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# The three links Purcell swimmer and some geometric problems related to periodic optimal controls

Piernicola Bettiol, Bernard Bonnard, Laetitia Giraldi, Pierre Martinon and  
Jérémy Rouot

**Abstract.** The maximum principle combined with numerical methods is a powerful tool to compute periodic optimal controls. We present some examples, in particular the three links Purcell swimmer and discuss some geometric problems related to such computations.

**Keywords.** Periodic optimal controls, Purcell swimmer, First and second order necessary optimality conditions.

**AMS classification.** 49K15, 93C10, 70Q05.

## 1 Introduction

The objective of this article is to analyze optimal control problems with periodic optimal control solutions where the state space  $\mathbf{x}$  decomposes into  $(\mathbf{x}_1, \mathbf{x}_2)$  and  $\mathbf{x}_2$  is associated to a periodic motion.

It is based on an example from micro-swimming : the three links Purcell swimmer introduced in [12] where periodic motions correspond to the deformation of its segments [1],[7]. The literature [6], [14], [15] presents first and second order optimality conditions, mainly in the fully periodic case corresponding to the condition of the maximum principle and conjugate-focal points conditions which can be easily numerically implemented.

The main contribution of this article is to compute a nilpotent approximation in the three links swimmer which corresponds to a SR-problem in dimension 5. This approximation is used to integrate the extremal trajectories using elliptic functions in order to compare such trajectories to the true optimal trajectories which are numerically integrated using Bocop toolbox [2].

Further studies will lead to a complete analysis of the system including second order optimality conditions for the nilpotent model and the complete system.

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## 2 Motivating examples

### 2.1 Lyapunov-Poincaré theorem and periodic trajectories for Hamiltonian system

A first motivation for our study is the well known Lyapunov-Poincaré theorem that we recall next, see [4].

**Proposition 2.1.** *Let  $\dot{x} = \vec{H}(x)$  be a smooth Hamiltonian vector field defined in an open set  $U \subset \mathbb{R}^{2n}$ . Assume  $\vec{H}(x_0) = 0$  and let  $A$  be the linearised system at  $x_0 \in U$  :  $A = \frac{\partial \vec{H}}{\partial x}(x_0)$  with spectrum  $\sigma(A) = \{\pm i\omega, \lambda_3, \dots, \lambda_{2n}\}, \omega > 0$ . Assume  $\lambda_j / (i\omega) \notin \mathbb{Z}$  for  $j = 3, 4, \dots, 2n$ . Then there exists a one parameter family of periodic trajectories emanating from  $x_0$ . Moreover, converging to  $x_0$ , the periods converge to  $\frac{2\pi}{\omega}$ .*

The theorem due to Lyapunov-Poincaré is a standard tool in celestial mechanics to generate a one parameter family of periodic trajectories with small amplitudes. The proof is based on the continuation method.

Another fruitful method to compute periodic trajectories was introduced by Poincaré in the N-body problems and is called the direct method. Solutions  $t \rightarrow q(t)$  of the N-body problem correspond to minima of the action  $S(q(t)) = \int_{t_0}^{t_1} L(q, \dot{q}) dt$  where  $L$  is the Lagrangian difference between the kinetic energy  $T$  and the potential  $V$ . Periodic trajectories can be constructed as limit of minimizing sequences. In the N-body problem they were computed by Poincaré-Gordon in each class of homotopy of  $\mathbb{R}^{3N} \setminus S$  where  $S$  is the colliding variety.

Those two examples justify variational framework to compute periodic trajectories and more precisely family of periodic trajectories depending upon parameters, e.g. the period. From the control point of view, the Euler-Lagrange equation corresponds to the necessary optimality conditions of the maximum principle and to distinguish the optimal one in the one parameter family lead to the problem of second-order conditions for periodic trajectories which were discussed in a series of articles, see for instance [14], [15], [6], which were tested in some academic examples.

### 2.2 Non academic examples

The importance of the study of periodic optimal control in control engineering can rely on the following two problems.

#### Optimization of the production in photobioreactors

A model is analysed in details in [8] to optimize the production under the influence of day/night cycles and lead to compute periodic optimal solutions with one day pre-scribed period.

## Periodic optimal control and micro-swimming

A recent but very important problem concerning optimal control appears in micro-swimming problem for which the state of the system (i.e., the parameters associated with swimmer)  $\mathbf{x}$  decomposes into  $(\mathbf{x}_1, \mathbf{x}_2)$  where  $\mathbf{x}_1$  represents the displacement of the body and  $\mathbf{x}_2$  the periodic motions of the articulated limbs to achieve the displacement. In this article, we shall develop the case of the three links Purcell swimmer [1]. A simplified mathematical model is the so-called Heisenberg model in SR-geometry. The system is described by

$$\dot{\mathbf{x}} = \sum_{i=1}^2 u_i F_i(\mathbf{x}) \quad (2.1)$$

with  $\mathbf{x} = (x, y, z)$ ,  $F_1 = \frac{\partial}{\partial x} + y \frac{\partial}{\partial z}$ ,  $F_2 = \frac{\partial}{\partial y} - x \frac{\partial}{\partial z}$  and the cost to minimize is the energy  $\int_0^T (u_1^2 + u_2^2) dt$ . This problem is equivalent to the Dido problem and the optimal solutions can be easily computed [3]. Projections of the optimal solutions in the  $(x, y)$ -plane are circles which allow the system to move in the  $z$ -direction.

## 3 The three links Purcell swimmer

### 3.1 Mathematical Model

**Purcell's 3-link swimmer.** The 3-link swimmer is modeled by the position of the center of the second stick  $\mathbf{x} = (x, y)$ , the angle  $\theta$  between the  $x$ -axis and the second stick (the orientation of the swimmer). The shape of the swimmer defined by the two relative angles  $\alpha_1$  and  $\alpha_2$  (see Fig 1). We also denote by  $L$  and  $L_2$  the length of the two external arms and central link.

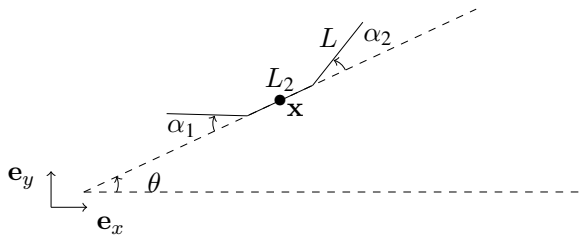


Figure 1. Purcell's 3-link swimmer.

**Dynamics via Resistive Force Theory.** We approximate the non local hydrodynamic forces exerted by the fluid on the swimmer with local drag forces depending linearly on the velocity. We denote by  $\mathbf{e}_i^{\parallel}$  and  $\mathbf{e}_i^{\perp}$  the unit vectors parallel and perpendicular to the  $i$ -th link, and we also introduce  $\mathbf{v}_i(s)$  the velocity of the point at distance  $s$  from

the extremity of the  $i$ -th link, that is

$$\begin{aligned}\mathbf{v}_1(s) &= \dot{\mathbf{x}} - \frac{L_2}{2}\dot{\theta}\mathbf{e}_2^\perp - s(\dot{\theta} - \dot{\alpha}_1)\mathbf{e}_1^\perp, \quad s \in [0, L], \\ \mathbf{v}_2(s) &= \dot{\mathbf{x}} - (s - \frac{L_2}{2})\dot{\theta}\mathbf{e}_2^\perp, \quad s \in [0, L_2], \\ \mathbf{v}_3(s) &= \dot{\mathbf{x}} + \frac{L_2}{2}\dot{\theta}\mathbf{e}_2^\perp + s(\dot{\theta} - \dot{\alpha}_2)\mathbf{e}_3^\perp, \quad s \in [0, L].\end{aligned}$$

The force  $\mathbf{f}_i$  acting on the  $i$ -th segment is taken as

$$\mathbf{f}_i(s) := -\xi \left( \mathbf{v}_i(s) \cdot \mathbf{e}_i^\parallel \right) \mathbf{e}_i^\parallel - \eta \left( \mathbf{v}_i(s) \cdot \mathbf{e}_i^\perp \right) \mathbf{e}_i^\perp, \quad (3.1)$$

where  $\xi$  and  $\eta$  are respectively the drag coefficients in the directions of  $\mathbf{e}_i^\parallel$  and  $\mathbf{e}_i^\perp$ . Neglecting inertia forces, Newton laws are written as

$$\begin{cases} \mathbf{F} = 0, \\ \mathbf{e}_z \cdot \mathbf{T}_x = 0, \end{cases} \quad (3.2)$$

where  $\mathbf{F}$  is the total force exerted on the swimmer by the fluid and  $\mathbf{e}_z = \mathbf{e}_x \wedge \mathbf{e}_y$ ,

$$\mathbf{F} = \int_0^L \mathbf{f}_1(s) ds + \int_0^{L_2} \mathbf{f}_2(s) ds + \int_0^L \mathbf{f}_3(s) ds, \quad (3.3)$$

and  $\mathbf{T}_x$  is the corresponding total torque computed with respect to the central point  $\mathbf{x}$ ,

$$\begin{aligned}\mathbf{T}_x &= \int_0^L (\mathbf{x}_1(s) - \mathbf{x}_1) \times \mathbf{f}_1(s) ds + \int_0^{L_2} (\mathbf{x}_2(s) - \mathbf{x}_1) \times \mathbf{f}_2(s) ds \\ &\quad + \int_0^L (\mathbf{x}_3(s) - \mathbf{x}_1) \times \mathbf{f}_3(s) ds.\end{aligned} \quad (3.4)$$

Since the  $\mathbf{f}_i(s)$  are linear in  $\dot{\mathbf{x}}$ ,  $\dot{\theta}$ ,  $\dot{\alpha}_1$ ,  $\dot{\alpha}_2$ , the system (3.2) can be rewritten as

$$\mathbf{A}(z) \cdot \begin{pmatrix} \dot{\mathbf{x}} \\ \dot{\theta} \end{pmatrix} - \mathbf{B}(z) \cdot \begin{pmatrix} \dot{\alpha}_1 \\ \dot{\alpha}_2 \end{pmatrix} = 0, \quad (3.5)$$

where  $z(t) := (\alpha_1, \alpha_2, x, y, \theta)(t)^T$ . The matrix  $\mathbf{A}$  is known as the "Grand Resistance Matrix" and is invertible (see [1]). Then the dynamics of the swimmer is finally expressed as an ODE system

$$\dot{\mathbf{z}}(t) = f(z, \dot{\alpha}_1, \dot{\alpha}_2) = F_1(z(t)) \dot{\alpha}_1(t) + F_2(z(t)) \dot{\alpha}_2(t), \quad (3.6)$$

where  $\begin{pmatrix} F_1(z) & F_2(z) \end{pmatrix} := \begin{pmatrix} \mathbb{I}_2 \\ \mathbf{A}^{-1}(z)\mathbf{B}(z) \end{pmatrix}$  with  $\mathbb{I}_2$  the  $2 \times 2$  identity matrix. The literal expression of the  $F_i$  is quite complicated (several pages).

### 3.2 Computations of the nilpotent approximation

Let us denote  $D = \text{span}\{F_1, F_2\}$ ,  $D_1 = D$ ,  $D_2 = \text{span}\{D_1 \cup [D_1, D_2]\}$  and  $D_3 = \text{span}\{D_2 \cup [D_1, D_2]\}$ . At the point  $x_0$  we have a  $(2, 3, 5)$ -distribution corresponding to the respective rank of  $D_1, D_2$  and  $D_3$ .

We write the control system as  $\dot{x} = Fu = \sum_{i=1}^2 u_i F_i$ .

#### Feedback group

The pseudo-group  $G = (\varphi, \beta)$  is defined by the actions :

- local diffeomorphism  $\varphi$  :

let  $\dot{x} = X(x)$  and  $x = \varphi(y)$ .

The action of  $\varphi$  on a vector field  $X$  is  $\dot{y} = (\varphi * X)(y) = \left[ \frac{\partial \varphi^{-1}}{\partial y} X \circ \varphi \right] (y)$ ,

$\varphi * F = (\varphi * F_1, \varphi * F_2)$ .

- feedback  $\beta$  :

$u = \beta(x)v$  where  $\beta$  is a  $2 \times 2$  invertible matrix.

The image of  $F$  is  $F\beta$ .

#### Computations

We introduce the following notations in 3.1  $x_1 = \alpha_1, x_2 = \alpha_2, (x_3, x_4) = \mathbf{x}, x_5 = \theta$  and we set  $L = 1, L_2 = 2, \xi = 1, \eta = 2$ .

The 2-jets of  $F_1$  et  $F_2$  at zero are

$$\begin{aligned} F_1(x) &= \frac{\partial}{\partial x_1} + \left( -\frac{1}{6} x_5 - \frac{4}{27} x_1 - \frac{2}{27} x_2 \right) \frac{\partial}{\partial x_3} \\ &+ \left( \frac{1}{6} - \frac{1}{12} x_5^2 - \frac{2}{27} x_5 x_2 - \frac{4}{27} x_5 x_1 - \frac{1}{27} x_1^2 - \frac{1}{27} x_1 x_2 - \frac{1}{36} x_2^2 \right) \frac{\partial}{\partial x_4} \\ &+ \left( -\frac{7}{27} + \frac{2}{81} x_1^2 - \frac{2}{81} x_1 x_2 - \frac{5}{162} x_2^2 \right) \frac{\partial}{\partial x_5} + O(|x|^3) \end{aligned}$$

$$\begin{aligned} F_2(x) &= \frac{\partial}{\partial x_2} + \left( \frac{1}{6} x_5 + \frac{4}{27} x_2 + \frac{2}{27} x_1 \right) \frac{\partial}{\partial x_3} \\ &+ \left( -\frac{1}{6} + \frac{1}{12} x_5^2 + \frac{4}{27} x_5 x_2 + \frac{2}{27} x_5 x_1 + \frac{1}{36} x_1^2 + \frac{1}{27} x_1 x_2 + \frac{1}{27} x_2^2 \right) \frac{\partial}{\partial x_4} \\ &+ \left( -\frac{7}{27} - \frac{5}{162} x_1^2 - \frac{2}{81} x_1 x_2 + \frac{2}{81} x_2^2 \right) \frac{\partial}{\partial x_5} + O(|x|^3) \end{aligned}$$

The normal form from [13] are

$$(\varphi * F_1) = \frac{\partial}{\partial x_1} + O(|x|^3), \quad (\varphi * F_2) = \frac{\partial}{\partial x_2} + x_1 \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4} + x_1^2 \frac{\partial}{\partial x_5} + O(|x|^3).$$

We introduce the weights 1 for  $x_1, x_2$ , 2 for  $x_3$  and 3 for  $x_4, x_5$ . If  $x_i$  is of order  $p$ ,  $\frac{\partial}{\partial x_i}$  is of order  $-p$  to define the nilpotent normal form of order  $-1$ .

We write  $\varphi = \varphi_N \circ \dots \circ \varphi_1 : \mathbb{R}^5 \rightarrow \mathbb{R}^5$ . At each step  $i$ , for  $i = 1, \dots, N$  of the computations we shall use  $N = 13$  steps :

- .  $x = (x_1, x_2, x_3, x_4, x_5)$  are the old local coordinates and  $y = (y_1, y_2, y_3, y_4, y_5)$  the new ones resulting from the change of variables  $\varphi_i$ ,
- .  $x_j = \varphi_i^{(j)}(y_j) : \mathbb{R} \rightarrow \mathbb{R}$  denoting the  $j^{\text{th}}$  component of  $\varphi_i$  for some  $j \in \{1, \dots, 5\}$ . The other components  $\varphi_i^{(k)}$ ,  $k \neq j$  are the identity transformations and are not precised.
- . the vector fields  $F_1, F_2$  resulting from the change of variables  $\varphi_i$ .

The computations steps are

Step 1.  $x_5 = y_5 - \frac{7}{27}y_1,$

$$\begin{aligned} F_1(y) &= \frac{\partial}{\partial y_1} + \left( -\frac{1}{6}y_5 - \frac{17}{162}y_1 - \frac{2}{27}y_2 \right) \frac{\partial}{\partial y_3} \\ &+ \left( \frac{1}{6} - \frac{1}{12}y_5^2 - \frac{2}{27}y_5y_2 - \frac{17}{162}y_5y_1 - \frac{37}{8748}y_1^2 - \frac{13}{729}y_1y_2 - \frac{1}{36}y_2^2 \right) \frac{\partial}{\partial y_4} \\ &+ \left( \frac{2}{81}y_1^2 - \frac{2}{81}y_1y_2 - \frac{5}{162}y_2^2 \right) \frac{\partial}{\partial y_5} + O(|y|^3) \\ F_2(y) &= \frac{\partial}{\partial y_2} + \left( \frac{1}{6}y_5 + \frac{4}{27}y_2 + \frac{5}{162}y_1 \right) \frac{\partial}{\partial y_3} \\ &+ \left( -\frac{1}{6} + \frac{1}{12}y_5^2 + \frac{4}{27}y_5y_2 + \frac{5}{162}y_5y_1 + \frac{31}{2187}y_1^2 - \frac{1}{729}y_1y_2 + \frac{1}{27}y_2^2 \right) \frac{\partial}{\partial y_4} \\ &+ \left( -\frac{7}{27} - \frac{5}{162}y_1^2 - \frac{2}{81}y_1y_2 + \frac{2}{81}y_2^2 \right) \frac{\partial}{\partial y_5} + O(|y|^3) \end{aligned}$$

Step 2.  $x_3 = y_3 - \frac{1}{6}y_5y_1 - \frac{17}{324}y_1^2 - \frac{2}{27}y_2y_1,$

$$\begin{aligned} F_1(y) &= \frac{\partial}{\partial y_1} \\ &+ \left( \frac{1}{6} - \frac{1}{12}y_5^2 - \frac{17}{162}y_5y_1 - \frac{37}{8748}y_1^2 - \frac{2}{27}y_5y_2 - \frac{13}{729}y_1y_2 - \frac{1}{36}y_2^2 \right) \frac{\partial}{\partial y_4} \\ &+ \left( \frac{2}{81}y_1^2 - \frac{2}{81}y_1y_2 - \frac{5}{162}y_2^2 \right) \frac{\partial}{\partial y_5} + O(|y|^3) \\ F_2(y) &= \frac{\partial}{\partial y_2} + \left( \frac{5}{81}y_1 + \frac{4}{27}y_2 + \frac{1}{6}y_5 \right) \frac{\partial}{\partial y_3} \\ &+ \left( -\frac{1}{6} + \frac{1}{12}y_5^2 + \frac{5}{162}y_5y_1 + \frac{31}{2187}y_1^2 + \frac{4}{27}y_5y_2 - \frac{1}{729}y_1y_2 + \frac{1}{27}y_2^2 \right) \frac{\partial}{\partial y_4} \\ &+ \left( -\frac{7}{27} - \frac{5}{162}y_1^2 - \frac{2}{81}y_1y_2 + \frac{2}{81}y_2^2 \right) \frac{\partial}{\partial y_5} + O(|y|^3) \end{aligned}$$

$$\text{Step 3.} \quad x_4 = y_4 + \frac{1}{6}y_1 - \frac{37}{3 \times 8748}y_1^3 - \frac{13}{1458}y_1^2y_2 - \frac{17}{324}y_5y_1^2 - \frac{1}{36}y_2^2y_1 - \frac{2}{27}y_5y_2y_1 - \frac{1}{12}y_5^2y_1,$$

$$F_1(y) = \frac{\partial}{\partial y_1} + \left( \frac{2}{81}y_1^2 - \frac{2}{81}y_1y_2 - \frac{5}{162}y_2^2 \right) \frac{\partial}{\partial y_5} + O(|y|^3)$$

$$\begin{aligned} F_2(y) &= \frac{\partial}{\partial y_2} + \left( \frac{5}{81}y_1 + \frac{4}{27}y_2 + \frac{1}{6}y_5 \right) \frac{\partial}{\partial y_3} \\ &+ \left( \frac{83}{8748}y_1^2 + \frac{17}{486}y_1y_2 + \frac{5}{81}y_5y_1 + \frac{1}{27}y_2^2 + \frac{4}{27}y_5y_2 + \frac{1}{12}y_5^2 - \frac{1}{6} \right) \frac{\partial}{\partial y_4} \\ &+ \left( -\frac{7}{27} - \frac{5}{162}y_1^2 - \frac{2}{81}y_1y_2 + \frac{2}{81}y_2^2 \right) \frac{\partial}{\partial y_5} + O(|y|^3) \end{aligned}$$

$$\text{Step 4.} \quad x_5 = y_5 + \frac{2}{24}y_1^3 - \frac{2}{162}y_1^2y_2 - \frac{5}{162}y_1y_2^2,$$

$$F_1(y) = \frac{\partial}{\partial y_1} + O(|y|^3)$$

$$\begin{aligned} F_2(y) &= \frac{\partial}{\partial y_2} + \left( \frac{5}{81}y_1 + \frac{4}{27}y_2 + \frac{1}{6}y_5 \right) \frac{\partial}{\partial y_3} \\ &+ \left( \frac{83}{8748}y_1^2 + \frac{1}{12}y_5^2 + \frac{4}{27}y_5y_2 + \frac{1}{27}y_2^2 - \frac{1}{6} + \frac{17}{486}y_1y_2 + \frac{5}{81}y_5y_1 \right) \frac{\partial}{\partial y_4} \\ &+ \left( -\frac{1}{54}y_1^2 + \frac{1}{27}y_1y_2 + \frac{2}{81}y_2^2 - \frac{7}{27} \right) \frac{\partial}{\partial y_5} + O(|y|^3) \end{aligned}$$

The next change of variables will let  $F_1$  invariant up to order  $O(|x|^3)$ .

$$\text{Step 5.} \quad x_5 = y_5 - \frac{7}{27}y_2,$$

$$F_1(y) = \frac{\partial}{\partial y_1} + O(|y|^3)$$

$$\begin{aligned} F_2(y) &= \frac{\partial}{\partial y_2} + \left( \frac{5}{81}y_1 + \frac{17}{162}y_2 + \frac{1}{6}y_5 \right) \frac{\partial}{\partial y_3} \\ &+ \left( \frac{1}{12}y_5^2 + \frac{17}{162}y_5y_2 + \frac{37}{8748}y_2^2 - \frac{1}{6} + \frac{83}{8748}y_1^2 + \frac{83}{4374}y_1y_2 + \frac{5}{81}y_5y_1 \right) \frac{\partial}{\partial y_4} \\ &+ \left( -\frac{1}{54}y_1^2 + \frac{1}{27}y_1y_2 + \frac{2}{81}y_2^2 \right) \frac{\partial}{\partial y_5} + O(|y|^3) \end{aligned}$$

$$\text{Step 6.} \quad x_3 = \frac{5}{81}y_3 + \frac{17}{324}y_2^2 + \frac{1}{6}y_5y_2,$$

$$F_1(y) = \frac{\partial}{\partial y_1} + O(|y|^3)$$

$$\begin{aligned} F_2(y) &= \frac{\partial}{\partial y_2} + y_1 \frac{\partial}{\partial y_3} \\ &+ \left( \frac{1}{12}y_5^2 + \frac{17}{162}y_5y_2 + \frac{37}{8748}y_2^2 - \frac{1}{6} + \frac{83}{8748}y_1^2 + \frac{83}{4374}y_1y_2 + \frac{5}{81}y_5y_1 \right) \frac{\partial}{\partial y_4} \\ &+ \left( -\frac{1}{54}y_1^2 + \frac{1}{27}y_1y_2 + \frac{2}{81}y_2^2 \right) \frac{\partial}{\partial y_5} + O(|y|^3) \end{aligned}$$



Step 7.  $x_4 = y_4 - y_3 y_2,$

$$F_1(y) = \frac{\partial}{\partial y_1} + O(|y|^3)$$

$$F_2(y) = \frac{\partial}{\partial y_2} + y_1 \frac{\partial}{\partial y_3} + \left( \frac{83}{8748} y_1^2 + \frac{4457}{4374} y_1 y_2 + \frac{5}{81} y_5 y_1 + \frac{37}{8748} y_2^2 + \frac{17}{162} y_5 y_2 + \frac{1}{12} y_5^2 + y_3 - \frac{1}{6} \right) \frac{\partial}{\partial y_4} + \left( -\frac{1}{54} y_1^2 + \frac{1}{27} y_1 y_2 + \frac{2}{81} y_2^2 \right) \frac{\partial}{\partial y_5} + O(|y|^3)$$

Step 8.  $x_4 = y_4 + \frac{37}{3 \times 8748} y_2^3 + \frac{17}{324} y_5 y_2^2 + \frac{1}{12} y_5^2 y_2 - \frac{1}{6} y_2,$

$$F_1(y) = \frac{\partial}{\partial y_1} + O(|y|^3)$$

$$F_2(y) = \frac{\partial}{\partial y_2} + y_1 \frac{\partial}{\partial y_3} + \left( \frac{83}{8748} y_1^2 + \frac{4457}{4374} y_1 y_2 + \frac{5}{81} y_5 y_1 + y_3 \right) \frac{\partial}{\partial y_4} + \left( -\frac{1}{54} y_1^2 + \frac{1}{27} y_1 y_2 + \frac{2}{81} y_2^2 \right) \frac{\partial}{\partial y_5} + O(|y|^3)$$

Step 9.  $x_4 = y_4 - 54 \times \frac{83}{8748} y_5,$

$$F_1(y) = \frac{\partial}{\partial y_1} + O(|y|^3)$$

$$F_2(y) = \frac{\partial}{\partial y_2} + y_1 \frac{\partial}{\partial y_3} + \left( \frac{83}{8748} y_1^2 + \frac{4457}{4374} y_1 y_2 + \frac{5}{81} y_5 y_1 + y_3 \right) \frac{\partial}{\partial y_4} + \left( -\frac{1}{54} y_1^2 + \frac{1}{27} y_1 y_2 + \frac{2}{81} y_2^2 \right) \frac{\partial}{\partial y_5} + O(|y|^3)$$

Step 10.  $x_4 = y_4 + \frac{2270}{2187} y_2 y_3 + \frac{5}{81} y_5 y_3 + \frac{83}{3 \times 6561} y_2^3,$

$$F_1(y) = \frac{\partial}{\partial y_1} + O(|y|^3)$$

$$F_2(y) = \frac{\partial}{\partial y_2} + y_1 \frac{\partial}{\partial y_3} - \frac{83}{2187} y_3 \frac{\partial}{\partial y_4} + \left( -\frac{1}{54} y_1^2 + \frac{1}{27} y_1 y_2 + \frac{2}{81} y_2^2 \right) \frac{\partial}{\partial y_5} + O(|y|^3)$$

Step 11.  $x_4 = -\frac{83}{2187} y_4,$

$$F_1(y) = \frac{\partial}{\partial y_1} + O(|y|^3)$$

$$F_2(y) = \frac{\partial}{\partial y_2} + y_1 \frac{\partial}{\partial y_3} + y_3 \frac{\partial}{\partial y_4} + \left( -\frac{1}{54} y_1^2 + \frac{1}{27} y_1 y_2 + \frac{2}{81} y_2^2 \right) \frac{\partial}{\partial y_5} + O(|y|^3)$$

Step 12.  $x_5 = y_5 + \frac{1}{27} y_3 y_2 + \frac{2}{3 \times 81} y_2^3,$

$$F_1(y) = \frac{\partial}{\partial y_1} + O(|y|^3)$$

$$F_2(y) = \frac{\partial}{\partial y_2} + y_1 \frac{\partial}{\partial y_3} + y_3 \frac{\partial}{\partial y_4} + \left( -\frac{1}{54} y_1^2 - \frac{1}{27} y_3 \right) \frac{\partial}{\partial y_5} + O(|y|^3)$$

Step 13.  $x_5 = -\frac{1}{54}y_5 - \frac{1}{27}y_4,$

$$F_1(y) = \frac{\partial}{\partial y_1} + O(|y|^3)$$

$$F_2(y) = \frac{\partial}{\partial y_2} + y_1 \frac{\partial}{\partial y_3} + y_3 \frac{\partial}{\partial y_4} + y_1^2 \frac{\partial}{\partial y_5} + O(|y|^3)$$

### 3.3 Integration of extremal trajectories

For two vector fields  $F$  and  $G$ , we use the following Lie bracket convention

$$[F, G](x) = \frac{\partial F}{\partial x}(x)G(x) - \frac{\partial G}{\partial x}(x)F(x)$$

Computing we have

$$F_1(x) = \frac{\partial}{\partial x_1}, \quad F_2(x) = \frac{\partial}{\partial x_2} + x_1 \frac{\partial}{\partial x_3} + x_3 \frac{\partial}{\partial x_4} + x_1^2 \frac{\partial}{\partial x_5},$$

$$[F_1, F_2](x) = -\frac{\partial}{\partial x_3} - 2x_1 \frac{\partial}{\partial x_5}, \quad [[F_1, F_2], F_1](x) = -2 \frac{\partial}{\partial x_5},$$

$$[[F_1, F_2], F_2](x) = \frac{\partial}{\partial x_4}.$$

All brackets of length superior than 3 are zero.

We introduce

$$H_1 = \langle p, F_1(x) \rangle = p_1, \quad H_2 = \langle p, F_2(x) \rangle = p_2 + p_3 x_1 + p_4 x_3 + p_5 x_1^2,$$

$$H_3 = \langle p, [F_1, F_2](x) \rangle = -p_3 - 2x_1 p_5, \quad H_4 = \langle p, [[F_1, F_2], F_1](x) \rangle = -2p_5,$$

$$H_5 = \langle p, [[F_1, F_2], F_2](x) \rangle = p_4.$$

We recall the following relation for the Poisson brackets of two lifting Hamiltonians  $H_F$  and  $H_G$  of vector fields  $F$  and  $G$ .

If

$$H_F = \langle p, F(x) \rangle, \quad H_G = \langle p, G(x) \rangle,$$

are the Hamiltonian lifts of vector fields of  $F$  and  $G$ , then we have

$$\{H_F, H_G\} = \langle p, [F, G](x) \rangle.$$

We consider the SR-Cartan flat case [13], [5]

$$\dot{x} = \sum_{i=1}^2 u_i F_i, \quad \min_u \int_0^T (u_1^2 + u_2^2) dt.$$

**Normal case.** The pseudo Hamiltonian is

$$\mathbf{H} = \sum_i u_i H_i - \frac{1}{2}(u_1^2 + u_2^2).$$

The Pontryagin Maximum Principle [11] gives  $u_i = H_i$ .

Hence, the true Hamiltonian is

$$H = \frac{1}{2}(H_1^2 + H_2^2).$$

Computing we have

$$\dot{H}_1 = dH_1(\vec{H}) = \{H_1, H_2\}H_2 = \langle p, [F_1, F_2](x) \rangle = H_2H_3,$$

$$\dot{H}_2 = -2p_5, \quad \dot{H}_3 = H_1H_4 + H_2H_5 = p_4,$$

$$\dot{H}_4 = 0 \quad \text{hence} \quad H_4 = c_4, \quad \dot{H}_5 = 0 \quad \text{hence} \quad H_5 = c_5.$$

Fixing the level energy,  $H_1^2 + H_2^2 = 1$  we set  $H_1 = \cos(\theta)$  and  $H_2 = \sin(\theta)$ .

$$\dot{H}_1 = -\sin(\theta)\dot{\theta} = H_2H_3 = \sin(\theta)H_3.$$

Hence  $\dot{\theta} = -H_3$  and

$$\ddot{\theta} = -(H_1c_4 + H_2c_5) = -c_4\cos(\theta) - c_5\sin(\theta) = -A\sin(\theta + \phi)$$

where  $A$  is a constant.

By identification, we get  $A\sin(\phi) = c_4$  and  $A\cos(\phi) = c_5$ .

Let  $\psi = \theta + \phi$ , we get

$$\frac{1}{2}\dot{\psi}^2 - A\cos(\psi) = B, \tag{3.7}$$

where  $B$  is a constant.

We have the two following cases :

*Oscillating case*

$$\dot{\psi}^2 = 4A \left( \frac{1}{2} + \frac{B}{2A} - \sin^2(\psi/2) \right).$$

We introduce  $\omega^2 = A$  and  $k^2 = \frac{1}{2} + \frac{B}{2A}$  with  $0 < k^2 < 1$ , and we obtain [9]

$$\sin(\psi/2) = k \operatorname{sn}(u, k), \quad \cos(\psi/2) = \operatorname{dn}(u, k)$$

where  $u = \omega t + \varphi_0$ .

$H_1$  and  $H_2$  are elliptic functions of the first kind. Therefore the system becomes

$$\begin{aligned} \dot{x}_1 &= H_1, & \dot{x}_2 &= H_2, & \dot{x}_3 &= H_2x_1, \\ \dot{x}_4 &= H_2x_3, & \dot{x}_5 &= H_2x_1^2. \end{aligned} \tag{3.8}$$

The next step is to compute the  $x$  variables using quadratures in the oscillating case. Since  $x(0) = 0$ , solutions depend upon 4 independent parameters  $H_i(0)$  for  $i = 1, \dots, 5$  coupled with the relation  $H_1^2(0) + H_2(0)^2 = 1$ .

We need the following formulae [9]

$$\begin{aligned}
\int \operatorname{dn}^2(u) du &= E(u), & \int \operatorname{cn}(u) \operatorname{dn}(u) du &= \operatorname{sn}(u), \\
\int \operatorname{cn}(u) du &= \frac{1}{k} \arctan \left( k \frac{\operatorname{sn}(u)}{\operatorname{dn}(u)} \right), & \int u \operatorname{sn}(u) \operatorname{dn}(u) du &= \frac{1}{k} \arctan \left( k \frac{\operatorname{sn}(u)}{\operatorname{dn}(u)} \right) - u \operatorname{cn}(u), \\
\int \operatorname{cn}(u) \operatorname{sn}(u) \operatorname{dn}(u) du &= -\frac{1}{2} \operatorname{cn}(u)^2, \\
\int E(u) \operatorname{sn}(u) \operatorname{dn}(u) du &= -E(u) \operatorname{cn}(u) + \frac{1}{2k} \arctan \left( k \frac{\operatorname{sn}(u)}{\operatorname{dn}(u)} \right) + \frac{1}{2} \operatorname{dn}(u) \operatorname{sn}(u), \\
\int \operatorname{cn}(u) \operatorname{dn}^2(u) du &= \frac{1}{2k} \arctan \left( k \frac{\operatorname{sn}(u)}{\operatorname{dn}(u)} \right) + \frac{1}{2} \operatorname{dn}(u) \operatorname{sn}(u), \\
\int E(u) \operatorname{dn}^2(u) du &= \frac{1}{2} E(u)^2, & \int \operatorname{cn}^2(u) du &= \frac{E(u) + (k^2 - 1)u}{k^2}, \\
\int \operatorname{sn}^4(u) du &= \frac{1}{3k^4} (k^2 \operatorname{sn}(u) \operatorname{dn}(u) \operatorname{cn}(u) + (2 + k^2)u - 2(k^2 + 1)E(u)), \\
\int E(u) \operatorname{dn}(u) \operatorname{sn}(u) \operatorname{cn}(u) du &= \frac{1}{6k^2} ((1 - 2k^2)E(u) + (k^2 - 1)u \\
&\quad + k^2 \operatorname{dn}(u) \operatorname{sn}(u) \operatorname{cn}(u) + 3k^2 \operatorname{sn}^2(u)E(u)), \\
\int E(u)^2 \operatorname{dn}(u) \operatorname{sn}(u) du &= -\operatorname{cn}(u)E(u)^2 + 2 \int E(u) \operatorname{dn}(u)^2 \operatorname{cn}(u) du \\
&= -\operatorname{cn}(u)E(u)^2 + E(u) \operatorname{dn}(u) \operatorname{sn}(u) - 1/3 \operatorname{cn}(u)(-3 + 2k^2 + k^2 \operatorname{sn}^2(u)) + \int E(u) \operatorname{cn}(u) du, \\
\int \operatorname{sn}^3(u) \operatorname{dn}(u) du &= -1/3 \operatorname{cn}(u)(2 + \operatorname{sn}^2(u)), \\
\int u \operatorname{dn}^2(u) \operatorname{cn}(u) du &= 1/2 (\operatorname{dn}(u) \operatorname{sn}(u) + \operatorname{cn}(u)) + \int u \operatorname{cn}(u) du, \\
\int u^2 \operatorname{dn}(u) \operatorname{sn}(u) du &= -u^2 \operatorname{cn}(u) + 2 \int u \operatorname{cn}(u) du, \\
\int u \operatorname{sn}(u) \operatorname{cn}(u) \operatorname{dn}(u) du &= \frac{1}{2} \left( \frac{E(u) + (k^2 - 1)u}{k^2} - u \operatorname{cn}^2(u) \right), \\
\int u E(u) \operatorname{sn}(u) \operatorname{dn}(u) du &= u(-E(u) \operatorname{cn}(u) + 1/2 \operatorname{dn}(u) \operatorname{sn}(u)) \\
&\quad + 1/2 \operatorname{cn}(u) + 1/2 \int u \operatorname{cn}(u) + \int E(u) \operatorname{cn}(u).
\end{aligned} \tag{3.9}$$

Parameterizing (3.8) with respect to  $u$  we have

$$\frac{dx_1}{du} = \cos(\theta(u)) = 2k \sin(\phi) \operatorname{sn}(u) \operatorname{dn}(u) + (2 \operatorname{dn}^2(u) - 1) \cos(\phi). \tag{3.10}$$

$$\frac{dx_2}{du} = \sin(\theta(u)) = 2k \cos(\phi) \operatorname{sn}(u) \operatorname{dn}(u) + (-2 \operatorname{dn}^2(u) + 1) \sin(\phi). \tag{3.11}$$

$$\begin{aligned}
\frac{dx_3}{du} &= \sin(\theta(u))x_1(u) = -4k^2 \sin(\phi) \cos(\phi) \operatorname{cn}(u) \operatorname{sn}(u) \operatorname{dn}(u) \\
&+ 2x_1(\phi_0)k \cos(\phi) \operatorname{sn}(u) \operatorname{dn}(u) + (-2 \operatorname{dn}^2(u) + 1)x_1(\phi_0) \sin(\phi) \\
&+ (-2u \operatorname{sn}(u) \operatorname{dn}(u) - 4 \operatorname{cn}(u) \operatorname{dn}^2(u) + 4 \operatorname{sn}(u) \operatorname{dn}(u)E(u) \\
&+ 2 \operatorname{cn}(u))k \cos^2(\phi) + (2u \operatorname{dn}^2(u) - 4 \operatorname{dn}^2(u)E(u) - u \\
&+ 2E(u)) \sin(\phi) \cos(\phi) \\
&+ (4 \operatorname{cn}(u) \operatorname{dn}^2(u) - 2 \operatorname{cn}(u))k.
\end{aligned} \tag{3.12}$$

$$\begin{aligned}
\frac{dx_4}{du} &= \sin(\theta(u))x_3(u) = 4k^4 \cos^3(\phi) \operatorname{sn}^4(u) - 8k^4 \cos(\phi) \operatorname{sn}^4(u) \\
&- 4k^2 x_1(\phi_0) \cos^2(\phi) \operatorname{cn}(u) \operatorname{sn}(u) \operatorname{dn}(u) + 2x_3(\phi_0)k \cos(\phi) \operatorname{sn}(u) \operatorname{dn}(u) \\
&+ 4k^3 \sin(\phi) \operatorname{sn}(u) \operatorname{dn}(u) + (-u^2 \operatorname{sn}(u) \operatorname{dn}(u) - 4u \operatorname{cn}(u) \operatorname{dn}^2(u) \\
&+ 4u \operatorname{sn}(u) \operatorname{dn}(u)E(u) + 8 \operatorname{cn}(u) \operatorname{dn}^2(u)E(u) - 4 \operatorname{sn}(u) \operatorname{dn}(u)E(u)^2 \\
&+ 2u \operatorname{cn}(u) - 4 \operatorname{cn}(u)E(u) - 2 \operatorname{sn}(u) \operatorname{dn}(u))k \sin(\phi) \\
&+ (4u \operatorname{cn}(u) \operatorname{sn}(u) \operatorname{dn}(u) - 8 \operatorname{cn}(u) \operatorname{sn}(u) \operatorname{dn}(u)E(u) \\
&+ 4 \operatorname{dn}^2(u) - 2)k^2 \cos^3(\phi) + (-2x_3(\phi_0) \operatorname{dn}^2(u) + x_3(\phi_0)) \sin(\phi) \\
&+ (u^2 \operatorname{sn}(u) \operatorname{dn}(u) + 4u \operatorname{cn}(u) \operatorname{dn}^2(u) - 4u \operatorname{sn}(u) \operatorname{dn}(u)E(u) \\
&- 8 \operatorname{cn}(u) \operatorname{dn}^2(u)E(u) + 4 \operatorname{sn}(u) \operatorname{dn}(u)E(u)^2 - 2u \operatorname{cn}(u) \\
&+ 4 \operatorname{cn}(u)E(u))k \sin^3(\phi) + (2u \operatorname{dn}^2(u) - 4 \operatorname{dn}^2(u)E(u) - u \\
&+ 2E(u))x_1(\phi_0) \cos^2(\phi) + (-u^2 \operatorname{dn}^2(u) + 4u \operatorname{dn}^2(u)E(u) \\
&- 4 \operatorname{dn}^2(u)E(u)^2 + 1/2 u^2 - 2uE(u) + 2 \operatorname{dn}^2(u) + 2E(u)^2 - 2) \cos^3(\phi) \\
&+ (u^2 \operatorname{dn}^2(u) - 4u \operatorname{dn}^2(u)E(u) + 4 \operatorname{dn}^2(u)E(u)^2 \\
&- 1/2 u^2 + 2uE(u) - 6 \operatorname{dn}^2(u) - 2E(u)^2 + 6) \cos(\phi) \\
&+ (2u \operatorname{sn}(u) \operatorname{dn}(u) + 4 \operatorname{cn}(u) \operatorname{dn}^2(u) - 4 \operatorname{sn}(u) \operatorname{dn}(u)E(u) \\
&- 2 \operatorname{cn}(u)) \cos(\phi)k x_1(\phi_0) \sin(\phi) + (-4 \operatorname{dn}^2(u) + 2)k^2 \cos(\phi) \\
&+ (4 \operatorname{sn}^3(u) \operatorname{dn}(u) - 4 \operatorname{sn}(u) \operatorname{dn}(u))k^3 \sin^3(\phi) \\
&+ (-2u \operatorname{dn}^2(u) + 4 \operatorname{dn}^2(u)E(u) + u - 2E(u))x_1(\phi_0).
\end{aligned} \tag{3.13}$$

$$\begin{aligned}
\frac{dx_5}{du} = & -8k^4 \sin^3(\phi) \operatorname{sn}^4(u) - 8k^2 x_1(\phi_0) \sin(\phi) \cos(\phi) \operatorname{cn}(u) \operatorname{sn}(u) \operatorname{dn}(u) \\
& + 2x_1(\phi_0)^2 k \cos(\phi) \operatorname{sn}(u) \operatorname{dn}(u) + (-8u \operatorname{cn}(u) \operatorname{sn}(u) \operatorname{dn}(u) \\
& + 16 \operatorname{cn}(u) \operatorname{sn}(u) \operatorname{dn}(u) E(u) - 8 \operatorname{dn}^2(u) + 4) k^2 \sin^3(\phi) \\
& + (4u \operatorname{dn}^2(u) - 8 \operatorname{dn}^2(u) E(u) - 2u + 4 E(u)) x_1(\phi_0) \sin(\phi) \cos(\phi) \\
& + (-2u^2 \operatorname{dn}^2(u) + 8u \operatorname{dn}^2(u) E(u) - 8 \operatorname{dn}^2(u) E(u)^2 \\
& + u^2 - 4u E(u) + 4 E(u)^2) \sin(\phi) + (-2 \operatorname{dn}^2(u) + 1) x_1(\phi_0)^2 \sin(\phi) \\
& + (8u \operatorname{cn}(u) \operatorname{sn}(u) \operatorname{dn}(u) - 16 \operatorname{cn}(u) \operatorname{sn}(u) \operatorname{dn}(u) E(u)) k^2 \sin(\phi) \\
& + (2u^2 \operatorname{sn}(u) \operatorname{dn}(u) + 8u \operatorname{cn}(u) \operatorname{dn}^2(u) - 8u \operatorname{sn}(u) \operatorname{dn}(u) E(u) \\
& - 16 \operatorname{cn}(u) \operatorname{dn}^2(u) E(u) + 8 \operatorname{sn}(u) \operatorname{dn}(u) E(u)^2 - 4u \operatorname{cn}(u) \\
& + 8 \operatorname{cn}(u) E(u)) k \cos^3(\phi) + (8 \operatorname{sn}^3(u) \operatorname{dn}(u) - 8 \operatorname{sn}(u) \operatorname{dn}(u)) k^3 \cos^3(\phi) \\
& + (-4u \operatorname{sn}(u) \operatorname{dn}(u) - 8 \operatorname{cn}(u) \operatorname{dn}^2(u) + 8 \operatorname{sn}(u) \operatorname{dn}(u) E(u) \\
& + 4 \operatorname{cn}(u)) x_1(\phi_0) k \cos^2(\phi) + (-8u \operatorname{cn}(u) \operatorname{dn}^2(u) + 16 \operatorname{cn}(u) \operatorname{dn}^2(u) E(u) \\
& + 4u \operatorname{cn}(u) - 8 \operatorname{cn}(u) E(u)) k \cos(\phi) + (-8 \operatorname{sn}^3(u) \operatorname{dn}(u) \\
& + 8 \operatorname{sn}(u) \operatorname{dn}(u)) k^3 \cos(\phi) + (2u^2 \operatorname{dn}^2(u) - 8u \operatorname{dn}^2(u) E(u) \\
& + 8 \operatorname{dn}^2(u) E(u)^2 - u^2 + 4u E(u) - 4 \operatorname{dn}^2(u) - 4 E(u)^2 + 4) \sin^3(\phi) \\
& + (8 \operatorname{cn}(u) \operatorname{dn}^2(u) - 4 \operatorname{cn}(u)) x_1(\phi_0) k.
\end{aligned} \tag{3.14}$$

**Proposition 3.1.** *The solution  $\mathbf{x}(u)$  of the system (3.8) can be expressed as a polynomial function of  $(u, \operatorname{sn}(u), \operatorname{cn}(u), \operatorname{dn}(u), E(u))$ .*

*Proof.* Integrating equations (3.10) to (3.14) thanks to formulae (3.9) gives the result.  $\square$

**Remark 3.2.** [9]

- $\operatorname{sn}, \operatorname{cn}$  are  $4K$ -periodics,
- $\operatorname{dn}$  is  $2K$ -periodic,
- $E(u) = \frac{E}{K}u + Z(u)$  where  $E, K$  are complete integrals and  $Z$  is the  $2K$ -periodic zeta function.

The final expressions of the solution  $(x_i(t))_{i=1,\dots,5}$  of (3.8).

- $x_1(t) = \omega [x_1(\phi_0) - 2k \sin(\phi) \operatorname{cn}(u) + (-u + 2 E(u)) \cos(\phi)].$
- $x_2(t) = \omega [x_2(\phi_0) - 2k \cos(\phi) \operatorname{cn}(u) + (u - 2 E(u)) \sin(\phi)].$
- $x_3(t) = \omega [x_3(\phi_0) + 2k \operatorname{sn}(u) \operatorname{dn}(u) - 2k x_1(\phi_0) \cos(\phi) \operatorname{cn}(u) + 2k^2 \sin(\phi) \cos(\phi) \\ + (-1/2 u^2 + 2u E(u) + 2 \operatorname{dn}^2(u) - 2 E(u)^2 - 2) \sin(\phi) \cos(\phi) \\ + (2u \operatorname{cn}(u) - 4 \operatorname{cn}(u) E(u)) \cos^2(\phi) k + (u - 2 E(u)) x_1(\phi_0) \sin(\phi)].$

- $$\begin{aligned}
x_4(t) = \omega & [x_4(\phi_0) + 4/3 k^3 \sin^3(\phi) \operatorname{cn}(u) - 2 x_3(\phi_0) k \cos(\phi) \operatorname{cn}(u) - 4 k^3 \sin(\phi) \operatorname{cn}(u) \\
& + (-u^2 \operatorname{cn}(u) + 4u \operatorname{cn}(u) E(u) + 4/3 \operatorname{cn}(u) \operatorname{dn}^2(u) - 4 \operatorname{cn}(u) (E(u))^2 \\
& - 4/3 \operatorname{cn}(u)) k \sin^3(\phi) + (x_3(\phi_0) u - 2 x_3(\phi_0) E(u)) \sin(\phi) + (-2u \operatorname{cn}(u) \\
& + 4 \operatorname{cn}(u) E(u)) k x_I(\phi_0) \cos(\phi) \sin(\phi) + (1/6 u^3 - u^2 E(u) - 2u \operatorname{dn}^2(u) \\
& + 2u (E(u))^2 + 4 \operatorname{dn}^2(u) E(u) - 4/3 (E(u))^3 + 2u - 4 E(u)) \cos^3(\phi) \\
& + (u^2 \operatorname{cn}(u) - 4u \operatorname{cn}(u) E(u) + 4 \operatorname{cn}(u) (E(u))^2 + 2 \operatorname{cn}(u)) k \sin(\phi) \\
& + (-1/2 u^2 + 2u E(u) + 2 \operatorname{dn}^2(u) - 2 (E(u))^2 - 2) x_I(\phi_0) \cos^2(\phi) \\
& + 2 \cos^2(\phi) x_I(\phi_0) k^2 + (-1/6 u^3 + u^2 E(u) - 2u (E(u))^2 + 4/3 (E(u))^3 + 2/3 u \\
& - 2/3 E(u)) \cos(\phi) + (-2u + 4 E(u)) k^2 \cos^3(\phi) + (-8/3 \operatorname{cn}(u) \operatorname{sn}(u) \operatorname{dn}(u) \\
& - 2/3 u + 4/3 E(u)) k^2 \cos(\phi) + (1/2 u^2 - 2u E(u) + 2 (E(u))^2) x_I(\phi_0)].
\end{aligned}$$
- $$\begin{aligned}
x_5(t) = \omega & [x_5(\phi_0) + 4 x_I(\phi_0) k \operatorname{sn}(u) \operatorname{dn}(u) - 8 \cos(\phi) k^3 \operatorname{cn}(u) \\
& + 8/3 \cos^3(\phi) k^3 \operatorname{cn}(u) - 2 x_I(\phi_0)^2 k \cos(\phi) \operatorname{cn}(u) + (4u - 8 E(u)) k^2 \sin^3(\phi) \\
& + (-u^2 + 4u E(u) + 4 \operatorname{dn}^2(u) - 4 (E(u))^2 - 4) x_I(\phi_0) \sin(\phi) \cos(\phi) \\
& + (-8/3 \operatorname{cn}(u) \operatorname{sn}(u) \operatorname{dn}(u) - 8/3 u + 16/3 E(u)) k^2 \sin(\phi) \\
& + (u - 2 E(u)) x_I(\phi_0)^2 \sin(\phi) + 4 k^2 x_I(\phi_0) \sin(\phi) \cos(\phi) + (4u \operatorname{cn}(u) \\
& - 8 \operatorname{cn}(u) E(u)) k x_I(\phi_0) \cos^2(\phi) + (-4u \operatorname{sn}(u) \operatorname{dn}(u) + 8 \operatorname{sn}(u) \operatorname{dn}(u) E(u) \\
& + 4 \operatorname{cn}(u)) k \cos(\phi) + (-2u^2 \operatorname{cn}(u) + 8u \operatorname{cn}(u) E(u) + 8/3 \operatorname{cn}(u) \operatorname{dn}^2(u) \\
& - 8 \operatorname{cn}(u) (E(u))^2 - 8/3 \operatorname{cn}(u)) k \cos^3(\phi) + (1/3 u^3 - 2u^2 E(u) \\
& - 4u \operatorname{dn}^2(u) + 4u (E(u))^2 + 8 \operatorname{dn}^2(u) E(u) - 8/3 (E(u))^3 + 8/3 u \\
& - \frac{20}{3} E(u)) \sin(\phi) + (-1/3 u^3 + 2u^2 E(u) + 4u \operatorname{dn}^2(u) - 4u (E(u))^2 \\
& - 8 \operatorname{dn}^2(u) E(u) + 8/3 (E(u))^3 - 4u + 8 E(u)) \sin^3(\phi)].
\end{aligned}$$

### *Rotating case*

We can perform the same computations as in the oscillating case.

## 3.4 Numerical results

We present now some numerical simulations on the three-link swimmer, performed with the toolbox BOCOP ([www.bocop.org](http://www.bocop.org), [2]). BOCOP implements a so-called direct transcription method, namely a time discretization that allows the state and control variables to be represented by vectors. The optimal control problem can therefore be rewritten as a finite dimensional optimization problem (i.e nonlinear programming), and solved by an interior point method (IPOPT). We recall below the optimal control

problem, formulated with the state  $\mathbf{z} = (\alpha_1, \alpha_2, x, y, \theta)$  and control  $u = (\dot{\alpha}_1, \dot{\alpha}_2)$

$$(OCP) \begin{cases} \text{Minimize } \int_0^T |u(t)|^2 dt \\ \dot{\mathbf{z}}(t) = \mathbf{F}_1(\mathbf{z}(t)) u_1(t) + \mathbf{F}_2(\mathbf{z}(t)) u_2(t) \\ \alpha_{1|2}(t) \in [-a, a] \\ x(0) = y(0) = 0, x(T) = x_f \\ y(T) = y(0), \theta(T) = \theta(0), \alpha_{1|2}(T) = \alpha_{1|2}(0) \end{cases}$$

For the simulations we set  $T = 10$  and  $x_f = 0.5$ , and the bounds  $a = 3$  were chosen large enough so that the solution is actually unconstrained. Using a midpoint time discretization with 500 steps, the optimization took less than one minute on a standard desktop computer. The state and control for the optimal trajectory are shown on Fig.2-3 and Fig.4. We observe that it is actually a sequence of three identical strokes. The phase portrait for the angles  $\alpha_1, \alpha_2$  is an ellipse, while the control satisfies the constant energy level with  $u_1^2 + u_2^2 \approx 4.76$ , see Fig.5. The adjoint variables (or more accurately, the multipliers associated to the discretized dynamics) are shown on Fig.6-7. For the energy criterion case, the main difference is that the constant energy level (satisfied by the trajectory) does not mean  $|u|^2 = C$  anymore, thus control is not circle-shaped.

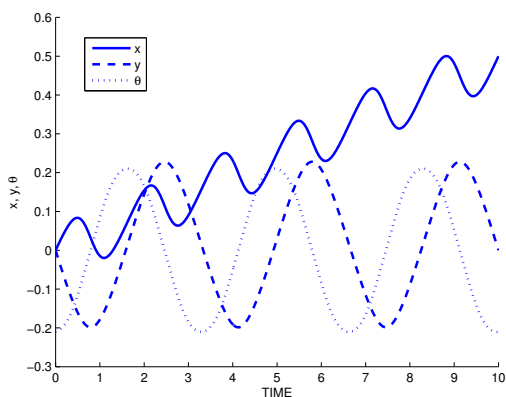
## 4 First and second order optimality conditions

First order necessary conditions for optimality (e.g. the Pontryagin Maximum Principle) and second order conditions play a crucial role in the selection and the characterization of solutions (minimizers) for problems in optimal control. Very general versions of first and second order optimality conditions are now available. However, here we restrict attention to optimal control problems, with a (partial) periodic endpoint constraint, of the form

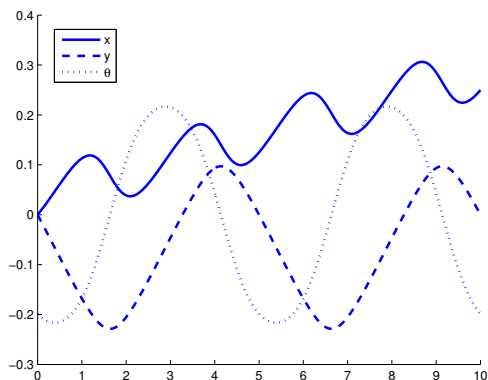
$$(P) \begin{cases} \text{Minimize } \int_0^T L(x(t), u(t)) dt \\ \text{subject to} \\ \dot{x}(t) = f(x(t), u(t)) \quad \text{a.e. } t \in [0, T], \\ u(t) \in U \quad \text{a.e. } t \in [0, T], \\ x'(0) = x'_0, x'(T) = x'_T, x''(0) = x''(T), \end{cases}$$

in which  $x = (x', x'') \in \mathbb{R}^k \times \mathbb{R}^{n-k}$  for some fixed integer  $0 \leq k < n$ ,  $x'_0, x'_T \in \mathbb{R}^k$  are given points,  $f(\cdot, \cdot) : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$  and  $L(\cdot, \cdot) : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$  are given functions of class  $\mathcal{C}^2$  with continuous second derivatives w.r.t.  $(x, u)$  variables, and  $U \subset \mathbb{R}^m$  is a given set. Observe that  $x''(\cdot)$  represents the periodic component of the state trajectory  $x(\cdot)$ .





$|u|^2$  criterion

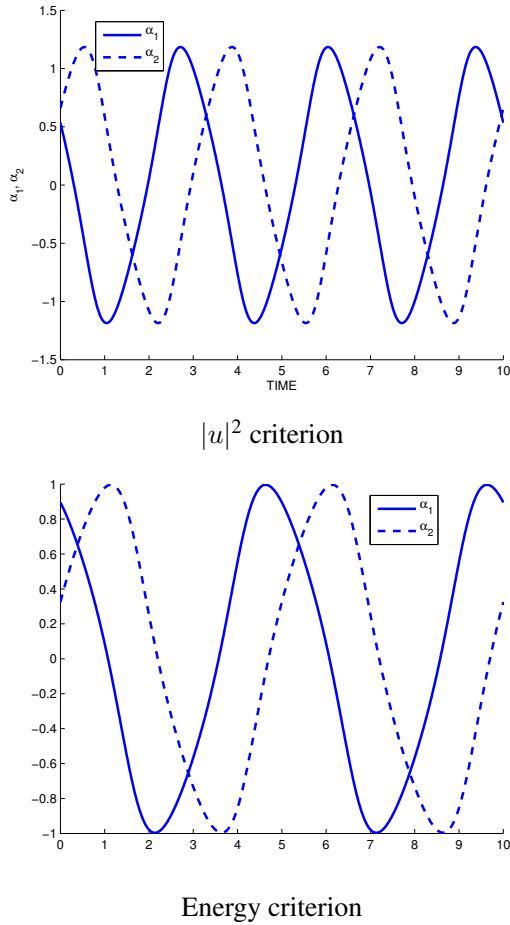


Energy criterion

Figure 2. Optimal trajectory: state variables  $x, y, \theta$

Take an optimal trajectory/control couple  $(\bar{x}(\cdot), \bar{u}(\cdot))$  for (P). The Pontryagin Maximum Principle (see e.g. [11]) asserts (under appropriate hypotheses) that there exist a vector-valued function  $p(\cdot) \in W^{1,1}([0, T]; \mathbb{R}^n)$  and  $\lambda \geq 0$  such that

- (i)  $(p(\cdot), \lambda) \neq (0, 0)$  (The Nontriviality Condition),
- (ii)  $-\dot{p}(t) = p^T(t) \frac{\partial f}{\partial x}(\bar{x}(t), \bar{u}(t)) - \lambda \frac{\partial L}{\partial x}(\bar{x}(t), \bar{u}(t))$  a.e.  
(The Adjoint System),
- (iii)  $\langle p(t), f(\bar{x}(t), \bar{u}(t)) \rangle - \lambda L(\bar{x}(t), \bar{u}(t)) = \max_{u \in U} \{ \langle p(t), f(\bar{x}(t), u) \rangle - \lambda L(\bar{x}(t), u) \}$  a.e.  
(The Weierstrass or ‘Maximization of the Hamiltonian’ Condition),
- (iv)  $p''(0) = p''(T)$  (The Transversality Condition).

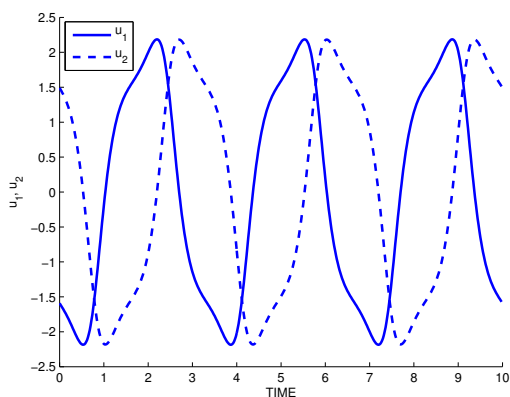
Figure 3. Optimal trajectory: state variables  $\alpha_1, \alpha_2$ 

Notice that the transversality condition (iv) involves only the component  $p''(\cdot)$  of the adjoint arc  $p(\cdot)$ , which is associated with the ‘periodic component’ of the minimizer  $\bar{x}(\cdot)$ , that is  $\bar{x}''(\cdot)$ .

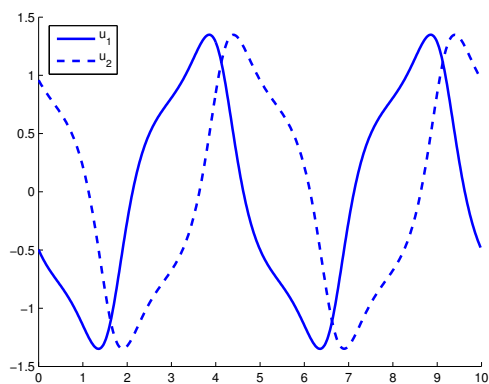
We shall consider the necessary conditions above in the normal form. ‘Normal’ means that the Maximum Principle is valid with the Lagrange multiplier  $\lambda$  (associated with the objective function) different from zero. In this case it is not restrictive to take  $\lambda = 1/2$ , by standard normalization. In these conditions the pseudo Hamiltonian (also referred to as ‘unmaximized’ Hamiltonian)  $\mathbf{H} : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^m$  is the function

$$\mathbf{H}(x, p, u) := \langle p, f(x, u) \rangle - \frac{1}{2}L(x, u).$$

Since  $f(\cdot, \cdot)$  is independent of the time variable  $t$ , the asserts of the Maximum Prin-



$|u|^2$  criterion



Energy criterion

Figure 4. Optimal trajectory: control variables

ciple above can be supplemented by a further condition: there exists  $r \in \mathbb{R}$  such that

$$\mathbf{H}(\bar{x}(t), p(t), \bar{u}(t)) = r \quad \text{for all } t.$$

(This condition is known as the *Second Erdmann Condition*.)

In addition, if  $\bar{u}(t)$  belongs to the interior of  $U$  (this holds true whenever we take  $U = \mathbb{R}^m$ ), condition (iii) above can be re-written in the form:

$$\frac{\partial \mathbf{H}}{\partial u}(\bar{x}(t), p(t), \bar{u}(t)) = 0.$$

Second order conditions for optimality in optimal control (and more in general higher-order conditions) play an important role in providing further tests to confirm

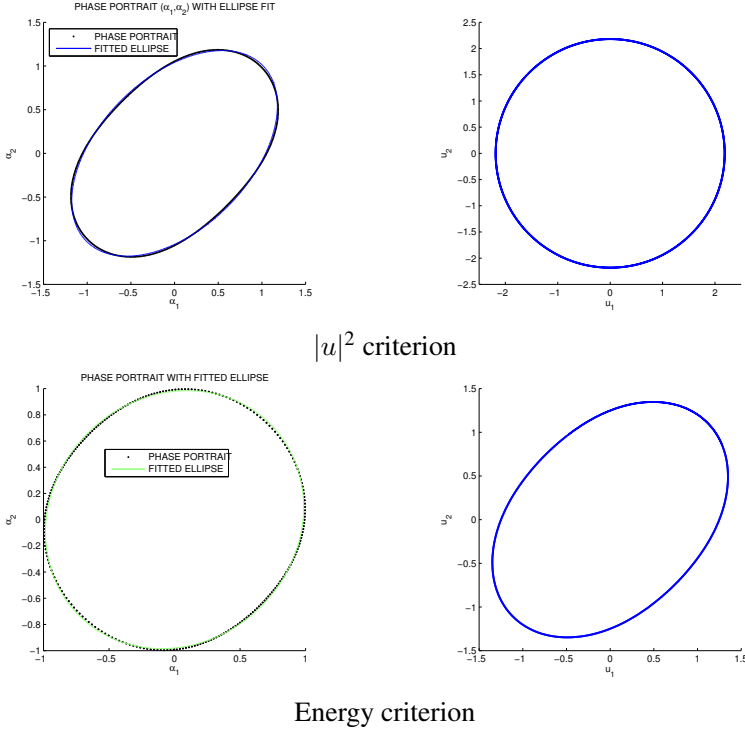


Figure 5. Phase portrait (ellipse) and controls

whether a given normal extremal is in fact a true (local) minimizer. Higher-order conditions are also used to establish finest necessary conditions. Second order optimality conditions are often expressed in terms of the Riccati equation (and its refinements):

$$\begin{cases} \dot{P} + PA + A^T P + Q - (B^T P + D^T)^T R^{-1} (B^T P + D^T) = 0 \\ P^T(\cdot) = P(\cdot), \end{cases} \quad (4.1)$$

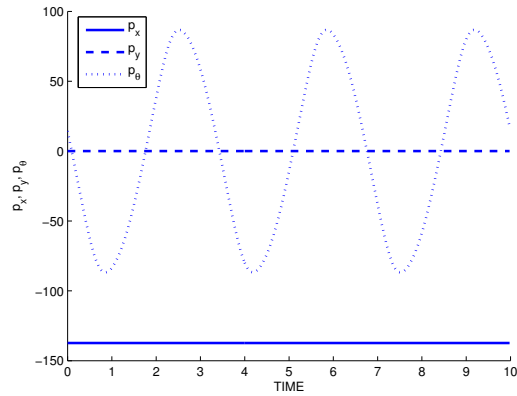
where

$$A(t) := \frac{\partial f}{\partial x}(\bar{x}(t), \bar{u}(t)), \quad B(t) := \frac{\partial f}{\partial u}(\bar{x}(t), \bar{u}(t))$$

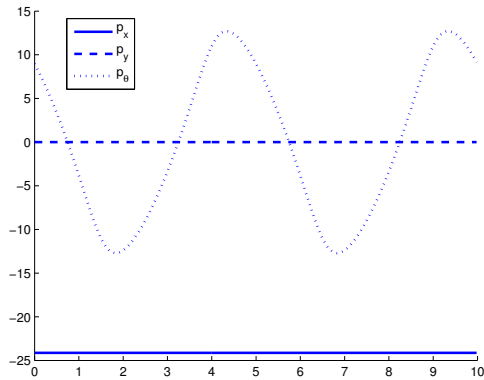
and

$$\begin{pmatrix} Q(t) & D(t) \\ D^T(t) & R(t) \end{pmatrix} := \begin{pmatrix} \frac{\partial^2 \mathbf{H}}{\partial x^2}(\bar{x}(t), p(t), \bar{u}(t)) & \frac{\partial^2 \mathbf{H}}{\partial x u}(\bar{x}(t), p(t), \bar{u}(t)) \\ \frac{\partial^2 \mathbf{H}^T}{\partial x u}(\bar{x}(t), p(t), \bar{u}(t)) & \frac{\partial^2 \mathbf{H}}{\partial u^2}(\bar{x}(t), p(t), \bar{u}(t)) \end{pmatrix}.$$

Second order sufficient conditions for (local) optimality have been extensively investigated to derive optimal solutions with the property to be (locally) unique. This cannot be the case of pure periodic (i.e. when  $k = 0$  in problem (P) above) optimal control



$|u|^2$  criterion



Energy criterion

Figure 6. Optimal trajectory: adjoint variables

problems, in which given any periodic trajectory/control pair, any time translation produces a new periodic trajectory/control pair with the same cost. Therefore there is a growing interest in studying second order conditions in a framework which comprises periodic optimal control problems, and testing them in examples coming from applications (cf. [14], [15]). A more general framework is proposed in [6], where second order optimality conditions for optimal control problems with non-unique minimizers are investigated ('non-unique' in the sense that an extremum is continuously embedded in a family of extrema with the same -constant- cost). This might happen for optimal control problems with partially periodic end-point constraints as (P) above.

A crucial issue in the second order theory (periodic, partially periodic, or non-periodic) for deriving sufficient conditions for optimality is establishing the existence of (sym-

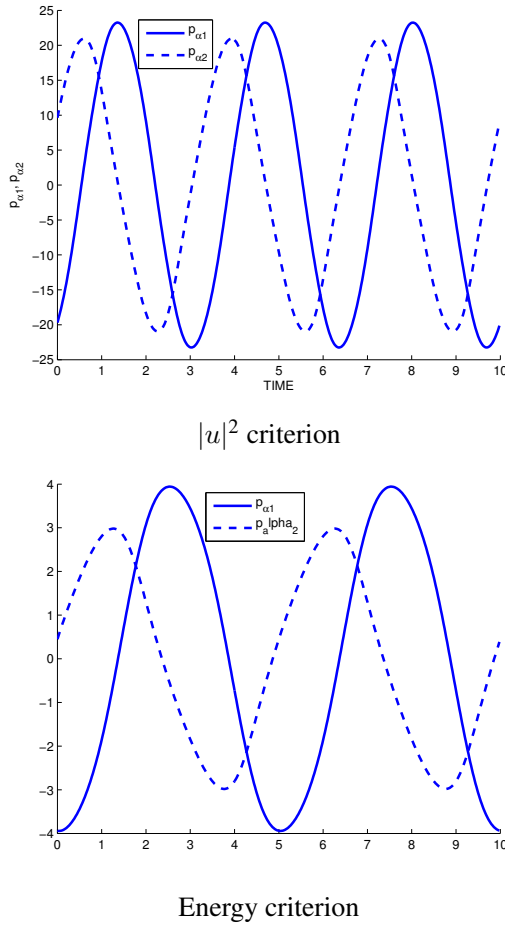


Figure 7. Optimal trajectory: adjoint variables  $p_{\alpha_1}, p_{\alpha_2}$

metric) solutions to the Riccati equation (4.1) on the whole interval  $[0, T]$ . This would mean that no conjugate time appears in the period  $[0, T]$ : the existence of a conjugate time is equivalent to the presence of time in which the solution to the Riccati equation blows up. The concept of conjugate time (and conjugate point) plays a crucial role in optimality conditions, and can be characterized in terms of the degeneracy of the exponential mapping or, equivalently, of the quadratic form associated with the second variation of the endpoint mapping.

In the present paper we focus on some aspects related to the maximum principle applied to a mathematical model of the three links Purcell swimmer. Combining numerical methods and a geometrical approach we investigate crucial features of this model, as integrability of extremals and periodicity of controls. Second order condi-

tions analysis (applied to this example) goes beyond the aim of the present manuscript and represent a primary objective of future work.

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