline{A}$ thus there exists n s.t. $f(n,x) \subseteq A \subseteq A'$ thus for all y, $y \in A'$, thus $A' = I$. □

Notice that the proof of Theorem 5 was nonuniform: We need to exhibit, for X maximal and $X \neq I$, some element a that is not in X, and this a might depend on X. In many examples, a can be chosen independently of X.

Theorem 7 (Uniform version)**.** *Let V be a quasivariety so that I (the whole set) is finitely presented. Then there exists a partial recursive function g so that if X is maximal, $X \neq I$, then $\overline{X} \leq_e^g X$.*

Proof. Let f be the partial recursive function so that $A \leq_e^f B$ iff B is a presentation of A.

By definition, there exists a finite set E s.t. any point containing all of E is equal to I. Say $E = \{e_1 \dots e_k\}$.

Thus $x \in \overline{X}$ iff for all $e \in E$, there exists n so that $f(n, s) \subseteq X \cup \{x\}$.

Let $g(n_1, n_2 \dots n_k, x) = \cup_{i \leq k} f(n_i, e_i) \setminus \{x\}$.

Then $x \in \overline{X}$ iff there exists $n_1 \dots n_k$ so that $g(n_1, n_2 \dots n_k, x) \subseteq X$.

□

Example 4.3. *Let V be the quasivariety of first order theories. $\{\exists x, x \neq x\}$ is a finite presentation of the inconsistent theory, i.e. I itself.*

Let V be the quasivariety of subshifts. $\{\epsilon\}$ is a finite presentation of the empty subshift, i.e. I itself: If we forbid the empty word ϵ to appear, we have essentially forbidden all words to appear.

Let V be the quasivariety of groups with n generators $g_1, g_2, \dots g_n$. Then $\{g_1, g_2, \dots g_n\}$ is a finite presentation of the trivial group, i.e. I itself.

As we saw earlier, the closed subset $\emptyset$ is not finitely presented in the quasivariety V of closed subsets of $\mathbb{R}$, as any finitely presented closed set contains neighborhoods of $\pm\infty$.

This means that, in the first three cases, the reduction is uniform. In particular there exists an algorithm that, given a minimal subshift of finite type S (resp. a finitely axiomatisable complete theory S, a f.g. simple group) computes the set of forbidden words of S (resp. the set of valid formulas in S, the word problem of G).

The reduction is not uniform in the last case: To enumerate the set of open rational intervals $]p, q[$ that contain x from the set of open rational intervals that do not contain x, we need an additional information about x, e.g. in the form of a finite interval that contains it. This phenomenon is well known in computable analysis.

We present here a slight generalization of the main theorem. Instead of requiring that X is maximal, we require that we have an exact description of all points above X:

Theorem 8. *Let $(S_n)_{n \in \mathbb{N}}$ be a recursively enumerable collection of finite subsets of I, and $(Y_n)_{n \in \mathbb{N}}$ be the points presented by S_n. Let $\mathcal{Y} = \{Y_n, n \in \mathbb{N}\}$.*

We say that X is maximal below $\mathcal{Y}$ if $X \notin \mathcal{Y}$, but every point larger than X is in $\mathcal{Y}$.

Then $\overline{X} \leq_e X$, uniformly.

Notice that it is not required that all elements of $\mathcal{Y}$ are larger than X. In the examples one is usually interested in a set $\mathcal{Y}$ of "well-known" points, and we look at all points that are above this set of well-known points.

In the theory of algebra, one would take for $\mathcal{Y}$ all finite algebras. Then a point X below $\mathcal{Y}$ is a point X that is not a finite algebra but for which every quotient of X is finite. A specific version of this theorem in this context was proven by Maltsev [30]: Every finitely generated and finitely presented algebra on which all congruences have finite index is recursive.

Example 4.4. *Let's call a subshift just-infinite if it is infinite but all its proper subshifts are finite. For example, the set of infinite words over the alphabet* $\{0,1\}$ *with at most one occurence of the symbol* 1 *is a just-infinite subshift. It is easy to see that a just-infinite subshift satisfy the property above. In particular, for a just-infinite subshift of finite type (or any effectively closed just-infinite subshift), the set of forbidden patterns is recursive.*

A group G *with generators* $a_1, \dots a_k$ *is finite iff there exists a size* p *s.t. all words of length* p *over the alphabet* $\{a_1^{\pm 1} \dots a_k^{\pm 1}\}$ *are equal to a word of smaller length. As a consequence, the set of all finite presentations of finite groups is recursively enumerable. A group for which all proper quotient are finite is called a* just-infinite *group. Said otherwise, a just-infinite group is maximal below the finite groups. We get from the previous theorem that if* G *is a just-infinite recursively presented group, then the word problem for* G *is recursive.*

Proof. Straightforward generalization of the previous theorem.

Let $(S_n)_n$ be the recursive collection of finite sets, and write $S_n = \{a_1^n, \dots, a_{h(n)}^n\}$, where $h(n)$ is the (computable) size of S_n.

Recall there is a function f so that $A \leq_e^f B$ iff B is a presentation of A.

Then $x \in \overline{X}$ iff the smallest point (or any point) containing $\overline{X} \cup \{x\}$ is one of the sets of $\mathcal{Y}$.

Thus $x \in \overline{X}$ iff $\exists p, \exists n_1 \dots n_{h(p)}, \cup_{i \leq h(p)} f(n_i, a_i^p) \setminus \{x\} \subseteq X$. □

5 Discriminable points

Discrimination is a generalization of maximal elements. The concept and the vocabulary comes from group theory, in particular [14]. The notion is already present in Kuznetsov[27], where the author defines a concept of a completely finitely presented algebra, which corresponds in our vocabulary to a point which is both finitely presented and finitely discriminated.

Definition 5.1. *Let* V *be a quasivariety.*

A set Y *is a discriminator for a point* $X \in V$ *if* $Y \cap X = \emptyset$ *and for every point* $X' \in V$ *s.t.* $X \subsetneq X'$*, we have* $X' \cap Y \neq \emptyset$*.*

X *is* finitely discriminable *if it admits a finite discriminator.* X *is* recursively discriminable *if it admits a recursively enumerable discriminator.*

So a discriminator Y is a set of objects that are not in X but s.t. every extension of X contains at least some element of Y.

Notice that by definition $\overline{X}$ is always a discriminator for X. If X is maximal, any *single* element in $\overline{X}$ acts as a discriminator for X. Said otherwise, every (nontrivial) maximal element is finitely discriminable.

When V is seen as a topological space, points X that are both finitely discriminable and finitely presented are isolated: If R is the presentation and Y the discriminator, then X is the only point of V that contains R and does not intersect Y. Isolated points have been particularly studied in the theory of groups [14].

Easy examples come from the following proposition:

Definition 5.2. *Let V be a quasivariety. A point $X \in V$ is* quasi-maximal *if there are only finitely many points in V above X.*

Proposition 4. *Quasi-maximal points are finitely discriminable.*

Proof. Choose for every point $X' \supsetneq X$ some $a_{X'} \in X' \setminus X$ and take $Y = \{a_{X'}, X' \supsetneq X\}$. □

There are examples of finitely discriminable points that are not quasi-maximal. See [25] for an example in the quasivariety of two-dimensional subshifts, or [14, Theorem 5.3] for an example in the quasivariety of groups.

On the other hand:

Proposition 5. *In the quasivariety of theories, finitely discriminable points are exactly the quasi-maximal points.*

Proof. Let T be a finitely discriminable theory. This means there exists $Y = \{\phi_1 \dots \phi_n\}$ s.t. no ϕ_i is in T and every extension of T contains some ϕ_i.

First, note that if $T \models \phi_i \to \phi_j$ for some $i \neq j$ then every extension that contains ϕ_i also contains ϕ_j so that $Y \setminus \{\phi_i\}$ is also a discriminator for T. We can therefore suppose wlog that if ϕ_i and ϕ_j are in Y, then $\phi_i \to \phi_j$ is not in T unless $i = j$.

We want to prove that T has only finitely many extensions. It is sufficent to prove that T has only finitely many *complete* extensions. Indeed, a theory is entirely characterized by its set of complete extensions: If T_1 and T_2 are two different consistent theories there exists φ s.t. $\varphi \in T_1$ and $\varphi \notin T_2$ (or conversely). As T_2 does not prove φ, the theory $T_2 \cup \{\neg\varphi\}$ is not inconsistent and therefore has a complete extension. This complete extension cannot be an extension of T_1.

First we examine the formula $\vartheta = \neg\phi_1 \vee \neg\phi_2 \vee \dots \neg\phi_n$. Suppose that $\vartheta \notin T$. Then by discriminability there exists i s.t. $T \cup \{\vartheta\} \models \phi_i$. But then $T \models \neg\phi_i \to \phi_i$ and therefore $T \models \phi_i$, which is impossible by discriminability. Therefore $\vartheta \in T$, that is $T \models \neg\phi_1 \vee \neg\phi_2 \vee \dots \vee \neg\phi_n$.

Next, fix some formula $\phi_i \in Y$, and let ψ be any formula. We look at $\phi_i \vee \psi$. There are two cases:

- $\phi_i \vee \psi \in T$. Therefore $T \cup \{\neg\phi_i\} \models \psi$.
- Otherwise by discriminability there exists j s.t. $T \cup \{\phi_i \vee \psi\} \models \phi_j$. This means that $T \cup \{\phi_i\} \models \phi_j$ and by our supposition this implies $\phi_i = \phi_j$. Therefore $T \cup \{\phi_i \vee \psi\} \models \phi_i$. In particular $T \models \psi \to \phi_i$ and therefore $T \cup \{\neg\phi_i\} \models \neg\psi$

We have therefore proven that for any formula ψ, either $T \cup \{\neg\phi_i\} \models \psi$ or $T \cup \{\neg\phi_i\} \models \neg\psi$. Therefore $T \cup \{\neg\phi_i\}$ is an axiomatisation of a complete theory (or is inconsistent).

This proves the result: if T' is a complete extension of T, then by the first point, some $\neg\phi_i$ should be true in T'. As $T \cup \{\neg\phi_i\}$ is an axiomatisation of a complete theory, this means that T' is actually the closure of $T \cup \{\neg\phi_i\}$. Therefore there are at most n complete extensions of T. □

We now go to the generalization of the theorem to recursively discriminable points:

Theorem 9. *Let V be a quasivariety and X a point in V. If X is recursively discriminable, then $\overline{X} \leq_e X$.*

Corollary 2. *If X is quasi-maximal and recursively presented, then X is recursive.*

This corollary was first obtained for subshifts in [38].

Proof. Let Y be the discriminator. Now $x \in \overline{X}$ iff the point presented by $X \cup \{x\}$ contains some element of Y.

Recall again that there exists a partial recursive function f s.t. $A \leq_e^f B$ iff B is a presentation of A. Thus $x \in \overline{X} \iff \exists y \in Y, \exists n \in N, f(n, y) \subseteq X \cup \{x\}$.

Let $Y = (y_m)_{m \in \mathbb{N}}$ a recursive enumeration of Y.

We now define $g(n, m, x) = f(n, y_m) \backslash \{x\}$. Then $x \in \overline{X} \iff \exists n, m, g(n, m, x) \subseteq X$. □

Corollary 3. *Let V be a quasivariety and X a point of V.*

X is recursive iff it is recursively presented and recursively discriminable.

This corollary was first obtained for groups in [14].

Proof. If X is recursive, then X is a presentation of X which is recursive, and $\overline{X}$ is a discriminator for X which is recursive.

Conversely, if X is recursively presented by Y, then X is recursively enumerable, as $X \leq_e Y$. As X is recursively discriminated, $\overline{X} \leq_e X$, thus $\overline{X}$ is recursively enumerable, and X is recursive. □

6 Difference between X and $\overline{X}$

All results in the previous sections show that all consequences of maximality or quasi-maximality are of the same form: from any enumeration of X we can compute some enumeration of $\overline{X}$. As the converse is usually not true, this means that in all these examples we get strictly more information from X than from $\overline{X}$.

In this last section, we will try to explain what is this information. For this we need to ask a bit more from the reduction:

Proposition 6. *Let $X \in V, X \neq I$ s.t. $\overline{X} \leq_e X$. Then there exists a total recursive function f s.t. $\overline{X} \leq_e^f X$.*

(That is, f can be chosen total).

Proof. $\overline{X} \leq_e^g X$ for some partial recursive function g. As g is partial recursive, there exists a total recursive function $h : \mathbb{N} \times \mathbb{N} \times I \to \{0, 1\}$ s.t $g(n, x)$ is defined iff there exists m s.t. $h(n, m, x) = 1$.

Let $a \notin X$. Define $f(n,m,x)$ by $f(n,m,x) = \{a\}$ if $h(n,m,x) = 0$ and $f(n,m,x) = g(n,x)$ otherwise. f is total and $x \in \overline{X} \iff \exists n,m\ f(n,m,x) \subseteq X$. □

Notice that if I is finitely presented by a set A, we can replace $\{a\}$ by A and obtain a uniform version of the theorem, i.e. f does not depend on X.

Definition 6.1. *Let $\overline{X} \leq_e^f X$ with f total. Let $g(x) = \min\{n \in \mathbb{N} | f(n,x) \subseteq X\}$.*

g is a partial map, as it is defined only on $\overline{X}$. We identify g with $G = \{(x, g(x)) | g(x) \textit{is defined}\}$

As $f(n,x)$ is the list of all possible witness that $x \in \overline{X}$, $g(x)$ therefore represents the very first real witness in the enumeration. Note that g depends of course on the enumeration f. We will now see that a bound on g is exactly the information we need to recover X from $\overline{X}$.

Proposition 7. *Let $\overline{X} \leq_e^f X$ with f total. Then $G \leq_e X$.*

Proof. Suppose we are given an enumeration of X, and at some point we conclude that $x \in \overline{X}$ because $f(n,x) \subseteq X$ for some n, and all elements of $f(n,x)$ are currently known to be in X. Then we know that $g(x) \leq n$. We then look at all sets $f(i,x)$ for $i < n$. At some point in our enumeration, we will know the status of all points in $\cup_{i<n} f(i,x)$, either because they were enumerated in X, or we were able to prove that they are in $\overline{X}$. Thus we will be able to determine the exact value of $g(x)$. □

Proposition 8. *Let h be a total function that dominates g: $h(x) \geq g(x)$ whenever $g(x)$ is defined. We identify h with the total set $H = \{(x, h(x)), x \in I\}$*

Then $X \leq_e H \oplus \overline{X}$

Proof. $x \in X \iff \forall n \leq h(x), f(n,x) \cap \overline{X} \neq \emptyset$. □

It is important to note that it is not g itself which is important, but any upper bound on g. Note also that the reduction in the theorem is stronger than enumeration reducibility.

The previous propositions may seem uninteresting, as the exact definition of g depends on the particular operator f that was used to prove that $\overline{X} \leq_e X$, and there does not seem to be any canonical way to associate some map g to every set X that satisfies that $\overline{X} \leq_e X$.

However, in many cases, we can say more. In particular, in the interesting case of a quasivariety V where I is finitely presented, f does not depend on X but only on V, so that g can be defined indeed in a canonical way

Example 6.1. *A minimal subshift S has a* quasiperiodicity *function (also called uniform recurrence function): There exists a function $g(n)$ s.t. every word w of size n that appear in S is contained in every word of size $g(n)$ that appear in S. For minimal subshifts, the previous propositions may thus be interpreted this way:*

- *The set of words that appear can be obtained from an enumeration of the set of words that do not appear*
- *The quasiperiodicity function can be obtained from an enumeration of the set of words that do not appear*
- *The set of words that do not appear can be obtained from an enumeration of both the quasiperiodicity function and the set of words that appear.*

In particular, a minimal subshift of finite type has a recursive quasiperiodicity function. This theorem was first established in [4]. This theorem is optimal in the sense that it is easy to find minimal subshifts which have a recursive quasiperiodicity function but which are not recursive, and minimal subshifts for which the set of words that appear is recursively enumerable but not recursive.

We now introduce a notion which seems new in group theory for simple groups:

Definition 6.2. *Let S be a finitely generated, simple group, with generators $a_1 \dots a_k$.*

For a word w over the generators $a_1 \dots a_k$, let $g(w)$ be the smallest p s.t. all generators a_i can be written as products of less than p elements of the form hwh^{-1} or $hw^{-1}h^{-1}$ for $h \in S$, and each h is a product of less than p generators. $g(w)$ is defined only when w is not the identity element on S.

We now define $g(n) = \max\{g(w) | w \in B_n, w \neq 1\}$, where B_n is the set of elements of G that can be written as a product of less than n generators.

$g(w)$ is well defined if $w \neq 1$. Indeed the set of all elements that can be written as above is a normal subgroup of S that is nontrivial (it contains w), and thus is equal to S.

Note that g depends on the choices of generators of S, but it is easy to see that different choices of G only changes the function upto a linear factor.

Then, in S:

- The complement of the word problem on S can be enumerated from an enumeration of the word problem on S.
- The function g can be computed from an enumeration of the word problem on S.
- The word problem on S can be enumerated from both the complement of the word problem on S and any bound t on g. Indeed $w = 1$ iff there exists a generator a s.t. all products of less than $t(|w|)$ terms of the form hwh^{-1}, where each h is the product of less than $t(|w|)$ generators, are different from a.

This new function g is therefore an equivalent of the quasi-periodicity function for subshifts.

Conclusion

A consequence of this work is the following: many results in algebra assert that if a finitely presented structure has some property P, then the structure is recursive. The usual way these results are done is by proving that having property P and being finitely presented imply that the structure is both recursively enumerable and co-recursively enumerable.

However these results can be divided in two parts:

- Either they are still valid when the structure is only recursively presented instead of finitely presented. In which case, as presented here, the result can usually be generalized to obtain a result that hold for any structure with property P. From the proof we also obtain that in this case a function g can be attached to each structure, that gives additional information about it. This is the case for example for minimal subshifts (where we can attach the *quasiperiodicity* function).

- Or they do not generalize to recursively presented structures, which means they really need the structure to be finitely presented to be able to prove that the structure is co-recursively enumerable. In which case it is not clear how these results can be generalized. It is for example the case for residually finite groups (finitely presented residually finite groups have a recursive word problem, but there are some recursively presented residually finite groups that are not [23]), or for the analog concept of subshifts whose periodic points are dense [22] (a subshift of finite type whose periodic points are dense has a recursive set of forbidden words, but this is not true for an effectively closed subshift whose periodic points are dense).

Open Questions

This article presents how a few results in logic and symbolic dynamics may be related once seen in the concept of universal algebra, and how they can be generalized for structures that are not recursively presented.

There exist other theorems which offer a striking similarity, but for which a general statement is not known, most proeminently Higman's embedding theorem and Boone-Higman's theorem. We focus here the discussion on the former theorem (The author claims he has a proof of an equivalent of the Boone-Higman theorem for subshifts, which will be found in a later paper).

Theorem 10 ([19])**.** *A finitely generated group can be embedded in a finitely presented groups iff it has a recursively enumerable set of defining relations.*

Theorem 11 ([26, 13])**.** *An arbitrary theory (with identity) is finitely axiomatisable using additional predicates iff it is recursively axiomatisable.*

Theorem 12 ([21], see also [2, 16])**.** *A subshift is the subaction of a (projection) of a shift of finite type iff it is effectively closed.*

The note by [27] also suggests an analog for universal algebras, and a similar theorem for semigroups also exist. Note also that the Relative Higman Embedding Theorem [20] also has an equivalent in the domain of subshifts [1].

Proofs of these theorems are tremendously combinatorial, as each proof needs to embed a Turing machine (or another computational device) into an algebraic system, and the methods to do this are quite different. However the fact remains that all these theorems have similar hypotheses and conclusions, so that either it is a striking coincidence, or something deep can be found here.

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A Appendix

A.1 Proof of Theorem 1

Theorem (Theorem 1)**.** *A Π^0_1 class $S \subseteq \{0,1\}^I$ is a quasivariety iff it contains I and is closed under (finite) intersections.*

Proof. One direction has already been stated as a fact above.

Now suppose S is a Π^0_1 class which contains I and is closed under (finite) intersection.

Let $\mathcal{F}$ be the collection of all partial maps $(f_i)_{i\in\mathbb{N}}$ where $f_i \in \{0,1\}^{F_i}$, with F_i finite, which disagree with every element of S.

By definition of a Π^0_1 class, $\mathcal{F}$ is recursively enumerable, and every element not in $\mathcal{F}$ agrees with at least one point of S.

Now let $\mathcal{F}'$ be the restriction of $\mathcal{F}$ to partial maps that takes value 0 in exactly one point. $\mathcal{F}'$ is also recursively enumerable, and the Π^0_1 class defined by $\mathcal{F}'$ is by definition a quasivariety V. It is clear that $S \subseteq V$, we now prove that they are equal.

For this, suppose $x \in V \setminus S$. Then there exists a map $f \in \mathcal{F} \setminus \mathcal{F}'$ that agrees with x.

As the whole set I is in S, no partial map taking only the value 1 can be in $\mathcal{F}$, hence f must take value 0 in at least one point.

Let A be the (possibily empty) set of positions where f takes value 1, and B the set of positions where f takes value 0. As $f \notin \mathcal{F}'$, $|B| \geq 2$.

For each $b \in B$ consider the map f_b defined on $A \cup \{b\}$ and taking value 1 on A and 0 on $\{b\}$.

Note that x agrees with every map f_b, and each such map takes value 0 in exactly one point. As a consequence, none of the map f_b is in $\mathcal{F}$ (otherwise it would be in $\mathcal{F}'$). Therefore, for each b, there exists a point $y_b \in S$ that agrees with f_b.

But then $\bigcap_b y_b$ is a point of S that agrees with f, a contradiction. $\square$