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Enumeration in Closure Spaces with Applications to Algebra

Emmanuel Jeandel

29th May 2015

Abstract

We show how the concept of enumeration reducibility from computability theory may be applied to obtain computability conditions on descriptions of substructures of algebras, more prominently on the language of forbidden words of a subshift, and on the word problem for finitely generated groups. Many of the results here are already known for recursively presented groups and effectively closed subshifts, but using enumeration reducibility we are able to generalize them to arbitrary objects. The proof is based on a topological framework similar to universal algebra, and generalizes in particular work by Kuznetsov and others.

There exist finitely presented groups with an uncomputable word problem [Nov55, Boo57a, Boo57b]. However finitely presented *simple* groups have a computable word problem [BH74].

Some subshifts of finite type have an uncomputable language [Rob71]. However *minimal* subshifts of finite type have a computable language [Hoc09, BJ08].

There is a pattern here. In fact, this pattern was already observed by Kuznetsov [Kuz58]: in the vocabulary of Maltsev [Mal61, Theorem 4.2.2]), every simple finitely presented algebra is constructive.

The goal of this paper is to reprove this statement using the vocabulary of topology rather than algebras. We will in particular be more general and drop the requirement that the algebra is finitely presented, or even recursively presented. We will prove for example that, for a (finitely generated) simple group, the complement of the word problem is enumeration reducible to the word problem. This implies the previous result: a finitely presented group has a recursively enumerable word problem, thus the complement of the word problem of a simple f.p. group, being enumeration reducible to a recursively enumerable set, is recursively enumerable, and thus the word problem is recursive.

This article uses two key concepts: the notion of a closure space [MP96] (more precisely of a Tarski space) and the notion of enumeration reducibility [FR59, Odi99].

Roughly speaking, a set A is enumeration reducible to a set B , if an enumeration of A can be obtained from any enumeration of B . We will see that this concept is of great interest in algebra for objects given by presentations, like

groups. We know for example that for every enumeration degree d , there exists a finitely generated group for which the word problem has enumeration degree d (the reduction is actually stronger, Dobritsa [Bel74, Theorem 2.4], see also [Zie76]) or that enumeration reducibility characterizes when a group G can be embedded into a group that is finitely presentend over a group H (C.F. Miller, see [HS88, Chapter 6]). This notion has also been used in a more general context by Belegradek [Bel96], and our first easy propositions about presentations are reflected in this article.

We note in passing that various other reductions have been used in conjunction with algebraic objects, in particular quasi-reducibility [DLN98] and Ziegler-reducibility [Zie80]. However many of our theorems have converses (see in particular Theorem 5) which suggest enumeration-reducibility is indeed the right notion in our context.

The set of all substructures of a given structure can be seen as a *closure space*, i.e. a space equipped with a closure operator C satisfying some natural properties. As we are more interested in algebra than in topology, we will use a vocabulary coming from algebra, and call this space a quasivariety V , by similarity with the corresponding notion from universal algebra. This notion is introduced in the first section.

In the case of finitely generated groups, this space is known as the space of marked groups [Gri85]. For groups, the points of the quasivariety V are (marked) groups, seen as normal subgroups of the free group. For shift spaces, the points of the quasivariety V correspond to subshifts, seen as factorial extensible languages of finite words.

A *presentation* of a point X is any set R so that X is the smallest point of V that contains R . The map from R to X is precisely the closure operator of the closure space. The next section introduces this concept and show computability relations between R and X .

The next three sections introduce various way of limiting the power of a point X in a quasivariety, and the consequence in terms of computability. The first one correspond to the concept of simple group, or minimal subshifts, and is coined “maximal points”. The next concept is borrowed from group theory [dCGP07] and provides a complete characterization of which objects are computable.

It is quite clear from the previous paragraphs that groups and subshifts are central examples of the theory. They will be running examples in all sections below. A third example from commutative algebra is also given, but less developed.

1 Definitions

Let I be a computable countable set, that we identify with the set of integers. In applications, I will be the set of words over a given finite alphabet, or the set of words in a free group.

In this article, we will always identify a subset $X \subseteq I$ and a point $x \in \{0, 1\}^I$.

We are interested in subsets $X \subseteq I$ that can be defined by some Horn formulas, i.e. by axioms of the type:

$$a \in X \wedge b \in X \wedge \dots \wedge c \in X \implies z \in X$$

In the vocabulary of Higman, these are called *identical implications*.

In what follows, we will be given such a collection of formulas, and we will look at the set of all X that satisfies such a collection.

Definition 1.1. *Let S be a recursive enumeration of finite sequences of elements of I .*

A word $x \in \{0, 1\}^I$ satisfies S if for all $(n_0, n_1, \dots, n_k) \in S$,

$$x_{n_1} = 1 \wedge x_{n_2} = 1 \wedge \dots \wedge x_{n_k} = 1 \implies x_{n_0} = 1$$

Equivalently, a set $X \subseteq I$ satisfies S if for all $(n_0, n_1, \dots, n_k) \in S$

$$n_1 \in X \wedge n_2 \in X \wedge \dots \wedge n_k \in X \implies n_0 \in X$$

The quasivariety V defined by S is the set of all words x (or all subsets $X \subseteq I$) that satisfy S .

The fact that the sequence S can be recursively enumerated is not mandatory, but happens in all interesting examples. This assumption can be dropped in almost all theorems, to obtain relativized versions of the theorems, by replacing all statements of the form $X \leq_e Y$ by $X \leq_e Y \oplus S$.

Example 1.1. *Let Σ be a finite alphabet. A subset Y of $\Sigma^{\mathbb{Z}}$ is called a subshift [LM95] if it is topologically closed and invariant under translation. Y is entirely characterized by the set X of all finite words that do not appear in any word of Y .*

In this way, the set of all subshifts over Σ is a quasivariety: A set X of words over Σ is (the forbidden language of) a subshift if it is extensible and factorial, that is:

- *For any letter a , if $w \in X$ then $aw \in X$*
- *For any letter a , if $w \in X$ then $wa \in X$*
- *If $wa \in X$ for all letters a , then $w \in X$.*
- *If $aw \in X$ for all letters a , then $w \in X$.*

The quasivariety V of subshifts contains two particular points: the point $X = \emptyset$ (which corresponds to the subshift $Y = \Sigma^{\mathbb{Z}}$) and the point $X = \Sigma^*$ (which corresponds to the subshift $Y = \emptyset$)

Example 1.2. Let n be an integer. The set of all groups with n generators may be seen as a quasivariety. Indeed, such a group G can be seen (up to isomorphism) as a quotient of the free group \mathbb{F}_n , or equivalently as a normal subgroup R of \mathbb{F}_n (the subgroup R corresponds to the word problem of G , i.e. all combinations of generators of G that are equal to the identity).

Indeed, a set $X \subseteq \mathbb{F}_n$ is a normal subgroup of \mathbb{F}_n (i.e. codes a group) if:

- If (nothing) then $1 \in X$
- If $g \in X$ then $g^{-1} \in X$
- If $g \in X, h \in X$ then $gh \in X$
- For any h , if $g \in X$ then $hgh^{-1} \in X$

For example, for $\mathbb{Z}^2 = \{a, b | ab = ba\}$, we have $a \notin X, b \notin X$ but $aba^{-1}b^{-1} \in X, aaba^{-1}b^{-1}a^{-1} \in X$, and more generally, X is exactly the set of words of the free group for which the number of occurrences of a is equal to the number of occurrences of a^{-1} , and the same for b and b^{-1} .

The quasivariety V of groups contains two particular points: the point $X = \{1\}$ (which corresponds to the group $G = \mathbb{F}_n$) and the point $X = \mathbb{F}_n$ (which corresponds to the one-element group).

This particular quasivariety is usually called the space of marked groups, see Grigorchuck [Gri85].

Example 1.3. The set of all ideals of $\mathbb{Q}[t]$ is a quasivariety (once some recursive coding of $\mathbb{Q}[t]$ into I is given)

Indeed, a set $X \subseteq \mathbb{Q}[t]$ is an ideal if

- If (nothing) then $0 \in X$
- If $a, b \in X$ then $a + b \in X$
- If $a \in X$ then $-a \in X$
- For all f , if $a \in X$ then $fa \in X$

The same can be done for any commutative ring as long as it is computable in some sense.

It is easy to see from the definitions that V can be given the structure of a topological space, by inheriting the natural (product/Tychonoff) topology on $\{0, 1\}^I$. As S is chosen to be computably enumerated, V is what is called a Π_1^0 set in computability theory [CR98, CR15]

It has also the structure of a complete semi-lattice, as evidenced by the following easy facts:

Fact 1.2. *Let V be a quasivariety. Any intersection of elements of V is again in V . In particular:*

- V contains a minimal element, the intersection of all elements of V
- V contains a maximal element, the set I itself.
- For any set $Y \subseteq I$ there exists a smallest element X of V that contains Y .

Note that a complete semi-lattice is also a complete lattice, where we define the meet of X and Y to be the smallest element of V that contains $X \cup Y$.

This is a characterization in the following sense:

Theorem 1. *A Π_1^0 class $S \subseteq \{0, 1\}^I$ is a quasivariety iff it contains I and is closed under (finite) intersections.*

We defer the proof of this theorem to the appendix, to not deviate from the narrative. Note however that this characterization does not mean that what we are investigating here are Π_1^0 complete lattices. What is interesting is not V in itself, but how V behaves as a subset of $\{0, 1\}^I$.

Let C be the map from $\{0, 1\}^I$ to V that sends a set R to the smallest set of V containing R . C is a *closure operator* in the sense that:

- $R \subseteq C(R)$
- $C(C(R)) = C(R)$
- $A \subseteq B \implies C(A) \subseteq C(B)$

This means V (or more precisely (I, V)) is a closure space, or a closure system [MP96]. It is actually a Tarski space [Tar30b, Tar30a]¹, as it is a finitarily closure space ($x \in C(R) \iff x \in C(S)$ for some finite $S \subseteq R$) and I is countable. The function C is called a *consequence operator* in the context of Tarski spaces. So what we are studying here are precisely computability properties of Π_1^0 -Tarski spaces. In these spaces, the map C is usually not computable, but as we will see in the next section, it is given by an enumeration operator.

This is the last time we mention the notion of closure space, and we will use vocabulary relevant to algebra rather than topology in the following. Table 1 gives a correspondence between the vocabularies.

¹[Tar30b] assumes further properties from the space, in particular that the set I itself is finitely presented (see below for definitions). All of our examples satisfy this assumption, and many, but not all, of our theorems, have it as an hypothesis.

Quasivarieties V	Π_1^0 -Tarski spaces (I, V)
X is a point of V	X is a (deductive) system
R is a presentation of $X \neq I$	R is consistent
$X \in V$ is maximal, $X \neq I$	X is maximally consistent
X is finitely presented	X is finitely axiomatizable/ X is compact
I is finitely presented	(I, V) is compact

Table 1: Dictionary between quasivarieties (as defined in this paper) and Tarski spaces

2 Presentations

Definition 2.1. A presentation of $X \in V$ is a set Y so that X is the smallest element of V containing Y . X is finitely presented if it admits a finite presentation. X is recursively presented if it admits a recursively enumerable presentation.

Example 2.1. A finitely (resp. recursively) presented group is exactly what is usually called a finitely (resp. recursively) presented group.

A finitely presented subshift is usually called a subshift of finite type. A recursively presented subshift is usually called an effectively closed subshift.

A finitely presented ideal is usually called a finitely generated ideal.

Presentations, and in fact everything below, is related to the notion of enumeration reducibility, that we define now:

Definition 2.2. Let A, B two subsets of I .

We say that A is enumeration reducible to B , written $A \leq_e B$, if there exists a computable function $f : I \times \mathbb{N} \rightarrow P_f(I) \cup \{\perp\}$, where $P_f(I)$ is (a recursive encoding of) the finite sets of I and $\perp \notin I$, so that

$$x \in A \iff \exists n, f(x, n) \subseteq B$$

When we want to emphasize the computable function f , we will say that A is enumeration-reducible via f , written $A \leq_e^f B$. (Note that A is uniquely defined given f and B).

The following statement says that a point X in a quasivariety is enumeration-reducible to any of its presentations. Thus any point has a smallest presentation (in terms of enumeration reducibility), which is X itself.

Proposition 2.3. Let V be a quasivariety.

Let Y be any presentation of X , then $X \leq_e Y$.

More precisely, there exists a computable function f (depending only on V) so that $X \leq_e^f Y$ iff Y is a presentation of X .

In particular a finitely/recursively presented point is recursively enumerable (as a subset of I).

(Finitely/recursively presented groups have a computably enumerable word problem. Subshifts of finite type and effectively closed subshifts have a computably enumerable set of forbidden words)

Proof. Let V be defined by a set S , and S' be the closure of S , i.e. all statements of the form

$$a \in X \wedge b \in X \wedge \dots \wedge c \in X \implies z \in X$$

that are logical consequences of statements of S .

It is easy to see that S' is also computably enumerable, and of course define the same variety V . Note that S' contains in particular all implications of the form “ $a \in X \implies a \in X$ ”.²

On input a , consider all finite sets F s.t. “ $\bigwedge_{i \in F} i \in X \implies a \in X$ ” is a statement of S' . All such finite sets can be enumerated, thus there is a computable function $f : I \times \mathbb{N} \rightarrow P_f(I)$ such that $f(a, \mathbb{N})$ is exactly the set of all F .

Then it is clear that $A \leq_e^f B$ iff B is a presentation of A . □

It is interesting to note that most of the statements we obtain have a converse, and this is the case here: A is enumeration reducible to B , iff B is a presentation of A in a suitable quasivariety.

Proposition 2.4. *Let $B \subseteq A$ so that $A \leq_e B$. Then there exists a quasivariety V such that B is a presentation of A .*

Proof. Let $A \leq_e^f B$.

Let V be the quasivariety defined by :

- If $f(n, x) \subseteq X$ then $x \in X$.

V is by definition a quasivariety. And it is clear by definition that the smallest point of V that contains B is A . □

3 Maximal elements

Definition 3.1. *An element X of a quasivariety V is maximal if $X \subseteq Y$, with $Y \in V$, implies $Y = X$ or $Y = I$.*

Example 3.1. *In the language of symbolic dynamics, this corresponds to minimal subshifts. In the language of groups, this corresponds to simple groups. In the language of ideals, this corresponds to a maximal ideal. In all three examples, X is implicitly supposed to be different from I .*

²It is easy to see that (if we drop the hypothesis for the base set S to be computably enumerated) the set of all quasivarieties V may be given itself the structure of a quasivariety VV , where each quasivariety $V \in VV$ is identified with the set $S(V)$ of all implications true in any point of the quasivariety. In this identification, $S(V) = S'$, and S above is just a particular presentation of V .

Our first (easy) theorem generalizes the theorem of Kuznetsov, without any hypothesis on the computability of the structure:

Theorem 2. *Let V be a quasivariety. If X is maximal then $\overline{X} \leq_e X$.*

(\overline{X} is the complement of X).

Proof. If $X = I$ it is clear. Otherwise, let $a \notin X$.

Then $x \in \overline{X}$ iff the smallest point containing both X and x contains a .

Recall there is a function f so that $A \leq_e^f B$ iff B is a presentation of A .

Thus $x \in \overline{X}$ iff there exists n so that $f(n, a) \subseteq X \cup \{x\}$.

Let $g(n, x) = f(n, a) \setminus \{x\}$.

Then $x \in \overline{X}$ iff there exists n so that $g(n, x) \subseteq X$. □

Corollary 3.2. *Let V be a quasivariety. If X is finitely (or recursively) presented and maximal, then X is computable.*

Indeed, X is recursively enumerable, as X is finitely presented, and \overline{X} is recursively enumerable, as it is enumeration reducible to X .

Theorem 3 (Uniform version). *Let V be a quasivariety so that I (the whole set) is finitely presented. Then there exists a computable function g so that if X is maximal, $X \neq I$, then $\overline{X} \leq_e^g X$.*

Proof. By definition, there exists a finite set A s.t. any point containing all of A is equal to I . Say $A = \{a_1 \dots a_k\}$.

Thus $x \in \overline{X}$ iff for all $a \in A$, there exists n so that $f(n, a) \subseteq X \cup \{x\}$

Let $g(n_1, n_2 \dots n_k, x) = \cup_{i \leq k} f(n_i, a_i) \setminus \{x\}$.

Then $x \in \overline{X}$ iff there exists $n_1 \dots n_k$ so that $g(n_1, n_2 \dots n_k, x) \subseteq X$. □

Corollary 3.3. *Let G be a finitely generated simple group. Then the complement of the word problem of G is enumeration reducible to the word problem of G . In particular [BH74]), if G is a finitely generated, finitely presented, simple group, then G has a computable word problem.*

Let S be a minimal subshift. Then the set of words that appear in S is enumeration-reducible to the set of words that do not appear in S . In particular [Hoc09, BJ08], if S is a minimal subshift of finite type, then S has a computable set of forbidden words.

In both cases, the problem is uniformly solvable: there exists an algorithm that, given a finite presentation of a simple group G (resp. a minimal subshift S) decides the word problem of G (resp. the set of admissible words of S).

Theorem 4. *Let $(S_n)_{n \in \mathbb{N}}$ be a computable collection of finite sets, and $(Y_n)_{n \in \mathbb{N}}$ be the points presented by S_n . Let $\mathcal{Y} = \{Y_n, n \in \mathbb{N}\}$*

We say that X is maximal below \mathcal{Y} if $X \notin \mathcal{Y}$, but every point larger than X is in \mathcal{Y} .

Then $\overline{X} \leq_e X$, uniformly.

An example of such a collection S_n is the set of all finite groups with k generators: A group G with generators a_1, \dots, a_k is finite iff there exists a size p s.t. all words of length p over the alphabet $\{a_1^{\pm 1} \dots a_k^{\pm 1}\}$ are equal to a word of smaller length.

We can use this to obtain a recursive list of finite presentations that contain only finite groups, and all presentations of finite groups arise this way. Thus the theorem apply to all groups for which all normal subgroups are of finite index (these groups are called *just infinite groups*).

More generally, every finitely generated and finitely presented algebra on which all congruences have finite index is computable [Mal61].

Proof. Straightforward generalization of the previous theorem.

Let $(S_n)_n$ be the computable collection of finite sets, and write $S_n = \{a_1^n, \dots, a_{h(n)}^n\}$, where $h(n)$ is the (computable) size of S_n .

Recall there is a function f so that $A \leq_e^f B$ iff B is a presentation of A .

Then $x \in \overline{X}$ iff the smallest point (or any point) containing $\overline{X} \cup \{x\}$ is one of the sets of \mathcal{Y} .

Thus $x \in \overline{X}$ iff $\exists p, \exists n_1 \dots n_{h(p)}, \cup_{i \leq h(p)} f(n_i, a_i^p) \setminus \{x\} \subseteq X$

□

As before, it is interesting to note that the main theorem has a converse:

Theorem 5. *Let $\overline{A} \leq_e A$. Then there exists a quasivariety V s.t. A is maximal.*

Proof. Let $\overline{A} \leq_e^f A$.

Thus $x \in \overline{A}$ iff there exists n s.t. $f(n, x) \subseteq A$.

Let V be the quasivariety defined by

- For all n , for all $x \in X$, for all $y \in I$, if $f(n, x) \subseteq X$ then $y \in X$.

A is in this variety: Indeed, there is no $x \in A$ and n so that $f(n, x) \subseteq A$, thus all premises are false.

It is clearly maximal: Let $A \subseteq A'$ and $a \in A' \setminus A$. Then $a \in \overline{A}$ thus there exists n s.t. $f(n, x) \subseteq A \subseteq A'$ thus for all $y, y \in A'$, thus $A' = I$.

□

4 Discriminable points

Discrimination is a generalization of maximal elements. The concept and the vocabulary comes from group theory, in particular [dCGP07]. The notion is already present in Kuznetsov [Kuz58], where the author defines a concept of a completely finitely presented algebra, which corresponds in our vocabulary to a point which is both finitely presented and finitely discriminated.

Definition 4.1. *Let V be a quasivariety.*

A set Y is a discriminator for a point X if $Y \cap X = \emptyset$ and for every point $X', X \subsetneq X', X' \cap Y \neq \emptyset$.

X is finitely discriminable if it admits a finite discriminator. X is recursively discriminable if it admits a recursively enumerable discriminator.

Theorem 6. *If X is recursively discriminable, then $\overline{X} \leq_e X$.*

Proof. Let Y be the discriminator.

Recall again that there exists a computable function f s.t. $A \leq_e^f B$ iff B is a presentation of A

Now $x \in \overline{X}$ iff the point presented by $X \cup \{x\}$ contains some element of Y.

Thus $x \in \overline{X} \iff \exists y \in Y, \exists n \in \mathbb{N}, f(n, y) \subseteq X \cup \{x\}$.

Let $Y = (y_m)_{m \in \mathbb{N}}$ a computable enumeration of Y.

We now define $g(n, m, x) = f(n, y_m) \setminus \{x\}$.

Then $x \in \overline{X} \iff \exists n, m, g(n, m, x) \subseteq X$. □

Corollary 4.2. *Let V be a quasivariety and X a point of V.*

X is computable iff it is recursively presented and recursively discriminated.

See [dCGP07] for the result for groups.

Proof. If X is computable, then X is a presentation of X which is recursive, and \overline{X} is a discriminator for X which is recursive.

Conversely, if X is recursively presented by Y, then X is recursively enumerable, as $X \leq_e Y$. As X is recursively discriminated, $\overline{X} \leq_e X$, thus \overline{X} is recursively enumerable, and X is computable. □

5 Difference between X and \overline{X}

Many of the previous results are of the same form: from an enumeration of X we can compute an enumeration of \overline{X} . It is easy to provide examples where the converse is not true. Hence X gives more information than \overline{X} . Here we try to quantify what is this information.

Definition 5.1. *Let $\overline{X} \leq_e^f X$. Let $g(x) = \min\{n \in \mathbb{N} \mid f(n, x) \subseteq X\}$.*

g is a partial map. We identify g with $G = \{(x, g(x)) \mid g(x) \text{ is defined}\}$

Proposition 5.2. *Let $\overline{X} \leq_e^f X$. Then $G \leq_e X$.*

Proof. Suppose we are given an enumeration of X, and at some point we conclude that $x \in \overline{X}$ because $f(n, x) \subseteq X$ for some n, and we have enumerated all elements of $f(n, x)$. Then we know that $g(x) \leq n$. We then look at all sets $f(i, x)$ for $i < n$. At some point in our enumeration, we will know the status of all points in $\cup_{i < n} f(i, x)$, either because they were enumerated in X, or we were able to prove that they are in \overline{X} . Thus we will be able to determine the exact value of $g(x)$. □

Proposition 5.3. *Let h be a total function that is greater than g: $h(x) \geq g(x)$ whenever x is defined.*

We identify h with the total set $H = \{(x, h(x)), x \in I\}$

Then $X \leq_e H \oplus \overline{X}$

Proof. $x \in X \iff \forall n \leq h(x), f(n, x) \cap \overline{X} \neq \emptyset.$ □

It is important to note that it is not g itself which is important, but any upper bound on g . Note also that the reduction in the theorem is stronger than enumeration reducibility.

The propositions are particularly interesting when we start from a quasivariety where I is finitely presented, so that for all maximal elements X , $\overline{X} \leq_e^f X$ for a given computable function f independent from X . In that case, it is possible to interpret the map g .

We start with subshifts. A minimal subshift S has a *quasiperiodicity* function (also called uniform recurrence function): There exists a function g s.t. every word w of size n that appear in S is contained in every word of size $g(n)$ that appear in S . For minimal subshifts, the previous theorems may thus be interpreted this way:

- The set of words that appear can be obtained from an enumeration of the set of words that do not appear
- The quasiperiodicity function can be obtained from an enumeration of the set of words that do not appear
- The set of words that do not appear can be obtained from an enumeration of both the quasiperiodicity function and the set of words that appear

We now introduce a similar concept for groups. Let S be a finitely generated, simple group, with generators $a_1 \dots a_k$. For $w \in S, w \neq 1$, let $g(w)$ be the smallest n s.t. all generators a_i can be written as products of less than n elements of the form hwh^{-1} or $hw^{-1}h^{-1}$ for $h \in S$.

g is well defined: Indeed the set of all elements that can be written this way is a normal subgroup of S that is nontrivial (it contains w), and thus is equal to S .

Note that g depends on the choices of generators of S , but it is easy to see that different choices of G only changes the function g upto a linear factor.

We now introduce another related function. Let S be a finitely generated, simple group, with generators $a_1 \dots a_k$. For a word w over the generators $a_1 \dots a_k$, let $g_1(w)$ be the smallest p s.t. all generators a_i can be written as products of less than p elements of the form hwh^{-1} or $hw^{-1}h^{-1}$ for $h \in S$, and each h is a product of less than p generators. $g_1(w)$ is defined only when w is not the identity element on S .

We now define $g'_1(n) = \max\{g_1(w) | w \in B_n, w \neq 1\}$, where B_n is the set of elements of G that can be written as a product of less than n generators.

Then, in S :

- The complement of the word problem on S can be enumerated from an enumeration of the word problem on S .
- The function g_1 can be computed from an enumeration of the word problem on S .

- The word problem on S can be enumerated from both the complement of the word problem on S and any bound t on g_1 . Indeed $w = 1$ iff there exists a generator a s.t. all products of less than $t(w)$ terms of the form hwh^{-1} , where each h is the product of less than $t(w)$ generators, are different from a .

Conclusion

A consequence of this work is the following: many results in algebra assert that if a finitely presented structure has some property P , then the structure is computable. The usual way these results are done is by proving that having property P and being finitely presented imply that the structure is both recursively enumerable and co-recursively enumerable.

However these results can be divided in two:

- Either they are still valid when the structure is only recursively presented instead of finitely presented. In which case, as presented here, the result can usually be generalized to obtain a result that hold for any structure with property P . From the proof we also obtain that in this case a function g can be attached to each structure, that gives additional information about it. This is the case for example for minimal subshifts (where we can attach the *quasiperiodicity* function) or simple groups (where we can attach a “simplicity” function)
- Or they do not generalize to recursively presented structures, which means they actually need the structure to be finitely presented to be able to prove that the structure is co-recursively enumerable. In which case it is not clear how these results can be generalized. It is for example the case for residually finite groups (finitely presented residually finite groups are computable, but there are some recursively presented residually finite groups that are not [Hub74]), or for the analog concept of subshifts whose periodic points are dense.

Open Questions

This article presents how a few results in group theory and symbolic dynamics may be related once seen in the concept of universal algebra, and how they can be generalized for structures that are not recursively presented.

There exist other theorems which offer a striking similarity, but for which a general statement is not known, most prominently Higman’s embedding theorem and Boone-Higman’s theorem. We concentrate here the discussion on the former theorem (The author claims he has a proof of an equivalent of the Boone-Higman theorem for subshifts, which will be found in a later paper).

Theorem 7 ([Hig61]). *A finitely generated group can be embedded in a finitely presented groups iff it has a recursively enumerable set of defining relations.*

Theorem 8 ([Kle52, CV58]). *An arbitrary theory (with identity) is finitely axiomatisable using additional predicates iff it is recursively axiomatisable.*

Theorem 9 ([Hoc09], see also [AS13, DRS10]). *A symbolic system is isomorphic to the subaction of a sofic shift iff it is effectively closed.*

The note by [Kuz58] also suggests an analog for universal algebras, and a similar theorem for semigroups also exist. Note also that the Relative Higman Embedding Theorem [HS88] also has an equivalent in the domain of subshifts [AS09].

Proofs of these theorems are tremendously combinatorial, as each proof needs to embed a Turing machine (or another computational device) into an algebraic system, and the methods to do this are quite different. However the fact remains that all these theorems have similar hypotheses and conclusions, so that either it is a striking coincidence, or something deep can be found here.

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A Appendix

Theorem (1). *A Π_1^0 class $S \subseteq \{0,1\}^I$ is a quasivariety iff it contains I and is closed under (finite) intersections.*

Proof. One direction has already been stated as a fact above.

Now suppose S is a Π_1^0 class which contains I and is closed under (finite) intersection.

Let \mathcal{F} be the collection of all partial maps $(f_i)_{i \in \mathbb{N}}$ where $f_i \in \{0,1\}^{F_i}$, with F_i finite, which disagree with every element of S .

By definition of a Π_1^0 class, \mathcal{F} is recursively enumerable, and every element not in \mathcal{F} agrees with at least one point of S .

Now let \mathcal{F}' be the restriction of \mathcal{F} to partial maps that takes value 0 in exactly one point. \mathcal{F}' is also recursively enumerable, and the Π_1^0 class defined by \mathcal{F}' is by definition a quasivariety V . It is clear that $S \subseteq V$, we now prove that they are equal.

For this, suppose $x \in V \setminus S$. Then there exists a map $f \in \mathcal{F} \setminus \mathcal{F}'$ that disagrees with x .

As the whole set I is in S , no partial map taking only the value 1 can be in \mathcal{F} , hence f must take value 0 in at least one point.

Let A be the (possibly empty) set of positions where f takes value 1, and B the set of positions where f takes value 0. As $f \notin \mathcal{F}'$, $|B| \geq 2$.

For each $b \in B$ consider the map f_b defined on $A \cup \{b\}$ and taking value 1 on A and 0 on $\{b\}$.

Note that x agrees with every map f_b , and each such map takes value 0 in exactly one point. As a consequence, none of the map f_b is in \mathcal{F} (otherwise it would be in \mathcal{F}'). Therefore, for each b , there exists a point $y_b \in S$ that agrees with f_b .

But then $\bigcap_b y_b$ is a point of S that agrees with f , a contradiction. \square